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CHARACTERIZATIONS OF SEVERAL CLASSES OF PRESERVERS

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Abstract: In this paper, we study a class of linear mappings that triple orthogonality preservers and characterize those linear mappings that preserve the spectrum on algebras of τ-measurable operators. First, we use the property $\mathbb{B}$ to characterize linear mappings that triple orthogonality preservers under slightly weaker assumptions, and obtain that such mappings are generalized Jordan derivations. For the study of linear mappings which preserve the τ-measurable operator spectrum, the result of spectrum-preserving in bounded operators is extended to unbounded operators.

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1 Introduction

Throughout this paper all algebras and vector spaces are over the complex field $\mathbb{C}$, and all algebras are associative with unity, unless indicated otherwise. Suppose that $\mathcal{A}$ is a complex Banach $*$-algebra, and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. We recall that a linear mapping $D : \mathcal{A} \to \mathcal{X}$ is a derivation whenever $D(ab) = D(a)b + aD(b)$, for every $a, b \in \mathcal{A}$. J. Ringrose [1] extends S. Sakai's theorem [2] on automatic continuity of derivations on C^*-algebras by proving that every derivation from a C^*-algebra $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule is continuous.

In Section 2, we first consider a mapping T from an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{X}$ that satisfies the following conditions:

$$ab = bc = 0 \Rightarrow aT(b)c + cT(b)a = 0,$$

and we give several applications of the conclusion.

Our results can be considered as extensions of some of the results in [3] and [4, 5] to more general classes of Banach algebras, as well as new applications of property $\mathbb{B}$ in the sense of [6] for new types of preservers, and complementary results for [7, 8].

Spectrum-preserving linear mappings are studied for the first time by G. Frobenius [9]. In [10], B. Aupetit studies spectrum-preserving mappings on von Neumann algebras. In

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Section 3, we shall consider the spectrum-preserving mappings on algebras of τ-measurable operators, which is a version of a theorem known for von Neumann algebras, but dealing with algebras of unbounded opeators. Theorem 3.6 deals with a factor of type II_1 and Theorem 3.7 with a finite von Neumann algebra. We prove such a mapping is a Jordan $*$-isomorphism.

2 Linear preserving mappings

We recall that a linear mapping G from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is said to be a generalized derivation if there exists $\xi \in \mathcal{X}^{**}$ satisfying

$$G(ab) = G(a)b + aG(b) - a\xi b, (a, b \in \mathcal{A}).$$

We shall say that a linear mapping G from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a Jordan derivation if $G(a \circ b) = G(a) \circ b + a \circ G(b)$ for every $a, b \in \mathcal{A}$, where the Jordan product is given by $a \circ b := \frac{1}{2}(ab + ba)$. G is called a generalized Jordan derivation if there exists $\xi \in \mathcal{X}^{**}$ such that the identity $G(a \circ b) = G(a) \circ b + a \circ G(b) - U_{a,b}(\xi)$, holds for every a, b in $\mathcal{A}$, where $U_{a,b}(z) := (a \circ z) \circ b + (b \circ z) \circ a - (a \circ b) \circ z$. If $\mathcal{A}$ is unital, every generalized (Jordan) derivation $D : \mathcal{A} \to \mathcal{X}$ with $D(1) = 0$ is a (Jordan) derivation, where $U_{a,b}(x) := \frac{1}{2}(axb + bxa)$.

We recall that every C^*-algebra is a JB^*-triple with respect to $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. Whenever we use a triple product on a C^*-algebra, it is always this triple product. More details about JB^*-triple can be found in [11].

We recall that elements a, b in a JB^*-triple $\mathcal{E}$ are said to be *orthogonal* ($a \perp b$ for short) if $L(a, b) = 0$, where $L(a, b)$ is the operator on $\mathcal{E}$ given by $L(a, b)x = \{a, b, x\}$. By [12, Lemma 1], we know that

$$a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0.$$

When a C^*-algebra $\mathcal{A}$ is regarded as a JB^*-triple, it is known that elements a, b in $\mathcal{A}$ are orthogonal if and only if $ab^* = 0 = b^*a$ ([13]). When $\mathcal{A}$ is a commutative C^*-algebra, $a \perp b$ if and only if $ab = 0$.

A complex Banach algebra $\mathcal{A}$ is said to have property $\mathbb{B}$ if for every continuous bilinear mapping $f : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ where $\mathcal{X}$ is an arbitrary Banach space, the condition that for all $x, y \in \mathcal{A}$,

$$xy = 0 \Rightarrow f(x, y) = 0,$$

implies that

$$f(xy, z) = f(x, yz) \quad \text{for all} \quad x, y, z \in \mathcal{A}.$$

It is shown in [6] that many important examples of Banach algebras, including C^*-algebras and group algebras $L^1(G)$ where G is a locally compact group, have property $\mathbb{B}$.

Recently, A. Essaleh and A. Peralta consider in [5] a linear preserver problem on maps which are triple derivable at orthogonal pairs. In this paper, we consider a weaker condition than that in [5].

In what follows, we denote by $\mathcal{A}_{sa}$ the hermitian elements of a Banach $*$-algebra $\mathcal{A}$.

Before giving the next lemma, we first give the following definition, which appears in [4].

Definition 2.1 Let $T : \mathcal{A} \to \mathcal{A}$ be a linear mapping on a C^*-algebra $\mathcal{A}$, and let z be an element in $\mathcal{A}$. We shall say that T is a triple derivation at z if $z = \{a, b, c\}$ in $\mathcal{A}$ implies that

$$T(z) = \{T(a), b, c\} + \{a, T(b), c\} + \{a, b, T(c)\}.$$

Lemma 2.1 Suppose that $\mathcal{A}$ is a unital C^*-algebra. Let $T : \mathcal{A} \to \mathcal{A}$ be a linear mapping satisfying

$$a \perp b \perp c \Rightarrow \{a, T(b), c\} = 0.$$

Then the identity

$$T(p) = pT(p) + T(p)p - pT(1)p \tag{2.1}$$

holds for every idempotent p in $\mathcal{A}$.

Proof Let $a = p, b = 1 - p^*, c = p$, where $p^2 = p$ in $\mathcal{A}$. According to the hypothesis, we have $pT(1 - p^*)^*p = \{p, T(1 - p^*), p\} = 0$, which gives $pT(1)^*p = pT(p^*)^*p$. By applying $*$ to both sides, we get $p^*T(1)p^* = p^*T(p^*)p^*$ for every idempotent p in $\mathcal{A}$. But p^* is an idempotent, so

$$pT(1)p = pT(p)p.$$

By a similar method, let $a = 1 - p^*, b = p, c = 1 - p^*$. Then $(1 - p^*)T(p)^*(1 - p^*) = 0$, so

$$T(p) = pT(p) + T(p)p - pT(p)p.$$

Since $pT(1)p = pT(p)p$,

$$T(p) = pT(p) + T(p)p - pT(1)p.$$

Definition 2.2 A Banach $\mathcal{A}$-bimodule $\mathcal{M}$ is said to be essential if it is equal to the closed linear span of the set of elements of the form $x \cdot m \cdot y$ with $x, y \in \mathcal{A}, m \in \mathcal{M}$.

Definition 2.3 Let $\mathcal{A}$ be a Banach algebra. A left approximate identity for $\mathcal{A}$ is a net $\{\rho_i\}_{i \in I}$ in $\mathcal{A}$ such that

$$\lim_i \rho_i x = x$$

for every $x \in \mathcal{A}$. A right approximate identity for $\mathcal{A}$ is defined similarly. An approximate identity for $\mathcal{A}$ is a net $\{\rho_i\}_{i \in I}$, which is both a left and a right approximate identity for

$\mathcal{A}$. A (left/right) approximate identity $\{\rho_i\}_{i\in I}$ is bounded if for some positive K we have $\|\rho_i\| \leq K$ for every $i \in I$.

Next we give the following proposition, which plays a crucial role in our paper.

Proposition 2.1 Suppose that $\mathcal{A}$ is a Banach algebra satisfying property $\mathbb{B}$ and having a bounded approximate identity $\{\rho_i\}_{i\in I}$. Let $\mathcal{M}$ be an essential Banach $\mathcal{A}$-bimodule, and let $T : \mathcal{A} \to \mathcal{M}$ be a continuous linear mapping satisfying

$$ab = bc = 0 \Rightarrow aT(b)c + cT(b)a = 0. \tag{2.2}$$

Then T is a generalized Jordan derivation.

Proof Fix $a, b \in \mathcal{A}$ with $ab = 0$. Define a continuous bilinear map $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ given by

$$\varphi(x, y) = aT(bx)y + yT(bx)a.$$

When $xy = 0$ in $\mathcal{A}$, we have $abx = 0 = bxy$. Hence $\varphi(x, y) = 0$ whenever $ab = 0, a, b \in \mathcal{A}$. According to the hypothesis

$$\varphi(xy, z) = \varphi(x, yz)$$

for any $x, y, z \in \mathcal{A}$, that is,

$$aT(bxy)z + zT(bxy)a = aT(bx)yz + yzT(bx)a,$$

for all $x, y, z, a, b \in \mathcal{A}$ with $ab = 0$. Fix $x, y, z \in \mathcal{A}$ and define a continuous bilinear mapping on $\mathcal{A}$ by

$$\Phi(a, b) = aT(bxy)z + zT(bxy)a - aT(bx)yz + yzT(bx)a.$$

Hence, $ab = 0$ in $\mathcal{A}$ implies $\Phi(a, b) = 0$. It follows from the hypothesis on $\mathcal{A}$ that

$$\Phi(ab, c) = \Phi(ab, c)$$

for any $a, b, c \in \mathcal{A}$, that is,

$$abT(cxy)z + zT(cxy)ab - abT(cx)yz - yzT(cx)ab$$

$$= aT(bcxy)z + zT(bcxy)a - aT(bcx)yz - yzT(bcx)a,$$

for all $x, y, z, a, b, c \in \mathcal{A}$.

First let $a = \rho_i$, we get

$$\rho_i bT(cxy)z + zT(cxy)\rho_i b - \rho_i bT(cx)yz - yzT(cx)\rho_i b$$

$$= \rho_i T(bcxy)z + zT(bcxy)\rho_i - \rho_i T(bcx)yz - yzT(bcx)\rho_i$$

which converges to

$$bT(cxy)z + zT(cxy)b - bT(cx)yz - yzT(cx)b$$

$$= T(bcxy)z + zT(bcxy) - T(bcx)yz - yzT(bcx)$$

with respect to the norm topology. On the other hand, let $x = \rho_i$, we get

$$bT(c\rho_i y)z + zT(c\rho_i y)b - bT(c\rho_i)yz - yzT(c\rho_i)b$$

$$= T(bc\rho_i y)z + zT(bc\rho_i y) - T(bc\rho_i)yz - yzT(bc\rho_i),$$

which converges to

$$bT(cy)z + zT(cy)b - bT(c)yz - yzT(c)b$$

$$= T(bcy)z + zT(bcy) - T(bc)yz - yzT(bc)$$

with respect to the norm topology.

In the next steps, we consider $z = \rho_i, c = \rho_i$.

Since T is a bounded linear mapping, $\{\rho_i\}_{i\in I}$ is bounded, $\{T\rho_i\}_{i\in I}$ is bounded too, and we can assume that $\{T\rho_i\}_{i\in I}$ converges to an element ξ in $\mathcal{M}^{**}$ with respect to the w^*-topology, then $\{y \cdot T(\rho_i) \cdot b\}_{i\in I}$ converges to $y \cdot \xi \cdot b$ with respect to the w^*-topology. Hence

$$2T(by) = T(b)y + bT(y) + T(y)b + yT(b) - y\xi b - b\xi y.$$

Since the right-hand-side of the above identity is symmetric on b and y, we deduce that $T(by) = T(yb)$.

$$T(by + yb) = T(b)y + yT(b) + T(y)b + bT(y) - y\xi b - b\xi y.$$

Hence T is a generalized Jordan derivation.

Corollary 2.2 Suppose that $\mathcal{A}$ is a commutative Banach algebra with the property $\mathbb{B}$ and having a bounded approximate identity $\{\rho_i\}_{i\in I}$. Let $\mathcal{M}$ be an essential Banach $\mathcal{A}$-bimodule. Then the following conditions are equivalent:

(1) $T : \mathcal{A} \to \mathcal{M}$ is a generalized Jordan derivation;

(2) $aT(b)c + cT(b)a = 0$ when $ab = bc = 0, a, b, c \in \mathcal{A}$.

Proof (2) $\Rightarrow$ (1) is clear from Proposition 2.1.

(1) $\Rightarrow$ (2) If T is a generalized Jordan derivation, then $T(b) = d(b) + \xi b$, for any $b \in \mathcal{A}$, where d is a Jordan derivation from $\mathcal{A}$ to $\mathcal{M}^{**}$, $\xi \in \mathcal{M}^{**}$. Hence

$$aT(b)c + cT(b)a = a(d(b) + \xi b)c + c(d(b) + \xi b)a.$$

For the Jordan derivations d, we have

$$d(abc + cba) = d(a)bc + ad(b)c + abd(c) + d(c)ba + cd(b)a + cbd(a)$$

for every a, b, c in $\mathcal{A}$. Hence, by the commutativity of $\mathcal{A}$, we get

$$ad(b)c + cd(b)a = 0$$

for all a, b, c with $ab = bc = 0$. By using the commutativity of $\mathcal{A}$, and $a\xi bc + c\xi ba = 0$. Hence

$$aT(b)c + cT(b)a = 0$$

when $ab = bc = 0, a, b, c \in \mathcal{A}$. The proof is completed.

Suppose that $\mathcal{A}$ is a $*$-algebra, an $\mathcal{A}$-bimodule $\mathcal{M}$ is called an $\mathcal{A}$-$*$-bimodule if $\mathcal{M}$ is equipped with a $*$-mapping from $\mathcal{M}$ into itself, such that

$$(\alpha m + \beta n)^* = \overline{\alpha} m^* + \overline{\beta} n^*, (am)^* = m^* a^*, (ma)^* = a^* m^* \text{ and } (m^*)^* = m$$

whenever a in $\mathcal{A}$, m, n in $\mathcal{M}$ and α, β in $\mathbb{C}$.

Corollary 2.3 Suppose that $\mathcal{A}$ is a commutative Banach $*$-algebra with property $\mathbb{B}$ and having a bounded approximate identity $\{\rho_i\}_{i\in I}$, $\mathcal{M}$ is an essential Banach $\mathcal{A}$-$*$-bimodule. Let T be a continuous linear mapping from $\mathcal{A}$ to $\mathcal{M}$ which satisfies

$$aT(b)^*c + cT(b)^*a = 0$$

whenever $a, b, c \in \mathcal{A}$ with $a \perp b \perp c$. Then T is a generalized Jordan derivation.

Proof If $\mathcal{A}$ is commutative, $a \perp b \perp c$ is equivalent to $ab^* = b^*c = 0$. Let $d = b^*$, then the conditions in the corollary can be replaced by

$$ad = dc = 0 \Rightarrow aT(d^*)^*c + cT(d^*)^*a = 0.$$

By defining $\tau(d) = T(d^*)^*$, we get that

$$a\tau(d)c + c\tau(d)a = 0,$$

when $ad = dc = 0$. So, we can apply Corollary 2.2 to deduce that τ is a generalized Jordan derivation. Hence, according to the definition of τ, T is a generalized Jordan derivation.

Remark 1 We can not deduce that T is a symmetric mapping or a $*$-mapping. If T is a inner derivation, it satisfies the above equation. However there are inner derivations which are not $*$-derivations. In particular, T need not to be a local triple derivation, since each local triple derivation preserves the adjoint ([14, Lemma 9]).

We give the following results which contain some new generalizations of [15, Proposition 3.4], [3, Theorem 2.11], [4, Lemma 2.8] with slightly weaker hypotheses.

Theorem 2.4 Suppose that $\mathcal{A}$ is a C^*-algebra, and let $T : \mathcal{A} \to \mathcal{A}$ be a bounded linear mapping. Then the following statements are equivalent:

(1) $\{a, T(b), c\} = 0$, when $a \perp b \perp c,\ a, b, c \in \mathcal{A}$;

(2) T is a generalized derivation.

Proof (1) $\Rightarrow$ (2). If $a, b, c \in \mathcal{A}_{sa}$ and $ab = bc = 0$. Hence $ab^* = b^*a = 0, bc^* = c^*b = 0$. So we obtain

$$aT(b)^*c + cT(b)^*a = 0.$$

Then applying $*$ to both sides. We get

$$aT(b)c + cT(b)a = 0.$$

By [4], so T is a generalized derivation.

(2) $\Rightarrow$ (1). By the definition of generalized derivation, $T(b) = D(b) + \xi b$ for all $b \in \mathcal{A}$, where D is a derivation from $\mathcal{A}$ to $\mathcal{A}^{**}$, $\xi \in \mathcal{A}^{**}$. If $ab^* = b^*a = 0, b^*c = cb^* = 0$, $D(a^*b) = D(a^*)b + a^*D(b) = 0$, then

$$b^*D(a^*)^* + D(b)^*a = 0,\ cb^*D(a^*)^* + cD(b)^*a = 0,$$

so

$$cD(b)^*a = 0.$$

By a similar calculation,

$$aD(b)^*c = 0.$$

So

$$aT(b)^*c + cT(b)^*a = a(D(b)^* + b^*\xi^*)c + c(D(b)^* + b^*\xi^*)a =$$

$$aD(b)^*c + ab^*\xi^*c + cD(b)^*a + cb^*\xi^*a = 0.$$

Hence T satisfies (1).

Before proving the next main result, we give the following lemma whose proof is contained in the proof of [3, Theorem 2.1].

Lemma 2.5 Suppose that $\mathcal{A}$ is a C^*-algebra, $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule, and let T be a linear mapping from $\mathcal{A}$ to $\mathcal{M}$. If there exists a $\xi \in \mathcal{M}^{**}$ such that T satisfies

$$T(a^2) = T(a)a + aT(a) - a\xi a,$$

for all $a \in \mathcal{A}_{sa}$, then T is a generalized Jordan derivation.

Proposition 2.2 Suppose that $\mathcal{A}$ is a C^*-algebra, $\mathcal{M}$ is an essential Banach $\mathcal{A}$-$*$-bimodule, and let $T : \mathcal{A} \to \mathcal{M}$ be a bounded linear mapping satisfying the following conditions:

$$aT(b)^*c + cT(b)^*a = 0,\ a \perp b \perp c,\ a, b, c \in \mathcal{A}. \tag{2.3}$$

Then T is a generalized Jordan derivation.

Proof Let $\mathcal{B}$ denote the abelian C^*-subalgebra of $\mathcal{A}$ generated by a self-adjoint element a of $\mathcal{A}$. According to Corollary 2.3, we see that $T|_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}$ is a generalized Jordan derivation. Hence

$$T(a^2) = T(a)a + aT(a) - a\xi a$$

where $\xi \in \mathcal{M}^{**}$. For any $a \in \mathcal{A}$, $a = a_1 + ia_2$, where $a_1, a_2 \in \mathcal{A}_{sa}$, according to Lemma 2.5, we obtain that

$$T(ab + ba) = T(a)b + aT(b) + T(b)a + bT(a) - a\xi b - b\xi a$$

for any $a, b \in \mathcal{A}$. So T is a generalized Jordan derivation.

We conclude this section with some results about homomorphisms.

Definition 2.4 Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. A Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is a linear mapping $T : \mathcal{A} \to \mathcal{B}$ such that

$$T(a \circ b) = T(a) \circ T(b) \quad (a, b \in \mathcal{A}),$$

where the symbol $\circ$ denotes the Jordan product on $\mathcal{A}$, i.e.

$$a \circ b = \frac{1}{2}(ab + ba) \quad (a, b \in \mathcal{A}).$$

We make full use of the powerful property $\mathbb{B}$ to characterize homomorphisms on unital Banach algebras satisfying this property.

Proposition 2.3 Suppose that $\mathcal{A}$ is a unital Banach algebra satisfying the property $\mathbb{B}$, let $T : \mathcal{A} \to \mathcal{A}$ be a continuous linear mapping satisfying

$$ab = bc = 0 \Rightarrow T(a)T(b)T(c) + T(c)T(b)T(a) = 0.$$

If $T(1) = 1$, then T is a Jordan homomorphism.

Proof Fix $a, b \in \mathcal{A}$ with $ab = 0$. Define a continuous bilinear mapping

$$\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A},$$

such that

$$\varphi(x, y) = T(a)T(bx)T(y) + T(y)T(bx)T(a).$$

When $xy = 0$, we have $abx = 0 = bxy$. Hence $\varphi(x, y) = 0$, when $xy = 0, a, b \in \mathcal{A}$. By property $\mathbb{B}$,

$$\varphi(x, 1) = \varphi(1, x),$$

for any $x \in \mathcal{A}$, that is, $T(a)T(bx) + T(bx)T(a) = T(a)T(b)T(x) + T(x)T(b)T(a)$.

Define a continuous bilinear mapping given by

$$\Phi(a, b) = aT(bx) + T(bx)a - aT(b)x - xT(b)a.$$

By the previous paragraph, $ab = 0 \Rightarrow \Phi(a, b) = 0$. So, by property $\mathbb{B}$, we get

$$\Phi(a, 1) = \Phi(1, a),$$

for any $a \in \mathcal{A}$.

So

$$T(a)T(x) + T(x)T(a) - T(a)T(1)T(x) - T(x)T(1)T(a)$$

$$= T(ax) + T(ax) - T(a)T(x) - T(x)T(a),$$

i.e.

$$2T(ax) = T(a)T(x) + T(a)T(x) + T(x)T(a) + T(x)T(a) -$$
$$T(a)T(1)T(x) - T(x)T(1)T(a).$$

Since $T(ax) = T(xa)$ and $T(1) = 1$,

$$T(ax + xa) = T(a)T(x) + T(a)T(x).$$

Hence T is a Jordan homomorphism.

3 Spectrum-preserving mappings

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ acting on a Hilbert space $\mathcal{H}$. We denote by $\mathcal{P}(\mathcal{M})$ the collection of all projections in $\mathcal{M}$, by $S(\mathcal{M}, \tau)$ the collection of all τ-measurable operators with respect to $\mathcal{M}$. More details about τ-measurable operators can be found in [16].

We recall the definition of the measure topology t_τ on the algebra $S(\mathcal{M}, \tau)$. For every $\epsilon, \delta > 0$, we define the set

$$U(\epsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \text{ there exists } P \in \mathcal{P}(\mathcal{M}) \text{ such that } \|X(I-P)\| \leq \epsilon, \tau(P) \leq \delta\}.$$

The topology generated by the sets $U(\epsilon, \delta), \epsilon, \delta > 0$, is called the measure topology t_τ on $S(\mathcal{M}, \tau)$. It is well known that the algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology is a complete metrizable topological algebra.

Definition 3.5 Suppose that T is a closed densely defined linear operator on a Hilbert space $\mathcal{H}$ with domain $D(T)$. The spectrum $\sigma(T)$ of T is the set of those complex numbers λ such that $T - \lambda I$ is not a one-to-one mapping of $D(T)$ onto $\mathcal{H}$.

Definition 3.6 Suppose that $\mathcal{A}, \mathcal{B}$ are algebras over the complex filed $\mathbb{C}$, and ϕ is a linear mapping from $\mathcal{A}$ to $\mathcal{B}$. If ϕ satisfies $\sigma(\phi(a)) = \sigma(a)$, for every $a \in \mathcal{A}$, we shall say ϕ is a spectrum-preserving linear mapping.

Proposition 3.4 If $h = h^* \in S(\mathcal{M}, \tau)$, then h is the limit of a sequence of linear combinations of mutually orthogonal projections in measure topology ($S(\mathcal{M}, \tau) = \overline{\mathcal{P}(\mathcal{M})}^{t_\tau}$).

Proof By [17, Theorem 5.6.18], h is affiliated with an abelian von Neumann subalgebra $\mathcal{R}$ of $\mathcal{M}$. Hence h belongs to the $S(\mathcal{R}, \tau|_\mathcal{R})$. For an abelian von Neumann algebra, it is well known that $\mathcal{R}$ can be uniformly approximated by finite linear combinations of mutually orthogonal projections ([2, Proposition1.3.1 and Lemma 1.7.5]) i.e. $\mathcal{R} = \overline{\mathcal{P}(\mathcal{R})}^{\|\cdot\|}$. With the consideration that $S(\mathcal{R}, \tau|_\mathcal{R}) = \overline{\mathcal{R}}^{t_{\tau}|_\mathcal{R}}$, it follows that $S(\mathcal{R}, \tau|_\mathcal{R}) = \overline{\overline{\mathcal{P}(\mathcal{R})}^{\|\cdot\|}}^{t_\tau|_\mathcal{R}} = \overline{\mathcal{P}(\mathcal{R})}^{t_\tau|_\mathcal{R}}$. Hence for any $h = h^* \in S(\mathcal{M}, \tau)$, there exists a von Neumann subalgebra $\mathcal{R}$ of $\mathcal{M}$ such that $h \in \overline{\mathcal{P}(\mathcal{R})}^{t_\tau|_\mathcal{R}}$.

Proposition 3.5 Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be finite von Neumann algebras, and $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ be a unital $*$-anti-homomorphism. If Φ is normal, then Φ is Cauchy-continuous for the measure topologies on $\mathcal{M}_1$ and $\mathcal{M}_2$.

Proof Let τ_2 be a normal tracial state on $\mathcal{M}_2$. Since Φ is normal, we note that $\tau_1 := \tau_2 \circ \Phi$ is a normal tracial state on $\mathcal{M}_1$. For $\epsilon, \delta > 0$, $A \in U(\tau_1, \epsilon, \delta)$, there is a projection E in $\mathcal{M}_1$ such that $\|AE\| \leq \epsilon$ and $\tau_1(I - E) \leq \delta$. First we note that if Φ is a $*$-anti-homomorphism, then $\Phi(F)\Phi(A)\Phi(E) = \Phi(EAF)$ for any $A, E, F \in \mathcal{M}_1$, and $\Phi(E)$ is a projection if E is a projection.

Let $E \in \mathcal{P}(\mathcal{M}_1), A \in \mathcal{M}_1$,

$$\begin{aligned}
\|\Phi(A)\Phi(E)\| &= \|(1 - \Phi(E))\Phi(A)\Phi(E) + \Phi(E)\Phi(A)\Phi(E)\| \\
&\leq \|(1 - \Phi(E))\Phi(A)\Phi(E)\| + \|\Phi(E)\Phi(A)\Phi(E)\| \\
&= \|\Phi(EA(1 - E))\| + \|\Phi(EAE)\| \\
&\leq \|AE(1 - E)\| + \|EAE\| \\
&\leq 2\|AE\| \\
&\leq 2\epsilon,
\end{aligned}$$

and $\tau_2(1 - \Phi(E)) = \tau_1(I - E) \leq \delta$. Consequently,

$$\Phi(U(\tau_1, \epsilon, \delta)) \subseteq U(\tau_2, 2\epsilon, \delta).$$

Thus if a net $\{A_i\}$ in $\mathcal{M}_1$ is Cauchy in measure topology, then the net $\{\Phi(A_i)\}$ in $\mathcal{M}_2$ is also Cauchy in measure topology. We conclude that Φ is Cauchy-continuous for the measure topologies on $\mathcal{M}_1$ and $\mathcal{M}_2$.

Theorem 3.6 Suppose that $\mathcal{M}$ is a factor of type II_1, and let ϕ be a spectrum-preserving linear mapping from $S(\mathcal{M}, \tau)$ onto itself. Then ϕ is a $*$-isomorphism or a $*$-anti-isomorphism.

Proof It can be easily seen that if ϕ satisfies $\sigma(\phi(a)) = \sigma(a)$, then ϕ is a positive mapping from $S(\mathcal{M}, \tau)$ onto itself. Hence ϕ is self-adjoint, i.e. $\phi(a^*) = \phi(a)^*$, for every $a \in S(\mathcal{M}, \tau)$, and if $a \in \mathcal{M}$, we can deduce that $\phi(a) \in \mathcal{M}$. It follows that the restriction of ϕ on $\mathcal{M}$, denoted by $\phi|_{\mathcal{M}}$, is a spectrum-preserving mapping. According to [10, Theorem 1.3], $\phi|_{\mathcal{M}}$ is a Jordan isomorphism. By [18, Corollary 11], $\phi|_{\mathcal{M}}$ is a $*$-isomorphism or a $*$-anti-isomorphism. Hence $\phi|_{\mathcal{M}}$ is normal. By [19, Theorem 4.9] and Proposition 3.5, $\phi|_{\mathcal{M}}$ is continuous in measure topology. Let $h = h^* \in S(\mathcal{M}, \tau)$, by Proposition 3.4, h is the limit of a sequence of linear combinations of orthogonal idempotents Consequently, by [10, Theorem 1.2], $\phi(h)$ is the limit of a sequence of linear combinations of orthogonal idempotents. By continuity of ϕ, taking the limits of these sequences we conclude that $\phi(h^2) = \phi(h)^2$. Taking h, k self-adjoint in $S(\mathcal{M}, \tau)$ we get

$$\begin{aligned}
(\phi(h + k))^2 = (\phi(h) + \phi(k))^2 &= \phi(h)^2 + \phi(k)^2 + \phi(h)\phi(k) + \phi(k)\phi(h) \\
&= \phi((h + k)^2) = \phi(h^2) + \phi(k^2) + \phi(hk + kh).
\end{aligned}$$

Thus $\phi(kh + kh) = \phi(h)\phi(k) + \phi(k)\phi(h)$, for every h, k self-adjoint elements. Let $x \in S(\mathcal{M}, \tau)$, then $x = h + ik$ where $h = (x + x^*)/2$ and $k = (x - x^*)/2i$ are self-adjoint elements. Hence

$$\begin{aligned}\phi(x^2) &= \phi(h^2 - k^2 + i(hk + kh)) = \phi(h^2) - \phi(k)^2 + i(\phi(h)\phi(k) + \phi(k)\phi(h)) \\ &= (\phi(h) + i\phi(k))^2 = \phi(x)^2.\end{aligned}$$

Hence, ϕ is a Jordan $*$-isomorphism. It follows that ϕ is a $*$-isomorphism or a $*$-anti-isomorphism.

Theorem 3.7 Suppose that $\mathcal{M}$ is a finite von Neumann algebra, and let ϕ be a spectrum-preserving linear mapping from $S(\mathcal{M}, \tau)$ onto itself. Then ϕ is a Jordan $*$-isomorphism.

Proof In the proof, we need the [18, Theorem 10] instead of [18, Corollary 11]. The remainder of the proof is similar to that of Theorem 3.6.

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几类保持映射的刻画

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摘要: 本文研究了一类三元组正交保持的线性映射并刻画了保持τ-可测算子谱的线性映射. 我们在更弱的条件下利用性质$\mathbb{B}$刻画了保持三元组正交的线性映射, 获得了这类映射是广义的Jordan导子的结果. 对于保持τ-可测算子谱的线性映射研究, 我们将有界算子中保谱的结果推广到无界算子.

关键词: C^*-代数; 导子; 性质$\mathbb{B}$; 谱

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