Vol. 44 (2024)
No. 1

数 学 杂 志
J. of Math. (PRC)

EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL PROBLEMS WITH p-LAPLACIAN OPERATOR AT RESONANCE

XUE Ting-ting, JIANG Yong-sheng, CAO Hong

(*School of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi 830000, China*)

Abstract: The paper studies the existence of positive solutions for fractional differential equations with p-Laplacian at resonance under two kinds of boundary conditions. Some new existence results are obtained by using Leggett-Williams norm-type theorem, which generalize the existing results.

Keywords: p-Laplacian operator; Leggett-Williams norm-type theorem; resonant; positive solution

2010 MR Subject Classification: 34A08; 34B15

Document code: A **Article ID:** 0255-7797(2024)01-0001-16

1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines. Recently, more and more scholars are interested in fractional differential equations, see [1-7]. For example, Arafa, Rida and Khalil [6] used the following fractional order model to describe the efficacy of anti-viral drugs in the treatment of human immunodeficiency virus type 1 (HIV-1):

$$\begin{cases} D^{\alpha_1}(x) = s - \mu x - \beta x z, \\ D^{\alpha_2}(y) = \beta x z - \varepsilon y, \\ D^{\alpha_3}(z) = c y - \gamma z, \end{cases}$$

where $D^{\alpha_1}, D^{\alpha_2}, D^{\alpha_3}$ are Caputo fractional derivatives with $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$, all parameters and variables are non-negative, x, y is the number of uninfected and infected CD4+ T-cells, respectively, z is the number of virions in plasma, s is the assumed constant rate of production of CD4+ T-cells, l is their per capita death rate, b is the rate of infection of CD4+ T-cells by virus, e is the per capita rate of disappearance of infected cells, and c

* **Received date:** 2022-03-23 **Accepted date:** 2022-04-25

Foundation item: Supported by Natural Science Foundation of Xinjiang Uygur Autonomous Region (2021D01B35), Natural Science Foundation of colleges and universities in Xinjiang Uygur Autonomous Region (XJEDU2021Y048) and Doctoral Initiation Fund of Xinjiang Institute of Engineering (2020xgy012302).

Biography: Xue Tingting(1987–), female, born at Yancheng, Jiangsu, associate professor, major in fractional differential equations. E-mail: xuett@cumt.edu.cn.

is the death rate of virus particles. Ates and Zegeling [7] investigated the fractional-order advection-diffusion reaction boundary value problems:

$$\begin{cases} \varepsilon^C D^\alpha u + \gamma u' + f(u) = S(x), \quad x \in [0,1], \\ u(0) = u_L,\ u(1) = u_R, \end{cases}$$

where $1 < \alpha \le 2$, $0 < \varepsilon \le 1$, $\gamma \in \mathbb{R}$, ${}^C D^\alpha$ is the Caputo fractional derivative.

In recent years, more and more attention are being paid to the existence of solutions for fractional p-Laplacian problems. And many important results have been achieved in this regard, see [8-14]. The p-Laplacian equation was derived from the following nonlinear diffusion equation proposed by Leibenson [15] in 1983, when studying the one-dimensional variable turbulent flow of gases through porous media

$$u_t = \frac{\partial}{\partial x}\left(\frac{\partial u^m}{\partial x}\left|\frac{\partial u^m}{\partial x}\right|^{\mu-1}\right), \quad m = n+1.$$

When $m > 1$, the above equation is the porous media equation, when $0 < m < 1$, the above equation is the fast diffusion equation, and when $m = 1$, the above equation is the heat equation, while when $m = 1, \mu \neq 1$, such equation often appears in the study of non-Newton fluids. Given the importance of such equations, the above equation is abstracted into the following p-Laplacian equation

$$(\phi_p(u'))' = f(t,u),$$

where $\phi_p(s) = |s|^{p-2}s\ (s \neq 0)$, $\phi_p(0) = 0$, $p > 1$. Note that when $p = 2$, p-Laplacian equation degenerates into a classical second-order differential equation. Naturally, in view of its significance in theory and practice, more and more people are concerned about the existence of solutions for fractional p-Laplacian problems. For example, Wang [16] studied p-Laplacian problems:

$$\begin{cases} D_{0+}^\gamma \varphi_p\left(D_{0+}^\alpha x(t)\right) = f(t, x(t)),\ 0 < t < 1, \\ x(0) = 0,\ x(1) = ax(\xi),\ D_{0+}^\alpha x(0) = 0,\ D_{0+}^\alpha x(1) = bD_{0+}^\alpha x(\eta), \end{cases}$$

where $1 < \alpha, \gamma \le 2$, $0 \le a, b \le 1$, $0 < \xi, \eta < 1$, D_{0+}^α is Riemann-Liouville fractional derivative. The existence results of positive solution for the problem were obtained by lower and upper solutions method. Tian [17] considered the following p-Laplacian problems:

$$\begin{cases} D_{0+}^\alpha \varphi_p\left(D_{0+}^\beta x(t)\right) + f(t, x(t)) = 0,\ 0 < t < 1, \\ x(0) = 0,\ D_{0+}^\gamma x(1) = \lambda D_{0+}^\gamma x(\xi),\ D_{0+}^\beta x(0) = 0, \end{cases}$$

where $0 < \alpha < 1$, $1 < \beta \le 2$, $0 < \gamma \le 1$, $0 < \xi < 1$, $1 + \gamma \le \beta$, $\lambda \in [0, \infty)$, D_{0+}^α is Riemann-Liouville fractional derivative. By using the fixed point theorem on the cone, the existence results of positive solution for this problem were obtained. Chen and Liu [18] discussed the following problems:

$$\begin{cases} {}^C D_{0+}^\beta \phi_p\left({}^C D_{0+}^\alpha x\right) = f(t, x),\ t \in [0,1], \\ x(0) = -x(1),\ {}^C D_{0+}^\alpha x(0) = -{}^C D_{0+}^\alpha x(1), \end{cases}$$

where $0 < \alpha,\ \beta \le 1$, $1 < \alpha,\ \beta \le 2$, ${}^{C}D_{0+}^{\alpha}$ is Caputo fractional derivative, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous, $\phi_p(\cdot)$ is p-Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s(s \neq 0, p > 1), \phi_p(0) = 0$. Note that, when $p = 2$, the nonlinear operator ${}^{C}D_{0+}^{\beta}\phi_p\left({}^{C}D_{0+}^{\alpha}\right)$ reduces to the linear operator. By Schaefer's fixed point theorem, the existence results of solutions for the problem were obtained.

An interesting and effective method used to prove the existence of positive solutions for fractional differential problems at resonance is Leggett-Williams norm-type theorem. Many existence results of positive solution for fractional boundary value problems at resonance with the linear derivative operator have been obtained, see literature [19-28]. However, as far as we know, only Jiang [29] studied the existence of positive solutions for the following fractional problems with p-Laplacian operator at resonance:

$$\begin{cases} {}^{C}D_{0+}^{\beta}\varphi_p\left({}^{C}D_{0+}^{\alpha}x(t)\right) = f\left(t, {}^{C}D_{0+}^{\alpha}x(t)\right),\ t \in (0,1), \\ {}^{C}D_{0+}^{\alpha}x(0) = {}^{C}D_{0+}^{\alpha}x(1),\ x^{(i)}(0) = 0,\ i = 0,1,2,\cdots,n-1, \end{cases}$$

where $0 < \beta < 1$, $n-1 < \alpha \le n$, ${}^{C}D_{0+}^{\alpha}$ is Caputo fractional derivative, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. By using Leggett-Williams norm-type theorem, the existence results of positive solutions for the problem with a nonlinear derivative operator at resonance were obtained.

Inspired by the above excellent results, first, this paper will study the existence of positive solutions for the following p-Laplacian boundary value problem

$${}^{C}D_{0+}^{\beta}\varphi_p({}^{C}D_{0+}^{\alpha}x(t)) = f(t, {}^{C}D_{0+}^{\alpha}x(t)),\ t \in (0,1), \tag{1.1}$$

$${}^{C}D_{0+}^{\alpha}x(1) = {}^{C}D_{0+}^{\alpha}x(\delta),\ x^{(i)}(0) = 0,\ i = 0,1,2,\cdots,n-1, \tag{1.2}$$

where $n-1 < \alpha \le n$, $0 < \beta < 1$, ${}^{C}D_{0+}^{\alpha}$, ${}^{C}D_{0+}^{\beta}$ are Caputo fractional derivatives, $0 < \delta < 1$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous.

On the other hand, we discuss problem (1.1) with the following integral boundary conditions:

$$x(0) = 0, \quad \varphi_p({}^{C}D_{0+}^{\alpha}x(1)) = \int_0^1 h(t)\varphi_p({}^{C}D_{0+}^{\alpha}x(t))dt, \tag{1.3}$$

where ${}^{C}D_{0+}^{\alpha}$, ${}^{C}D_{0+}^{\beta}$ are Caputo fractional derivatives, $0 < \alpha,\ \beta < 1$, $h(t) \ge 0$, $\int_0^1 h(t)dt = 1$, $\varphi_p(s) = s|s|^{p-2}$, $p > 1$, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous.

Let us emphasize the contribution of our article: firstly, as far as we know, there is no paper on the existence of positive solutions for fractional p-Laplacian boundary value problems (1.1)(1.2) and (1.1)(1.3) at resonance, so our article enriches some existing articles. Secondly, our article serves as a further development for the result of [29]. When $\delta = 0$, the results of [29] will be a special case of our result.

2 Preliminaries

To facilitate understanding, this section introduces some concepts and lemmas related to this article. For more details, please refer to the references hereunder (see [20,30,31]).

Definition 2.1 ([30]) Let X, Y be real Banach spaces, and $L : \mathrm{dom}L \subset X \to Y$ be a linear map. If $\dim \mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$ and $\mathrm{Im}L$ is a closed subset in Y, then the map L is a Fredholm operator with index zero. If there exists the continuous projections $P : X \to X$ and $Q : Y \to Y$ satisfying $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$, then $L\mid_{\mathrm{dom}L\cap\mathrm{Ker}P}$: $\mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is reversible. We denote the inverse of this map by K_P, i.e. $K_P = L_P^{-1}$ and $K_{P,Q} = K_P\,(I - Q)$. Moreover, since $\dim \mathrm{Im}Q = \mathrm{codim}\mathrm{Im}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$. It is known that the operator equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx,$$

where $N : X \to Y$ be a nonlinear operator. If Ω is an open bounded subset of X and $\mathrm{dom}L \cap \Omega \neq \varnothing$, then the map N is L-compact on $\overline{\Omega}$ when $QN : \overline{\Omega} \to Y$ is bounded and $K_P\,(I - Q)\,N : \overline{\Omega} \to X$ is compact.

Let C be a cone in X. Then C induces a partial order in X by $x \leq y$ iff $y - x \in C$.

Lemma 2.1 ([20]) Let C be a cone in X. Then for every $u \in C\backslash\{0\}$ there exists a positive number $\sigma(u)$ such that $\|x + u\| \geq \sigma(u)\,\|x\|$ for all $x \in C$. Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_P(I - Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.2 ([20]) Let C be a cone in X and Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega_1} \subset \Omega_2$ and $C \cap (\overline{\Omega_2}\backslash\Omega_1) \neq \emptyset$. Assume that the following conditions are satisfied:

(1) $L : \mathrm{dom}L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on every bounded subset of X.

(2) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [C \cap \partial\Omega_2 \cap \mathrm{dom}L] \times (0, 1)$.

(3) γ maps subsets of $\overline{\Omega_2}$ into bounded subsets of C.

(4) $\deg([I - (P + JQN)\gamma]|_{\mathrm{Ker}L}, \mathrm{Ker}L \cap \Omega_2, 0) \neq 0$.

(5) there exists $u_0 \in C\backslash\{0\}$ such that $\|x\| \leq \sigma(u_0)\,\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x\}$ for some $\mu > 0$ and $\sigma(u_0)$ are such that $\|x + u_0\| \geq \sigma(u_0)\,\|x\|$ for every $x \in C$.

(6) $(P + JQN)\gamma(\partial\Omega_2) \subset C$.

(7) $\Psi_\gamma(\overline{\Omega_2}\backslash\Omega_1) \subset C$.

Then the equation $Lx = Nx$ has at least one solution in $C \cap (\overline{\Omega_2}\backslash\Omega_1)$.

Definition 2.2 ([31]) The Riemann-Liouville fractional integral of order $\alpha(\alpha > 0)$ for the function $x : (0, +\infty) \to \mathbb{R}$ is defined as

$$I_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}x(s)\mathrm{d}s,$$

provided that the right-hand side integral is defined on $(0, +\infty)$.

Definition 2.3 ([31]) The Captuo fractional derivative of order $\alpha(\alpha > 0)$ for the function $x : (0, +\infty) \to \mathbb{R}$: is defined as

$${}^{C}D_{0+}^{\alpha}x(t) = I_{0+}^{n-\alpha}\frac{d^n x(t)}{dt^n} = \frac{1}{\Gamma(n - \alpha)}\int_0^t (t - s)^{n-\alpha-1}x^{(n)}(s)\mathrm{d}s,$$

where $n=[\alpha]+1$, provided that the right-hand side integral is defined on $(0,+\infty)$.

Lemma 2.3 ([31]) Assume $x\in L[0,1],\alpha>\beta\geq 0,\ \alpha>1$, then ${}^CD^{\beta}_{0+}I^{\alpha}_{0+}x(t)=I^{\alpha-\beta}_{0+}x(t),\ {}^CD^{\beta}_{0+}I^{\beta}_{0+}x(t)=x(t)$.

Lemma 2.4 ([31]) Let $n-1<\alpha\leq n$, if ${}^CD^{\alpha}_{0+}x(t)\in C[0,1]$, then $I^{\alpha}_{0+}\,{}^CD^{\alpha}_{0+}x(t)=x(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$, where $c_i\in\mathbb{R},\ i=0,1,\cdots,n-1,\ n=[\alpha]+1$.

3 The existence of positive solution for problem (1.1)(1.2)

Since ${}^CD^{\beta}_{0+}[\varphi_p({}^CD^{\alpha}_{0+}(\cdot))]$ is a nonlinear operator, so we can't solve problem (1.1)(1.2) by Lemma 2.2. Hence, we provide the following lemma.

Lemma 3.1 $u(t)$ is a solution of the following problem:

$$\begin{cases}{}^CD^{\beta}_{0+}u(t)=f(t,\varphi_q(u(t))),\ t\in(0,1),\\ u(1)=u(\delta),\end{cases}\tag{3.1}$$

if and only if $x(t)$ is a solution of problem (1.1)(1.2), where $x(t)=I^{\alpha}_{0+}\varphi_q(u(t)),\ \frac{1}{p}+\frac{1}{q}=1$.

Proof If $u(t)$ is a solution of problem (3.1) and $x(t)=I^{\alpha}_{0+}\varphi_q(u(t))$, then $u(t)=\varphi_p({}^CD^{\alpha}_{0+}x(t))$ and $x^{(i)}(0)=0,\ i=\overline{0,n-1}$. Replacing $u(t)$ with $\varphi_p({}^CD^{\alpha}_{0+}x(t))$ in problem (3.1), we can find that $x(t)$ is a solution of problem (1.1)(1.2).

On the other hand, if $x(t)$ is a solution of problem (1.1)(1.2) and $u(t)=\varphi_p({}^CD^{\alpha}_{0+}x(t))$, substituting $u(t)$ for $\varphi_p({}^CD^{\alpha}_{0+}x(t))$ in problem (1.1)(1.2), we can find that $u(t)$ satisfies problem (3.1).

Let $X=Y=C[0,1]$ with the norm $\|u\|=\max\limits_{t\in[0,1]}|u(t)|$. Set a cone $C=\{u(t)\in X|u(t)\geq 0,\ t\in[0,1]\}$. Define operators $L:\mathrm{dom}L\subset X\to Y$ and $N:X\to Y$ as follows:

$$Lu(t)={}^CD^{\beta}_{0+}u(t),\ Nu(t)=f(t,\varphi_q(u(t))),\tag{3.2}$$

where $\mathrm{dom}L=\{u(t)|u(t),\ {}^CD^{\beta}_{0+}u(t)\in X,\ u(1)=u(\delta)\}$. So problem (3.1) can be written by $Lu=Nu,\ u\in\mathrm{dom}L$.

For simplicity of notation, we set

$$l(s)=\begin{cases}(1-s)^{\beta-1}-(\delta-s)^{\beta-1}, & 0\leq s\leq\delta<1,\\ (1-s)^{\beta-1}, & 0\leq\delta\leq s<1,\end{cases}$$

and

$$G_1(t,s)=\begin{cases}\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(1-s)^{\beta}}{\Gamma(\beta+1)}+\frac{\beta(\Gamma(\beta+2)+1-(\beta+1)t^{\beta})}{(1-\delta^{\beta})\Gamma(\beta+2)}l(s),\ 0\leq s<t\leq 1\\ -\frac{(1-s)^{\beta}}{\Gamma(\beta+1)}+\frac{\beta(\Gamma(\beta+2)+1-(\beta+1)t^{\beta})}{(1-\delta^{\beta})\Gamma(\beta+2)}l(s),\ 0\leq t\leq s\leq 1.\end{cases}$$

We denote

$$K_1=\min\{1,\ \frac{1-\delta^{\beta}}{\beta\max\limits_{s\in[0,1]}l(s)},\ \frac{1}{\max\limits_{t,s\in[0,1]}G_1(t,s)}\}.$$

Thus, one has

$$1-\frac{K_1\beta l(s)}{1-\delta^{\beta}}\geq 0,\ 1-K_1G_1(t,s)\geq 0.\tag{3.3}$$

First, we give the main results of existence of positive solution for problem (1.1)(1.2).

Theorem 3.1 Suppose the following conditions hold.

(H_1) There exists a constant $R_0 > 0$ such that $f(t,u) < 0,\ t \in [0,1],\ u > R_0$.

(H_2) There exist nonnegative functions $a(t), b(t) \in C[0,1]$ with

$$\max_{t\in[0,1]} \int_0^t (t-s)^{\beta-1} a(s)ds := A < +\infty, \max_{t\in[0,1]} \int_0^t (t-s)^{\beta-1} b(s)ds := B < \frac{\Gamma(\beta)}{2},$$

such that

$$|f(t,u)| \le a(t) + b(t)\varphi_p(|u|),\ \forall t \in [0,1].$$

(H_3) $f(t,u) \ge -K_1\varphi_p(u),\ t \in [0,1],\ u > 0$.

(H_4) There exist $r > 0,\ t_0 \in [0,1]$ and $M_0 \in (0,1)$ such that

$$G_1(t_0,s)f(s,u) \ge \frac{1-M_0}{M_0}\varphi_p(u),\ s \in [0,1),\ M_0 r \le u \le r.$$

Then problem (1.1)(1.2) has at least one positive solution.

Next, we give some important lemmas related to Theorem 3.1.

Lemma 3.2 Let L be defined by (3.2), then

$$\mathrm{Ker}L = \{u \in X | u(t) = c,\ c \in \mathbb{R},\ \forall t \in [0,1]\}, \tag{3.4}$$

$$\mathrm{Im}L = \{y \in Y | \int_0^1 l(s)y(s)ds = 0\}. \tag{3.5}$$

Proof By Lemma 2.4, we can obtain (3.4). If $y \in \mathrm{Im}L$, there exists $u \in \mathrm{dom}L$ such that $y = Lu \in Y$. From Lemma 2.4, we have

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds + c,\ c \in \mathbb{R}.$$

Combined with boundary conditions of problem (3.1), we get

$$\int_0^1 (1-s)^{\beta-1} y(s)ds = \int_0^\delta (\delta-s)^{\beta-1} y(s)ds,$$

that is, $\int_0^1 l(s)y(s)ds = 0$.

On the other hand, if $\int_0^1 l(s)y(s)ds = 0$ for $y \in Y$, let $u(t) = I_{0+}^\beta y(t)$, then $u \in \mathrm{dom}L$ and ${}^C D_{0+}^\beta u(t) = y(t)$. Hence, $y \in \mathrm{Im}L$.

Lemma 3.3 Let L be defined by (3.2), then L is a Fredholm operator of index zero. The linear projection operators $P : X \to Y$ and $Q : Y \to Y$ can be defined as follows:

$$Pu(t) = \int_0^1 u(t)dt,\ Qy(t) = \frac{\beta}{1-\delta^\beta} \int_0^1 l(s)y(s)ds,\ \forall t \in [0,1],$$

and $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ is defined as

$$K_P y(t) = \int_0^1 k(t,s)y(s)ds,\ \forall t \in [0,1],$$

where

$$k(t,s)=\begin{cases}\dfrac{1}{\Gamma(\beta)}(t-s)^{\beta-1}-\dfrac{1}{\Gamma(\beta+1)}(1-s)^{\beta}, & 0\le s\le t\le 1,\\ -\dfrac{1}{\Gamma(\beta+1)}(1-s)^{\beta}, & 0\le t\le s\le 1.\end{cases}$$

Proof Clearly, $\mathrm{Im}P=\mathrm{Ker}L$ and $Pu^2=Pu$. By $u=(u-Pu)+Pu$, we have $X=\mathrm{Ker}P+\mathrm{Ker}L$. By a simple calculation, we obtain $\mathrm{Ker}L\cap\mathrm{Ker}P=\{0\}$. Hence, $X=\mathrm{Ker}L\oplus\mathrm{Ker}P$. It is clear that $\mathrm{Im}L\subset\mathrm{Ker}Q$. On the other hand, if $y(t)\in\mathrm{Ker}Q\subset Y$, then

$$Q^2y=Q(Qy)=Qy\cdot\frac{\beta}{1-\delta^{\beta}}\int_0^1 l(s)ds=Qy.$$

If $y\in Y$, let $y=(y-Qy)+Qy$, where $y-Qy\in\mathrm{Ker}Q$, $Qy\in\mathrm{Im}Q$. It follows from $\mathrm{Ker}Q=\mathrm{Im}L$ and $Q^2y=Qy$ that $\mathrm{Im}Q\cap\mathrm{Im}L=\{0\}$. Then, we obtain $Y=\mathrm{Im}L\oplus\mathrm{Im}Q$. Thus, $\dim\mathrm{Ker}L=\dim\mathrm{Im}Q=\mathrm{codim}\mathrm{Im}L=1<\infty$. It implies that L is a Fredholm operator of index zero.

For $y\in\mathrm{Im}L$, we have $K_Py\in\mathrm{dom}L\cap\mathrm{Ker}P$ and $LK_Py=y$. On the other hand, if $u\in\mathrm{dom}L\cap\mathrm{Ker}P$, by Lemma 2.4, one has

$$\begin{aligned}K_PLu(t)&=\frac{1}{\Gamma(\beta)}[\int_0^t(t-s)^{\beta-1}Lu(s)ds-\frac{1}{\beta}\int_0^1(1-s)^{\beta}Lu(s)ds]\\&=I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t)-I_{0+}^{\beta+1C}D_{0+}^{\beta}u(t)|_{t=1}=u(t)+c-I_{0+}^{\beta+1C}D_{0+}^{\beta}u(1).\end{aligned}$$

So, $\int_0^1K_PLu(t)dt=\int_0^1u(t)dt+c-I_{0+}^{\beta+1C}D_{0+}^{\beta}u(1)$. It follows from $u\in\mathrm{Ker}P$ and $K_PLu\in\mathrm{Ker}P$ that $c=I_{0+}^{\beta+1C}D_{0+}^{\beta}u(1)$. Hence, we have $K_PLu=u$, $u\in\mathrm{dom}L\cap\mathrm{Ker}P$.

Lemma 3.4 $QN:X\to Y$ is continuous and bounded and $K_P(I-Q)N:\overline{\Omega}\to X$ is compact, where $\Omega\subset X$ is bounded.

Proof By the continuity of f, we see that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. That is, there exist constants $M_1,M_2>0$ such that $|(I-Q)Nu)|\le M_1$ and $|K_P(I-Q)Nu)|\le M_2$, $\forall u\in\overline{\Omega}$, $t\in[0,1]$. Thus, one need only prove that $K_P(I-Q)N(\overline{\Omega})\subset X$ is equicontinuous. Let $K_{P,Q}=K_P(I-Q)N$, for $0\le t_1<t_2\le 1$, $u\in\overline{\Omega}$, we get

$$\begin{aligned}&|K_{P,Q}u(t_2)-K_{P,Q}u(t_1)|\\=&\frac{1}{\Gamma(\beta)}|\int_0^{t_2}(t_2-s)^{\beta-1}(I-Q)Nu(s)ds-\int_0^{t_1}(t_1-s)^{\beta-1}(I-Q)Nu(s)ds|\\=&\frac{1}{\Gamma(\beta)}|\int_0^{t_1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}](I-Q)Nu(s)ds+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}(I-Q)Nu(s)ds|\\\le&\frac{M_1}{\Gamma(\beta)}[\int_0^{t_1}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]ds+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds\\=&\frac{M_1}{\Gamma(\beta+1)}[t_1^{\beta}-t_2^{\beta}+2(t_2-t_1)^{\beta}].\end{aligned}$$

It follows from the uniform continuity of t^{β} and t on $[0,1]$ that $K_P(I-Q)N(\overline{\Omega})$ are equicontinuous on $[0,1]$. By Arzela-Ascoli theorem, we show that $K_P(I-Q)N(\overline{\Omega})$ is compact.

Lemma 3.5 If the condition (H_1) and (H_2) hold, the set

$$\Omega_0 = \{u(t) | Lu(t) = \lambda Nu(t),\ u(t) \in C \cap \mathrm{dom}L,\ \lambda \in (0,1)\}$$

is bounded.

Proof For $u(t) \in \Omega_0$, we have $QNu(t) = 0$. By (H_1) and $QNu(t) = 0$, there exists $t_0 \in [0,1]$ such that $\varphi_q(u(t_0)) \le R_0$, i.e. $u(t_0) \le \varphi_p(R_0)$. By $u(t) = I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t) + c$, one has

$$|c| \le |u(t)| + |I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t)| \le |u(t_0)| + |I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t_0)| \le \varphi_p(R_0) + |I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t_0)|,$$

and

$$\|u\| \le \varphi_p(R_0) + |I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t_0)| + |I_{0+}^{\beta}\,{}^{C}D_{0+}^{\beta}u(t)|. \tag{3.6}$$

From $Lu = \lambda Nu$, we get ${}^{C}D_{0+}^{\beta}u(t) = \lambda f(t,\ \varphi_q(u(t)))$. By ($H_2$) and $\lambda \in (0,1)$, one has

$$\begin{aligned}
&\|u\| \le \varphi_p(R_0) + |I_{0+}^{\beta} f(t,\ \varphi_q(u(t)))|_{t_0}| + |I_{0+}^{\beta} f(t,\ \varphi_q(u(t)))| \\
&\le \varphi_p(R_0) + \frac{1}{\Gamma(\beta)}\int_0^{t_0}(t_0-s)^{\beta-1}|f(s,\ \varphi_q(u(t)))|ds + \frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}|f(s,\ \varphi_q(u(s)))|ds \\
&\le \varphi_p(R_0) + \frac{1}{\Gamma(\beta)}\int_0^{t_0}(t_0-s)^{\beta-1}[a(s)+b(s)u(s)]ds + \frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}[a(s)+b(s)u(s)]ds \\
&\le \varphi_p(R_0) + \frac{2}{\Gamma(\beta)}(A + B\|u\|).
\end{aligned}$$

Thus,

$$\|u\| \le \frac{\varphi_p(R_0) + \frac{2A}{\Gamma(\beta)}}{1 - \frac{2B}{\Gamma(\beta)}} := N < +\infty.$$

Hence, Ω_0 is bounded.

Proof of Theorem 3.1 Set

$$\Omega_1 = \{u \in X | M_0\|u\| < |u(t)| < r < R,\ t \in [0,1]\},\ \Omega_2 = \{u \in X |\ \|u\| < R\},$$

where $R = \max\{\varphi_p(R_0), N\} + 1$. It is clear that Ω_1 and Ω_2 are open bounded sets of X, $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \backslash \Omega_1) \neq \phi$. By Lemma 3.2, 3.3, 3.4 and 3.5, we know that L is a Fredholm operator of index zero and the conditions (1), (2) of Lemma 2.2 are fulfilled.

Define $\gamma : X \to C$ as $\gamma u(t) = |u(t)|,\ u(t) \in X$ and $J : \mathrm{Im}Q \to \mathrm{Ker}L$ as $J(c) = c,\ c \in \mathbb{R}$. Then $\gamma : X \to C$ is a retraction and (3) of Lemma 2.2 holds. For $u(t) \in \mathrm{Ker}L \cap \partial\Omega_2$, then $u(t) \equiv c$. Let

$$H(c,\lambda) = c - \lambda|c| - \frac{\lambda\beta}{1-\delta^{\beta}}\int_0^1 l(s)f(s,\varphi_q(|c|))ds,$$

where $\lambda \in [0,1]$. Suppose $H(c,\lambda) = 0$, by (H_3), we have

$$\begin{aligned}
c &= \lambda|c| + \frac{\lambda\beta}{1-\delta^{\beta}}\int_0^1 l(s)f(s,\varphi_q(|c|))ds \ge \lambda|c| - \frac{\lambda\beta}{1-\delta^{\beta}}\int_0^1 l(s)K_1|c|ds \\
&= \lambda|c|(1-K_1) \ge 0.
\end{aligned}$$

Thus $H(c,\lambda)=0$ implies $c\geq 0$. Clearly, $H(R,0)\neq 0$. Moreover, if $H(R,\lambda)=0,\ \lambda\in(0,1]$,we get

$$0\leq R(1-\lambda)=\frac{\lambda\beta}{1-\delta^{\beta}}\int_0^1 l(s)f(s,\varphi_q(R))ds,$$

which contradicts to condition (H_1). Hence $H(u,\lambda)\neq 0$ for $u\in \mathrm{Ker}L\cap\partial\Omega_2,\ \lambda\in[0,1]$. Therefore,

$$\begin{aligned}&\deg([I-(P+JQN)\gamma]|_{\mathrm{Ker}L},\ \mathrm{Ker}L\cap\Omega_2,\ 0)\\ =&\deg(H(x,1),\ \mathrm{Ker}L\cap\Omega_2,\ 0)=\deg(H(x,0),\ \mathrm{Ker}L\cap\Omega_2,\ 0)\\ =&\deg(I,\ \mathrm{Ker}L\cap\Omega_2,\ 0)=1\neq 0.\end{aligned}$$

Then, (4) of Lemma 2.2 holds.

Set $u_0(t)=1,\ t\in[0,1]$, then $u_0\in C\backslash\{0\},\ C(u_0)=\{u\in C|u(t)>0,\ t\in[0,1]\}$. Take $\sigma(u_0)=1$ and $u\in C(u_0)\cap\partial\Omega_1$, then $M_0r\leq u(t)\leq r,\ t\in[0,1]$. By ($H_4$), we have

$$\begin{aligned}\Psi u(t_0)=&\int_0^1 u(s)ds+\frac{\beta}{1-\delta^{\beta}}\int_0^1 l(s)f(s,\varphi_q(u(s)))ds\\ &+\int_0^1 k(t_0,s)[f(s,\varphi_q(u(s)))-\frac{\beta}{1-\delta^{\beta}}\int_0^1 l(\tau)f(\tau,\varphi_q(u(\tau)))d\tau]ds\\ =&\int_0^1 u(s)ds+\int_0^1 G_1(t_0,s)f(s,\varphi_q(u(s)))ds\\ \geq&\int_0^1 u(s)ds+\int_0^1\frac{1-M_0}{M_0}u(s)ds\\ \geq& M_0r+(1-M_0)r=r=\|u\|.\end{aligned}$$

So, $\|u\|\leq\sigma(u_0)\|\Psi u\|$, for $u\in C(u_0)\cap\partial\Omega_1$. Hence, (5) of Lemma 2.2 holds.

For $u(t)\in\partial\Omega_2,\ t\in[0,1]$, by ($H_3$) and (3.3), we get

$$\begin{aligned}(P+JQN)\gamma(u)=&\int_0^1|u(s)|ds+\frac{\beta}{1-\delta^{\beta}}\int_0^1 l(s)f(s,\varphi_q(|u(s)|))ds\\ \geq&\int_0^1(1-\frac{\beta K_1}{1-\delta^{\beta}}l(s))|u(s)|ds\geq 0.\end{aligned}$$

Thus, $(P+JQN)\gamma(\partial\Omega_2)\subset C$. So (6) of Lemma 2.2 holds.

For $u(t)\in\overline{\Omega}_2\backslash\Omega_1,\ t\in[0,1]$, by ($H_3$) and (3.3), we have

$$\begin{aligned}\Psi_\gamma(u(t))=&\int_0^1|u(s)|ds+\int_0^1 G_1(t,s)f(s,\varphi_q(|u(s)|))ds\\ \geq&\int_0^1|u(s)|ds-K_1\int_0^1 G_1(t,s)|u(s)|ds\\ =&\int_0^1(1-K_1G_1(t,s))|u(s)|ds\geq 0.\end{aligned}$$

Hence, (7) of Lemma 2.2 holds.

By Lemma 2.2, we see that the equation $Lu = Nu$ has a positive solution u. By Lemma 3.1, problem (1.1)(1.2) has at least one positive solution.

4 The existence of positive solution for problem (1.1)(1.3)

Since ${}^{C}D_{0+}^{\beta}[\varphi_p({}^{C}D_{0+}^{\alpha}(\cdot))]$ is a nonlinear operator, so we can't solve problem (1.1)(1.3) by Lemma 2.2. Hence, we provide the following lemma.

Lemma 4.1 $u(t)$ is a solution of the following problem:

$$\begin{cases} {}^{C}D_{0+}^{\beta}u(t) = f(t, \varphi_q(u(t))),\ t \in (0,1), \\ u(1) = \displaystyle\int_0^1 h(t)u(t)dt, \end{cases} \tag{4.1}$$

if and only if $x(t)$ is a solution of problem (1.1)(1.3), where $x(t) = I_{0+}^{\alpha}\varphi_q(u(t))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof The proof process is similar to Lemma 3.1, which is omitted here.

Let $X = Y = C[0,1]$ with the norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$. Take a cone $C = \{u(t) \in X | u(t) \geq 0,\ t \in [0,1]\}$. Define operators $L : \mathrm{dom}L \subset X \to Y$ and $N : X \to Y$ as follows:

$$Lu(t) = {}^{C}D_{0+}^{\beta}u(t),\ Nu(t) = f(t, \varphi_q(u(t))), \tag{4.2}$$

where $\mathrm{dom}L = \{u(t) | u(t),\ {}^{C}D_{0+}^{\beta}u(t) \in X,\ u(1) = \int_0^1 h(t)u(t)dt\}$. Then problem (4.1) can be written by $Lu = Nu,\ u \in \mathrm{dom}L$. For the simplicity of notation, let

$$G_2(t,s) = \begin{cases} \dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} - \dfrac{(1-s)^{\beta}}{\Gamma(\beta+1)} + \dfrac{\beta(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+2)})}{1 - \int_0^1 h(t)t^{\beta}dt} \\ \times [(1-s)^{\beta-1} - \displaystyle\int_s^1 h(t)(t-s)^{\beta-1}dt],\ 0 \leq s < t < 1, \\ -\dfrac{(1-s)^{\beta}}{\Gamma(\beta+1)} + \dfrac{\beta(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+2)})}{1 - \int_0^1 h(t)t^{\beta}dt}[(1-s)^{\beta-1} - \displaystyle\int_s^1 h(t)(t-s)^{\beta-1}dt],\ 0 \leq t \leq s < 1, \end{cases}$$

$$K_2 = \min\{1,\ \frac{1 - \int_0^1 h(t)t^{\beta}dt}{\beta \max\limits_{t,s\in[0,1]}[(1-s)^{\beta-1} - \int_s^1 h(t)(t-s)^{\beta-1}dt]},\ \frac{1}{\max\limits_{t,s\in[0,1]} G_2(t,s)}\}.$$

Thus, one has

$$1 - \frac{K_2\beta}{1 - \int_0^1 h(t)t^{\beta}dt}[(1-s)^{\beta-1} - \int_s^1 h(t)(t-s)^{\beta-1}dt] \geq 0,\ 1 - K_2G_2(t,s) \geq 0. \tag{4.3}$$

Below, we first give the main results of existence of positive solution for problem (1.1)(1.3).

Theorem 4.1 Assume that the conditions (H_1)-(H_2) hold. And the following conditions are satisfied.

(H_5) $f(t,u) \geq -K_2\varphi_p(u),\ t \in [0,1],\ u > 0.$

(H_6) There exist $r > 0,\ t_0 \in [0,1]$ and $M_0 \in (0,1)$ such that

$$G_2(t_0,s)f(s,u) \geq \frac{1-M_0}{M_0}\varphi_p(u),\ s \in [0,1),\ M_0 r \leq u \leq r.$$

Then problem (1.1)(1.3) has at least one positive solution.

Next, we give some important lemmas related to Theorem 4.1.

Lemma 4.2. Let L be defined by (4.2), then

$$\mathrm{Ker}L = \{u \in X | u(t) = c,\ c \in \mathbb{R},\ \forall t \in [0,1]\},$$

$$\mathrm{Im}L = \{y \in Y | \int_0^1 [(1-s)^{\beta-1} - \int_s^1 h(t)(t-s)^{\beta-1}dt]y(s)ds = 0\}.$$

The linear projection operators $P : X \to Y$ and $Q : Y \to Y$ can be defined as follows:

$$Pu(t) = \int_0^1 u(t)dt,$$

$$Qy(t) = \frac{\beta}{1 - \int_0^1 h(t)t^\beta dt}\int_0^1 [(1-s)^{\beta-1} - \int_s^1 h(t)(t-s)^{\beta-1}dt]y(s)ds,\ \forall t \in [0,1],$$

and $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ is defined as

$$K_P y(t) = \int_0^1 k(t,s)y(s)ds,\ \forall t \in [0,1],$$

where

$$k(t,s) = \begin{cases} \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} - \frac{1}{\Gamma(\beta+1)}(1-s)^\beta, & 0 \leq s \leq t \leq 1, \\ -\frac{1}{\Gamma(\beta+1)}(1-s)^\beta, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof The proof is similar to that of Lemma 3.2, 3.3 and is omitted.

Lemma 4.3 $QN : X \to Y$ is continuous and bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact, where $\Omega \subset X$ is bounded.

Proof The proof is similar to that of Lemma 3.4 and is omitted.

Lemma 4.4 If the condition (H_1) and (H_2) hold, the set

$$\Omega_0 = \{u(t) | Lu(t) = \lambda Nu(t),\ u(t) \in C \cap \mathrm{dom}L,\ \lambda \in (0,1)\}$$

is bounded.

Proof The proof is similar to that of Lemma 3.5 and is omitted.

Proof of Theorem 4.1 Set

$$\Omega_1 = \{u \in X | M_0 \|u\| < |u(t)| < r < R,\ t \in [0,1]\},\ \Omega_2 = \{u \in X | \|u\| < R\},$$

where $R=\max\{\varphi_p(R_0),N\}+1$. It is clear that Ω_1 and Ω_2 are open bounded sets of X, $\overline{\Omega}_1\subset\Omega_2$ and $C\cap(\overline{\Omega}_2\backslash\Omega_1)\neq\phi$. By Lemma 4.2, 4.3 and 4.4, we know that L is a Fredholm operator of index zero and the conditions (1), (2) of Lemma 2.2 are fulfilled.

Define $\gamma:X\to C$ as $\gamma u(t)=|u(t)|,\ u(t)\in X$ and $J:\mathrm{Im}Q\to\mathrm{Ker}L$ as $J(c)=c,\ c\in\mathbb{R}$. Then $\gamma:X\to C$ is a retraction and (3) of Lemma 2.2 holds.

For $u(t)\in\mathrm{Ker}L\cap\partial\Omega_2$, then $u(t)\equiv c$. Let

$$H(c,\lambda)=c-\lambda|c|-\frac{\lambda\beta}{1-\int_0^1h(t)t^\beta dt}\int_0^1[(1-s)^{\beta-1}-\int_s^1h(t)(t-s)^{\beta-1}dt]f(s,\varphi_q(|c|))ds,$$

where $\lambda\in[0,1]$. Suppose $H(c,\lambda)=0$, by (H_5), we have

$$\begin{aligned}c&=\lambda|c|+\frac{\lambda\beta}{1-\int_0^1h(t)t^\beta dt}\int_0^1[(1-s)^{\beta-1}-\int_s^1h(t)(t-s)^{\beta-1}dt]f(s,\varphi_q(|c|))ds\\&\geq\lambda|c|-\frac{\lambda\beta}{1-\int_0^1h(t)t^\beta dt}\int_0^1[(1-s)^{\beta-1}-\int_s^1h(t)(t-s)^{\beta-1}dt]K_2|c|ds\\&=\lambda|c|(1-K_2)\geq0.\end{aligned}$$

Thus $H(c,\lambda)=0$ implies $c\geq0$. Clearly, $H(R,0)\neq0$. Moreover, if $H(R,\lambda)=0,\ \lambda\in(0,1]$,we get

$$0\leq R(1-\lambda)=\frac{\lambda\beta}{1-\int_0^1h(t)t^\beta dt}\int_0^1[(1-s)^{\beta-1}-\int_s^1h(t)(t-s)^{\beta-1}dt]f(s,\varphi_q(R))ds,$$

which contradicts to condition (H_1). Hence $H(u,\lambda)\neq0$ for $u\in\mathrm{Ker}L\cap\partial\Omega_2,\ \lambda\in[0,1]$. Therefore,

$$\begin{aligned}&\deg([I-(P+JQN)\gamma]|_{\mathrm{Ker}L},\ \mathrm{Ker}L\cap\Omega_2,\ 0)\\&=\deg(H(x,1),\ \mathrm{Ker}L\cap\Omega_2,\ 0)=\deg(H(x,0),\ \mathrm{Ker}L\cap\Omega_2,\ 0)\\&=\deg(I,\ \mathrm{Ker}L\cap\Omega_2,\ 0)=1\neq0.\end{aligned}$$

Then, (4) of Lemma 2.2 holds.

Set $u_0(t)=1,\ t\in[0,1]$, then $u_0\in C\backslash\{0\},\ C(u_0)=\{u\in C|u(t)>0,\ t\in[0,1]\}$. Take $\sigma(u_0)=1$ and $u\in C(u_0)\cap\partial\Omega_1$, then $M_0r\leq u(t)\leq r,\ t\in[0,1]$. By ($\mathrm{H}_6$), we have

$$\begin{aligned}\Psi u(t_0)=&\int_0^1u(s)ds+\frac{\beta}{1-\int_0^1h(t)t^\beta dt}\int_0^1[(1-s)^{\beta-1}-\int_s^1h(t)(t-s)^{\beta-1}dt]f(s,\varphi_q(u(s)))ds\\&+\int_0^1k(t_0,s)\{f(s,\varphi_q(u(s)))-\frac{\beta}{1-\int_0^1h(t)t^\beta dt}\times\\&\int_0^1[(1-\tau)^{\beta-1}-\int_\tau^1h(t)(t-\tau)^{\beta-1}dt]f(\tau,\varphi_q(u(\tau)))d\tau\}ds\\&=\int_0^1u(s)ds+\int_0^1G_2(t_0,s)f(s,\varphi_q(u(s)))ds\geq\int_0^1u(s)ds+\int_0^1\frac{1-M_0}{M_0}u(s)ds\\&\geq M_0r+(1-M_0)r=r=\|u\|.\end{aligned}$$

So, $\|u\| \le \sigma(u_0)\,\|\Psi u\|$, for $u \in C(u_0) \cap \partial\Omega_1$. Hence, (5) of Lemma 2.2 holds.

For $u(t) \in \partial\Omega_2,\ t \in [0,1]$, by (H_5) and (4.3), we get

$$\begin{aligned}
&(P+JQN)\gamma(u)\\
&= \int_0^1 |u(s)|ds + \frac{\beta}{1-\int_0^1 h(t)t^\beta dt}\int_0^1 [(1-s)^{\beta-1} - \int_s^1 h(t)(t-s)^{\beta-1}dt] f(s,\varphi_q(|u(s)|))ds\\
&\ge \int_0^1 \{1 - \frac{\beta K_2}{1-\int_0^1 h(t)t^\beta dt}[(1-s)^{\beta-1} - \int_s^1 h(t)(t-s)^{\beta-1}dt]\}|u(s)|ds \ge 0.
\end{aligned}$$

Thus, $(P+JQN)\gamma(\partial\Omega_2) \subset C$. So (6) of Lemma 2.2 holds.

For $u(t) \in \overline{\Omega}_2\backslash\Omega_1,\ t \in [0,1]$, by (H_5) and (4.3), we have

$$\begin{aligned}
\Psi_\gamma(u(t)) &= \int_0^1 |u(s)|ds + \int_0^1 G_2(t,s) f(s,\varphi_q(|u(s)|))ds\\
&\ge \int_0^1 |u(s)|ds - K_2\int_0^1 G_2(t,s)|u(s)|ds = \int_0^1 (1-K_2G_2(t,s))|u(s)|ds \ge 0.
\end{aligned}$$

Hence, (7) of Lemma 2.2 holds. By Lemma 2.2, we see that the equation $Lu = Nu$ has a positive solution u. By Lemma 4.1, problem (1.1)(1.3) have at least one positive solution.

5 Example

Example 5.1 Consider the following problem

$$\begin{cases} {}^CD_{0+}^{\frac12}\varphi_2({}^CD_{0+}^{\frac12}x(t)) = \dfrac14 - \dfrac{1}{20}|{}^CD_{0+}^{\frac12}x(t)|^{\frac12}, \quad t \in (0,1),\\ x(0) = 0, \quad {}^CD_{0+}^{\frac12}x(1) = {}^CD_{0+}^{\frac12}x(\dfrac12), \end{cases} \tag{5.1}$$

where $\alpha = \beta = \delta = \frac12,\ p = 2,\ q = 2,\ f(t,\ {}^CD_{0+}^{\frac12}x(t)) = \frac14 - \frac{1}{20}|{}^CD_{0+}^{\frac12}x(t)|^{\frac12}$.

By Lemma 3.1, we have

$$\begin{cases} {}^CD_{0+}^{\frac12}u(t) = \dfrac14 - \dfrac{1}{20}|u(t)|^{\frac12},\\ u(1) = u(\dfrac12). \end{cases} \tag{5.2}$$

So, we get

$$l(s) = \begin{cases} \dfrac{1}{\sqrt{1-s}} - \dfrac{1}{\sqrt{\frac12 - s}}, & 0 \le s \le \dfrac12,\\ \dfrac{1}{\sqrt{1-s}}, & \dfrac12 \le s < 1. \end{cases}$$

$$G_1(t,s) = \begin{cases} \dfrac{1}{\Gamma(\frac12)}(t-s)^{-\frac12} - \dfrac{1}{\Gamma(\frac32)}(1-s)^{\frac12} + \dfrac{\frac12(\Gamma(\frac52)+1-\frac32 t^{\frac12})}{(1-\frac12^{\frac12})\Gamma(\frac52)}l(s), & 0 \le s < t \le 1,\\ -\dfrac{1}{\Gamma(\frac32)}(1-s)^{\frac12} + \dfrac{\frac12(\Gamma(\frac52)+1-\frac32 t^{\frac12})}{(1-\frac12^{\frac12})\Gamma(\frac52)}l(s), & 0 \le t \le s \le 1. \end{cases}$$

Take $R_0 = 25,\ a(t) = \frac{1}{4},\ b(t) = \frac{1}{20},\ K_1 \approx 0.23641,\ M_0 = 0.7,\ r = 0.04,\ t_0 = 0$. By simple calculations, we can see that

$$\begin{aligned}
&f(t,u) = \frac{1}{4} - \frac{1}{20}|u|^{\frac{1}{2}} < 0,\ u > 25,\\
&|f(t,u)| \le a(t) + b(t)\varphi_p(|u|),\\
&A = \max_{t\in[0,1]} \int_0^t (t-s)^{-\frac{1}{2}} \cdot \frac{1}{4} ds = \frac{1}{2} < +\infty,\\
&B = \max_{t\in[0,1]} \int_0^t (t-s)^{-\frac{1}{2}} \cdot \frac{1}{20} ds = \frac{1}{10} < \frac{\Gamma(\frac{1}{2})}{2},\\
&f(t,u) \ge -0.23641u,\ u > 0,\\
&G_1(t_0,s)f(s,u) \ge 0.375 - 0.4583u > \frac{0.3}{0.7}u,\ 0.028 \le u \le 0.04,\ s \in [0,1).
\end{aligned}$$

So the conditions (H_1)-(H_4) of Theorem 3.1 hold. By Theorem 3.1, we can conclude that problem (5.1) has at least one positive solution.

Example 5.2 Consider the following problem

$$\begin{cases} {}^C D_{0+}^{\frac{1}{2}} \varphi_2({}^C D_{0+}^{\frac{1}{2}} x(t)) = \frac{1}{4} - \frac{1}{20}|{}^C D_{0+}^{\frac{1}{2}} x(t)|^{\frac{1}{2}}, \quad t \in (0,1),\\ x(0) = 0, \quad \varphi_2({}^C D_{0+}^{\frac{1}{2}} x(1)) = \int_0^1 \varphi_2({}^C D_{0+}^{\frac{1}{2}} x(t)) dt, \end{cases} \tag{5.3}$$

where $\alpha = \beta = \frac{1}{2},\ p = 2,\ q = 2,\ f(t,\ {}^C D_{0+}^{\frac{1}{2}} x(t)) = \frac{1}{4} - \frac{1}{20}|{}^C D_{0+}^{\frac{1}{2}} x(t)|^{\frac{1}{2}},\ h(t) = 1 > 0,\ \int_0^1 h(t)dt = 1$.

By Lemma 4.1, we have

$$\begin{cases} {}^C D_{0+}^{\frac{1}{2}} u(t) = \frac{1}{4} - \frac{1}{20}|u(t)|^{\frac{1}{2}},\\ u(1) = \int_0^1 u(t)dt. \end{cases} \tag{5.4}$$

So, we get

$$G_2(t,s) = \begin{cases} -\frac{2}{\sqrt{\pi}}(1-s)^{\frac{1}{2}} + \frac{3}{2}[(1-s)^{-\frac{1}{2}} - 2(1-s)^{\frac{1}{2}}](1 - \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{4}{3\sqrt{\pi}}),\ 0 \le t \le s < 1,\\ \frac{(t-s)^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{2(1-s)^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{3}{2}[(1-s)^{-\frac{1}{2}} - 2(1-s)^{\frac{1}{2}}](1 - \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{4}{3\sqrt{\pi}}), 0 \le s < t < 1. \end{cases}$$

Take $R_0 = 25,\ a(t) = \frac{1}{4},\ b(t) = \frac{1}{20},\ K_2 \approx 0.14356,\ M_0 = 0.6,\ r = 0.02,\ t_0 = 0$. By simple calculations, we can see that all conditions of Theorem 4.1 are satisfied. Hence, by Theorem 4.1, we can conclude that problem (5.3) has at least one positive solution.

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具有p-Laplacian算子的分数阶问题共振正解的存在性

薛婷婷，姜永胜，曹　虹

(新疆工程学院数理学院，新疆 乌鲁木齐 830000)

摘要：本文研究了具有p-拉普拉斯算子的分数阶微分方程在两种边界条件下的共振正解存在的问题. 利用Leggett-Williams范型定理的方法，获得了一些新的存在性结果，推广了该类问题已有的研究结果.

关键词：p-Laplacian算子; Leggett-Williams范数型定理; 共振; 正解

MR(2010)**主题分类号**：34A08; 34B15　　**中图分类号**：O175.8