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INEQUALITIES FOR EIGENVALUES OF THE SUB-LAPLCAIN ON THE ENGEL GROUP

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Abstract: The Engel groups are one important kind of simply connected nilpotent Lie groups in sub-Riemannian geometry. In this paper, we investigate the Dirichlet eigenvalue problem of the sub-Laplacian $\Delta_{\mathbb{E}}$ on a bounded domain Ω of the Engel group $\mathbb{E}=(\mathbb{R}^4,\circ,\{\delta_\lambda\})$ as follows

$$\begin{cases} (-\Delta_{\mathbb{E}})^3 u=\lambda u, & \text{in } \Omega,\\ u=\dfrac{\partial u}{\partial \nu}=\dfrac{\partial^2 u}{\partial \nu^2}=0, & \text{on } \partial\Omega, \end{cases}$$

where ν is the outwards unit normal vector field of $\partial\Omega$. We establish some universal inequalities for eigenvalues of this problem.

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1 Introduction

A sub-Riemannian manifold is a manifold endowed with a distribution and a fiber inner product on that distribution. It becomes a Riemannian manifold when the distribution under consideration is the entire tangential plexus. Sub-Riemannian manifolds have a wide range of applications, which are closely related to geometric cybernetics, CR manifolds, image processing and nonholonomic mechanical systems(see [1–7]).

With the deepening of the study of sub-Riemannian geometry, the importance of Carnot groups gradually emerged. Carnot groups play a role, for sub-Riemannian manifolds, analogous to that played by Euclidean vector spaces for Riemannian manifolds. Many scholars have also obtained some important results in this regard (cf. [3, 4, 5, 7]). As an important topic in geometry and analysis, people have obtained some interesting results on the spectrum of Laplace operator. It is natural to consider whether one can extend the results for Riemannian manifolds to sub-Riemannian manifolds.

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The Heisenberg group $\mathbb{H}^n$ is a classical example of Carnot groups. In 2003, Niu and Zhang [8] considered the following eigenvalue problem of the sub-Laplacian $\Delta_{\mathbb{H}^n}$ on a bounded domain Ω of $\mathbb{H}^n$

$$\begin{cases} (-\Delta_{\mathbb{H}^n})^k u = \lambda u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the outwards unit normal vector field of $\partial\Omega$. They proved that when k is odd, it holds

$$\lambda_{m+1} - \lambda_m \leq \frac{1}{m^2 n^2} \sum_{i=1}^{m} \lambda_i^{\frac{1}{k}} \left[(2n+4)k \sum_{i=1}^{m} \lambda_i^{\frac{k-1}{k}} + C_1(n,k) \sum_{i=1}^{m} (\lambda_i + \lambda_i^{\frac{k-2}{k}}) \right], \tag{1.2}$$

and when k is even, it holds

$$\lambda_{m+1} - \lambda_m \leq \frac{1}{m^2 n^2} \sum_{i=1}^{m} \lambda_i^{\frac{1}{k}} \left[(2n+4)k \sum_{i=1}^{m} \lambda_i^{\frac{k-1}{k}} + C_2(n,k) \sum_{i=1}^{m} \lambda_i^{\frac{k-1}{k}} \right], \tag{1.3}$$

where $C_1(n,k)$ and $C_2(n,k)$ are the constants depending on n and k. In 2010, Ilias and Makhoul [9] established some Yang-type inequalities for problem (1.1): for any odd $k \geq 3$, it holds

$$\begin{aligned} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq & \frac{1}{n} \left\{ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \left[(2k(n+k-1)) \lambda_i^{\frac{k-1}{k}} + C_1(n,k)(\lambda_i + \lambda_i^{\frac{k-2}{k}}) \right] \right\}^{\frac{1}{2}} \\ & \times \left[\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \lambda_i^{\frac{1}{k}} \right]^{\frac{1}{2}}, \end{aligned} \tag{1.4}$$

and for any even $k \geq 4$, it holds

$$\begin{aligned} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq & \frac{1}{n} \left\{ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \left[(2kn + 4(k-1)) \lambda_i^{\frac{k-1}{k}} + C_2(n,k) \lambda_i^{\frac{k-2}{k}} \right] \right\}^{\frac{1}{2}} \\ & \times \left[\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \lambda_i^{\frac{1}{k}} \right]^{\frac{1}{2}}, \end{aligned} \tag{1.5}$$

where $C_1(n,k)$ and $C_2(n,k)$ are the constants depending on n and k. In 2017, Du, Wu, Li and Xia [10] consider the following eigenvalue problem of the biharmonic sub-Laplacian on a bounded domain Ω on a Carnot group $\mathbb{G}$ with an d-dimensional sub-bundle

$$\begin{cases} (-\Delta_{\mathbb{G}})^2 u = \lambda u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

They obtained the following inequality for eigenvalues of problem (1.2)

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \left(\frac{8d+2}{d^2} \right)^{\frac{1}{2}} \left[\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}}. \tag{1.7}$$

Since the Heisenberg group $\mathbb{H}^n$ is a 2-step Carnot group, its generators are interchangeable with other layers. However, for some more general Carnot groups, their generators are not interchangeable with any other layer except the last layer. Hence it is difficult to directly apply the method of [8] to problem (1.1) on Carnot groups with any order.

In this paper, we consider the Engel groups. As a 3-step Carnot group, the generators of an Engel group $\mathbb{E}$ are not interchangeable with the second layer. In recent years, the research on Engel groups has made some achievements. For example, Ardentov and Sachkov [11] considered the left invariant sub-Riemannian problem on Engel groups, which plays an important role in the motion system of mobile robots with trailers. Here we investigate the Dirichlet eigenvalue problem of the sub-Laplacian $\Delta_{\mathbb{E}}$ on a bounded domain Ω of the Engel group $\mathbb{E} = (\mathbb{R}^4, \circ, \{\delta_\lambda\})$ as follows

$$\begin{cases} (-\Delta_{\mathbb{E}})^3 u = \lambda u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \dfrac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where ν is the outwards unit normal vector field of $\partial\Omega$. Set

$$S^{3,2}(\Omega) = \left\{ f : f, X_i(f), X_i^2(f), X_i^3(f) \in L^2(\Omega), i = 1, 2 \right\}.$$

The subspace $S_0^{3,2}(\Omega)$ of $S^{3,2}(\Omega)$ is defined by

$$S_0^{3,2}(\Omega) = \left\{ f \in S^{3,2}(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial v}|_{\partial\Omega} = \frac{\partial^2 f}{\partial v^2}|_{\partial\Omega} = 0 \right\}.$$

Then we know that $(\Delta_{\mathbb{E}})^3$ is a self-adjoint operator acting on $S_0^{3,2}(\Omega)$ with a discrete spectrum. Thus problem (1.8) has a discrete spectrum

$$0 < \lambda_1 \le \lambda_2 \le \dots \lambda_m \le \dots \to +\infty,$$

where each eigenvalue is repeated with its multiplicity (see [12]).

In this paper, we establish the following results for problem (1.8).

Theorem 1.1 Let Ω be a bounded domain on an Engel group $\mathbb{E}$. Denote by λ_i the i-th eigenvalue of problem (1.8). Then we have

$$\begin{aligned} \sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^2 \le & \left[\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^2 (140\lambda_i + 18\lambda_i^{\frac{2}{3}} + 8\lambda_i^{\frac{1}{3}}) \right]^{\frac{1}{2}} \\ & \times \left[\sum_{i=1}^m (\lambda_{m+1} - \lambda_i) \lambda_i^{\frac{1}{3}} \right]^{\frac{1}{2}}. \end{aligned} \tag{1.9}$$

Theorem 1.2 Under the assumptions of Theorem 1.1, we have

$$\lambda_{m+1} - \lambda_m \le \frac{1}{m^2} \sum_{i=1}^m \lambda_i^{\frac{1}{3}} \sum_{i=1}^m \left(140\lambda_i + 18\lambda_i^{\frac{2}{3}} + 8\lambda_i^{\frac{1}{3}} \right). \tag{1.10}$$

2 Preliminaries

In this section, we give some preliminary knowledge about the Engle groups and establish some necessary lemmas.

A r-step Carnot group $\mathbb{G}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$, such that

$$\begin{cases} [\mathfrak{g}_1,\mathfrak{g}_{i-1}] = \mathfrak{g}_i, & \text{if } 2 \le i \le r, \\ [\mathfrak{g}_i,\mathfrak{g}_j] = \{0\}, & \text{if } 2 \le i, j \le r. \end{cases}$$

If $\dim V_1 = d$, we also say that $\mathbb{G}$ has d generators. The vector fields $X_1, \dots, X_d$ are called the generators of $\mathbb{G}$, whereas any basis of $\mathrm{span}\{X_1, \dots, X_d\}$ is called a system of generators of $\mathbb{G}$. A sub-Laplacian on $\mathbb{G}$ is the second order differential operator defined by

$$\mathfrak{L} = \sum_{i=1}^{d} Y_i^2,$$

where $Y_1, \dots, Y_d$ is a basis of $\mathrm{span}\{X_1, \dots, X_d\}$. In special, $\Delta_{\mathbb{G}} = \sum_{i=1}^d X_i^2$ is the canonical sub-Laplacian on $\mathbb{G}$. The vector operator $\nabla_{\mathbb{G}} = (X_1, \dots, X_d)$ is called the horizontal $\mathbb{G}$-gradient.

The Engel algebra $\mathfrak{h}$ is the finite dimensional Lie algebra with a basis $(X_1, \dots, X_d)$, where the only non-vanishing commutator relationship among the generators are

$$[X_2, X_1] = X_3, \quad [X_3, X_1] = [X_3, X_2] = X_4, \quad [X_4, X_k] = 0, \quad k = 1, 2, 3. \tag{2.1}$$

The Engel algebra $\mathfrak{h}$ is of step 3. In fact, the Engel algebra is stratified as follows

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3,$$

where $\mathfrak{h}_1 = \mathrm{span}\{X_1, X_2\}$, $\mathfrak{h}_2 = \mathrm{span}\{X_3\}$ and $\mathfrak{h}_3 = \mathrm{span}\{X_4\}$. Thus the Engel group $\mathbb{E}$ is a simply connected nilpotent Lie group associated to the Engel algebra $\mathfrak{h}$. We can represent the Engel group $\mathbb{E} = (\mathbb{R}^4, \circ, \{\delta_\lambda\})$ by means of graded coordinates associated to the basis $(X_1, \dots, X_d)$. For any (x_1, x_2, x_3, x_4), $(y_1, y_2, y_3, y_4) \in \mathbb{E}$, it holds

$$(x_1, x_2, x_3, x_4) \circ (y_1, y_2, y_3, y_4) = \left\{ \begin{array}{c} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + f(x_1, x_2, y_1, y_2) \\ x_4 + y_4 + g(x_1, x_2, x_3, y_1, y_2, y_3) \end{array} \right\}^T,$$

where f and g are two polynomials. Moreover, the homogeneous dilations on $\mathbb{E}$ are

$$\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4),$$

where $\lambda > 0$. The polynomials can be different in different application scenarios (see [13, 14, 15, 16]). Depending on different f and g, the rappresentation of the basis (X_1, X_2, X_3, X_4)

on the graded coordinates is given as follows

$$\begin{cases} X_1(x_1,x_2,x_3,x_4) = \dfrac{\partial}{\partial x_1} + \dfrac{\partial f}{\partial x_3}\dfrac{\partial}{\partial x_3} + \dfrac{\partial g}{\partial x_4}\dfrac{\partial}{\partial x_4}, \\ X_2(x_1,x_2,x_3,x_4) = \dfrac{\partial}{\partial x_2} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial}{\partial x_3} + \dfrac{\partial g}{\partial x_2}\dfrac{\partial}{\partial x_4}, \\ X_3(x_1,x_2,x_3,x_4) = \dfrac{\partial}{\partial x_3} + \dfrac{\partial g}{\partial x_3}\dfrac{\partial}{\partial x_4}, \\ X_4(x_1,x_2,x_3,x_4) = \dfrac{\partial}{\partial x_4}. \end{cases}$$

The horizontal $\mathbb{E}$-gradient $\nabla_{\mathbb{E}}$ on the Engle group $\mathbb{E}$ is defined by $\nabla_{\mathbb{E}} u_i = (X_1u_i, X_2u_i)$. The sub-Laplacian on $\mathbb{E}$ is defined by

$$\Delta_{\mathbb{E}} = X_1^2 + X_2^2.$$

For simplicity's sake, we denote $-\Delta_{\mathbb{E}}$ by $\mathbb{L}$.

In order to prove the main theorems of this paper, we first give the following lemmas.

Lemma 2.1 Let Ω be a bounded domain on the Engel group $\mathbb{E}$. Denote by u_i the i-th orthonormal eigenfunction of problem (1.8). For $p = 1, 2$, we have

$$\left(\int_\Omega u_i \mathbb{L}^p u_i\right)^{\frac{1}{p}} \le \left(\int_\Omega u_i \mathbb{L}^{p+1} u_i\right)^{\frac{1}{p+1}}. \tag{2.2}$$

Proof Using Holder inequality, we have

$$\int_\Omega u_i \mathbb{L} u_i \le \left(\int_\Omega u_i^2\right)^{\frac{1}{2}} \left(\int_\Omega (\mathbb{L}u_i)^2\right)^{\frac{1}{2}} = \left(\int_\Omega u_i \mathbb{L}^2 u_i\right)^{\frac{1}{2}}. \tag{2.3}$$

Then it is from (2.3) that

$$\begin{aligned} \int_\Omega u_i \mathbb{L}^2 u_i &= -\int_\Omega \nabla_{\mathbb{E}} u_i \nabla_{\mathbb{E}}^3 u_i \\ &\le \left(\int_\Omega |\nabla_{\mathbb{E}} u_i|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla_{\mathbb{E}}^3 u_i|^2\right)^{\frac{1}{2}} \\ &= \left(\int_\Omega u_i \mathbb{L} u_i\right)^{\frac{1}{2}} \left(\int_\Omega u_i \mathbb{L}^3 u_i\right)^{\frac{1}{2}} \\ &\le \left(\int_\Omega u_i \mathbb{L}^2 u_i\right)^{\frac{1}{4}} \left(\int_\Omega u_i \mathbb{L}^3 u_i\right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 2.1 is proved.

Lemma 2.2 Under the same assumptions of Lemma 2.1, we have

$$\int_\Omega X_4 \mathbb{L} u_i X_4 u_i \le \lambda_i^{\frac{1}{3}} \int_\Omega (X_4 u_i)^2. \tag{2.4}$$

Proof Using Holder inequality, and noticing that

$$\int_\Omega (X_4 \mathbb{L}^2 u_i)^2 = \int_\Omega X_4 \mathbb{L}^4 u_i X_4 u_i = \lambda_i \int_\Omega X_4 \mathbb{L} u_i X_4 u_i,$$

we deduce

$$\begin{aligned}\int_\Omega X_4\mathbb{L}u_iX_4u_i &\le \left[\int_\Omega (X_4\mathbb{L}u_i)^2\right]^{\frac{1}{2}}\left[\int_\Omega (X_4u_i)^2\right]^{\frac{1}{2}}\\ &= \left(\int_\Omega X_4\mathbb{L}^2u_iX_4u_i\right)^{\frac{1}{2}}\left[\int_\Omega (X_4u_i)^2\right]^{\frac{1}{2}}\\ &\le \left[\int_\Omega (X_4\mathbb{L}^2u_i)^2\right]^{\frac{1}{4}}\left[\int_\Omega (X_4u_i)^2\right]^{\frac{3}{4}}\\ &= \lambda_i^{\frac{1}{4}}\left(\int_\Omega X_4\mathbb{L}u_iX_4u_i\right)^{\frac{1}{4}}\left[\int_\Omega (X_4u_i)^2\right]^{\frac{3}{4}}.\end{aligned} \tag{2.5}$$

It yields (2.4). The proof of Lemma 2.2 is finished.

3 Proofs of the Main Results

In this section, we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 For $i = 1, \dots, m$ and $j = 1, 2$, take the trial functions

$$\varphi_{ix_j} = x_ju_i - \sum_{l=1}^m a_{ilx_j}u_l,$$

where $a_{ilx_j} = \int_\Omega x_ju_iu_l$. It is easy to find that φ_{ix_j} is orthogonal to $u_1, \dots, u_m$. According to the Rayleigh-Ritz principle, it holds

$$\lambda_{m+1} \le \frac{\int_\Omega \varphi_{ix_j}\mathbb{L}^3\varphi_{ix_j}}{\int_\Omega \varphi_{ix_j}^2}, \quad \text{for } j = 1, 2. \tag{3.1}$$

Since $\int_\Omega u_l\varphi_{ix_j} = 0$ and $\displaystyle\int_\Omega u_l\mathbb{L}^3(x_ju_i) = \int_\Omega x_ju_i\mathbb{L}^3u_l = \lambda_la_{ilx_j}$, we obtain

$$\begin{aligned}(\lambda_{m+1}-\lambda_i)\int_\Omega \varphi_{ix_j}^2 &\le \int_\Omega \varphi_{ix_j}\mathbb{L}^3(x_ju_i) - \lambda_i\int_\Omega x_ju_i\varphi_{ix_j}\\ &= \int_\Omega x_ju_i\left[\mathbb{L}^3(x_ju_i)-\lambda_ix_ju_i\right] - \sum_{i=1}^m a_{ilx_j}\int_\Omega u_l\left[\mathbb{L}^3(x_ju_i)-\lambda_ix_ju_i\right]\\ &= \int_\Omega x_ju_i\left[\mathbb{L}^3(x_ju_i)-\lambda_ix_ju_i\right] - \sum_{i=1}^m a_{ilx_j}^2(\lambda_l-\lambda_i).\end{aligned} \tag{3.2}$$

Noticing that

$$\begin{aligned}X_1^2(x_1u_i) =&2X_1u_i + x_1X_1^2u_i, & X_2^2(x_1u_i) &= x_1X_2^2u_i,\\ X_1^2(x_2u_i) =&x_2X_1^2u_i, & X_2^2(x_2u_i) &= 2X_2u_i + x_2X_2^2u_i,\end{aligned} \tag{3.3}$$

we deduce

$$\mathbb{L}(x_ju_i) = -2X_ju_i - x_jX_1^2u_i - x_jX_2^2u_i = x_j\mathbb{L}u_i - 2X_ju_i, \tag{3.4}$$

$$\mathbb{L}^2(x_ju_i) = \mathbb{L}(x_j\mathbb{L}u_i - 2X_ju_i) = x_j\mathbb{L}^2u_i - 2\mathbb{L}X_ju_i - 2X_j\mathbb{L}u_i. \tag{3.5}$$

Therefore, we can get

$$\begin{aligned}\mathbb{L}^3(x_ju_i) =&\mathbb{L}(x_j\mathbb{L}^2u_i - 2\mathbb{L}X_ju_i - 2X_j\mathbb{L}u_i)\\ =&x_j\mathbb{L}^3u_i - 2\mathbb{L}^2X_ju_i - 2\mathbb{L}X_j\mathbb{L}u_i - 2X_j\mathbb{L}^2u_i.\end{aligned} \tag{3.6}$$

Then we have

$$\int_\Omega x_ju_i\left[\mathbb{L}^3(x_ju_i) - \lambda_ix_ju_i\right] = -2\int_\Omega x_ju_i(\mathbb{L}^2X_j + \mathbb{L}X_j\mathbb{L} + X_j\mathbb{L}^2)u_i. \tag{3.7}$$

According to the properties in (2.1), it is not difficult to find that

$$\mathbb{L}X_1 = X_1\mathbb{L} - 2X_2X_3 - X_4, \tag{3.8}$$

$$\mathbb{L}X_2 = X_2\mathbb{L} + 2X_1X_3 + X_4, \tag{3.9}$$

$$\mathbb{L}X_3 = X_3\mathbb{L} + 2X_1X_4 + 2X_2X_4. \tag{3.10}$$

Then, for $j = 1, 2$, we obtain

$$\begin{aligned}&\int_\Omega x_ju_i(\mathbb{L}^2X_j + \mathbb{L}X_j\mathbb{L} + X_j\mathbb{L}^2)u_i\\ =&\int_\Omega X_ju_i\mathbb{L}^2(x_ju_i) + \int_\Omega X_j\mathbb{L}u_i\mathbb{L}(x_ju_i) + \int_\Omega x_ju_iX_j\mathbb{L}^2u_i\\ =&\int_\Omega X_ju_i(x_j\mathbb{L}^2u_i - 2X_j\mathbb{L}u_i - 2\mathbb{L}X_ju_i) + \int_\Omega(x_j\mathbb{L}u_i - 2X_ju_i)X_j\mathbb{L}u_i + \int_\Omega x_ju_iX_j\mathbb{L}^2u_i\\ =&\int_\Omega(x_j\mathbb{L}^2u_iX_ju_i + x_j\mathbb{L}u_iX_j\mathbb{L}u_i + x_ju_iX_j\mathbb{L}^2u_i) + 4\int_\Omega\mathbb{L}u_iX_j^2u_i - 2\int_\Omega\mathbb{L}X_ju_iX_ju_i,\end{aligned} \tag{3.11}$$

Moreover, since

$$\begin{aligned}&\int_\Omega(x_j\mathbb{L}^2u_iX_ju_i + x_j\mathbb{L}u_iX_j\mathbb{L}u_i + x_ju_iX_j\mathbb{L}^2u_i)\\ =&-\int_\Omega\left(u_i\mathbb{L}^2u_i + x_ju_iX_j\mathbb{L}^2u_i\right) - \int_\Omega\left[(\mathbb{L}u_i)^2 + x_j\mathbb{L}u_iX_j\mathbb{L}u_i\right] - \int_\Omega u_i\mathbb{L}^2u_i - \int_\Omega x_iX_ju_i\mathbb{L}^2u_i\\ =&-3\int_\Omega u_i\mathbb{L}^2u_i - \int_\Omega(x_j\mathbb{L}^2u_iX_ju_i + x_j\mathbb{L}u_iX_j\mathbb{L}u_i + x_ju_iX_j\mathbb{L}^2u_i),\end{aligned} \tag{3.12}$$

we obtain

$$\int_\Omega(x_j\mathbb{L}^2u_iX_ju_i + x_j\mathbb{L}u_iX_j\mathbb{L}u_i + x_ju_iX_j\mathbb{L}^2u_i) = -\frac{3}{2}\int_\Omega u_i\mathbb{L}^2u_i. \tag{3.13}$$

Therefore, substituting (3.11) and (3.13) into (3.7), we obtain

$$\int_\Omega x_ju_i\left[\mathbb{L}^3(x_ju_i) - \lambda_ix_ju_i\right] \le \int_\Omega(3u_i\mathbb{L}^2u_i - 8\mathbb{L}u_iX_j^2u_i + 4\mathbb{L}X_ju_iX_ju_i). \tag{3.14}$$

On the other hand, from $-2\int_\Omega x_ju_iX_ju_i = 2\int_\Omega u_i^2 + 2\int_\Omega x_ju_iX_ju_i$, we get

$$-2\int_\Omega x_ju_iX_ju_i = 1. \tag{3.15}$$

Set $t_{jli} = \int_\Omega u_i X_j u_l$. It is easy to find that $t_{jli} = -t_{jil}$. Then we have

$$-2\int_\Omega \varphi_{ix_j} X_j u_i = 2\int_\Omega u_i X_j(x_j u_i) - 2\sum_{l=1}^m a_{ilx_j}\int_\Omega u_i X_j u_l = 1 + 2\sum_{l=1}^m a_{ilx_j} t_{jil}. \tag{3.16}$$

Multiplying both sides of (3.16) by $(\lambda_{m+1}-\lambda_i)^2$, we get

$$\begin{aligned}(\lambda_{m+1}-\lambda_i)^2(1+2\sum_{l=1}^m a_{ilx_j}t_{jil}) &= -2(\lambda_{m+1}-\lambda_i)^2\int_\Omega \varphi_{ix_j}(X_j u_i - \sum_{l=1}^m t_{jil}u_l)\\ &\le \delta(\lambda_{m+1}-\lambda_i)^3\int_\Omega \varphi_{ix_j}^2 + \frac{\lambda_{m+1}-\lambda_i}{\delta}\int_\Omega (X_j u_i - \sum_{l=1}^m t_{jil}u_l)^2.\end{aligned}$$

Then, using (3.2), we get

$$\begin{aligned}&(\lambda_{m+1}-\lambda_i)^2(1+2\sum_{l=1}^m a_{ilx_j}t_{jil})\\ \le&\delta(\lambda_{m+1}-\lambda_i)^2\int_\Omega x_j u_i\left[\mathbb{L}^3(x_j u_i) - \lambda_i x_j u_i\right] + \frac{(\lambda_{m+1}-\lambda_i)}{\delta}\int_\Omega (X_j u_i)^2\\ &-\delta(\lambda_{m+1}-\lambda_i)\sum_{l=1}^m(\lambda_l-\lambda_i)^2 a_{ilx_j}^2 - \frac{1}{\delta}(\lambda_{m+1}-\lambda_i)\sum_{l=1}^m t_{jil}^2.\end{aligned} \tag{3.17}$$

Substituting

$$(\lambda_{m+1}-\lambda_i)^2(1+2\sum_{l=1}^m a_{ilx_j}t_{jil}) \ge (\lambda_{m+1}-\lambda_i)^2 + 2(\lambda_{m+1}-\lambda_i)\sum_{l=1}^m(\lambda_l-\lambda_i)a_{ilx_j}t_{jil}$$

and

$$\delta\sum_{l=1}^m(\lambda_l-\lambda_i)^2 a_{ilx_j}^2 + \frac{1}{\delta}\sum_{l=1}^m t_{jil}^2 \ge -2\sum_{l=1}^m(\lambda_l-\lambda_i)a_{ilx_j}t_{jil}$$

into (3.17), we deduce

$$\begin{aligned}(\lambda_{m+1}-\lambda_i)^2 \le&\delta(\lambda_{m+1}-\lambda_i)^2\int_\Omega x_j u_i\left[\mathbb{L}^3(x_j u_i)-\lambda_i x_j u_i\right]\\ &+\frac{1}{\delta}(\lambda_{m+1}-\lambda_i)\int_\Omega (X_j u_i)^2.\end{aligned} \tag{3.18}$$

Taking sum on i from 1 to m, j from 1 to 2, and using (3.14), we obtain

$$\begin{aligned}2\sum_{i=1}^m(\lambda_{m+1}-\lambda_i)^2 \le&\delta\sum_{i=1}^m(\lambda_{m+1}-\lambda_i)^2\sum_{j=1}^2\int_\Omega\left(3u_i\mathbb{L}^2u_i - 8\mathbb{L}u_iX_j^2u_i + 4\mathbb{L}X_ju_iX_ju_i\right)\\ &+\frac{1}{\delta}\sum_{i=1}^m(\lambda_{m+1}-\lambda_i)\sum_{j=1}^2\int_\Omega (X_j u_i)^2.\end{aligned} \tag{3.19}$$

Now we estimate the term $\int_\Omega (\mathbb{L}X_1u_iX_1u_i + \mathbb{L}X_2u_iX_2u_i)$. From (3.8) and (3.9), we get

$$\begin{aligned}\int_\Omega (\mathbb{L}X_1u_iX_1u_i + \mathbb{L}X_2u_iX_2u_i) &= \int_\Omega (X_1\mathbb{L}u_iX_1u_i - 2X_2X_3u_iX_1u_i - X_4u_iX_1u_i)\\ &\quad + \int_\Omega (X_2\mathbb{L}u_iX_2u_i + 2X_1X_3u_iX_2u_i + X_4u_iX_2u_i)\\ &= \int_\Omega \left[u_i\mathbb{L}^2u_i + X_4u_iX_2u_i - X_4u_iX_1u_i + 2(X_3u_i)^2\right]\\ &= \int_\Omega u_i\mathbb{L}^2u_i + \int_\Omega (X_4u_iX_2u_i - X_4u_iX_1u_i) + 2\int_\Omega (X_3u_i)^2,\end{aligned} \tag{3.20}$$

Using mean value inequality, we have

$$\begin{aligned}\int_\Omega (X_3u_i)^2 &= -\int_\Omega X_1u_iX_2X_3u_i + \int_\Omega X_2u_iX_1X_3u_i\\ &\le \frac{1}{2}\int_\Omega [(X_1u_i)^2 + (X_2X_3u_i)^2] + \frac{1}{2}\int_\Omega [(X_2u_i)^2 + (X_1X_3u_i)^2]\\ &= \frac{1}{2}\int_\Omega u_i\mathbb{L}u_i + \frac{1}{2}\int_\Omega \mathbb{L}X_3u_iX_3u_i\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\int_\Omega (X_4u_iX_2u_i - X_4u_iX_1u_i) &\le \frac{1}{2}\int_\Omega [(X_4u_i)^2 + (X_1u_i)^2] + \frac{1}{2}\int_\Omega [(X_4u_i)^2 + (X_2u_i)^2]\\ &= \frac{1}{2}\int_\Omega u_i\mathbb{L}u_i + \int_\Omega (X_4u_i)^2.\end{aligned} \tag{3.22}$$

Moreover, it is easy to verify that

$$2\int_\Omega X_1X_4u_iX_3u_i = 2\int_\Omega X_2X_4u_iX_3u_i = \int_\Omega (X_4u_i)^2. \tag{3.23}$$

Then, using (3.10) and (3.23), we obtain

$$\begin{aligned}\int_\Omega \mathbb{L}X_3u_iX_3u_i &= \int_\Omega (X_3\mathbb{L}u_iX_3u_i + 2X_1X_4u_iX_3u_i + 2X_2X_4u_iX_3u_i)\\ &= \int_\Omega X_3\mathbb{L}u_iX_3u_i + 2\int_\Omega (X_4u_i)^2.\end{aligned} \tag{3.24}$$

Therefore, substituting (3.21), (3.22) and (3.24) into (3.20), we get

$$\begin{aligned}\int_\Omega (\mathbb{L}X_1u_iX_1u_i + \mathbb{L}X_2u_iX_2u_i) \le \int_\Omega u_i\mathbb{L}^2u_i + \frac{3}{2}\int_\Omega u_i\mathbb{L}u_i + \int_\Omega X_3\mathbb{L}u_iX_3u_i\\ + 3\int_\Omega (X_4u_i)^2.\end{aligned} \tag{3.25}$$

Using mean value inequality and Lemma 2.2 , for any positive ε and δ, we have

$$\begin{aligned}\int_\Omega (X_4u_i)^2 &= \int_\Omega X_1X_4u_iX_3u_i + \int_\Omega X_2X_4u_iX_3u_i\\ &\le \frac{\delta}{2}\int_\Omega X_4\mathbb{L}u_iX_4u_i + \frac{1}{\delta}\left(\frac{1}{2\varepsilon}\int_\Omega \mathbb{L}X_3u_iX_3u_i + \frac{\varepsilon}{2}\int_\Omega u_i\mathbb{L}u_i\right)\\ &\le \frac{\delta}{2}\lambda_i^{\frac{1}{3}}\int_\Omega (X_4u_i)^2 + \frac{1}{\delta}\left(\frac{1}{2\varepsilon}\int_\Omega \mathbb{L}X_3u_iX_3u_i + \frac{\varepsilon}{2}\int_\Omega u_i\mathbb{L}u_i\right).\end{aligned}$$

It yields

$$(1-\frac{\delta}{2}\lambda_i^{\frac{1}{3}})\int_\Omega (X_4u_i)^2 \le \frac{1}{2\delta\varepsilon}\int_\Omega \mathbb{L}X_3u_iX_3u_i + \frac{\varepsilon}{2\delta}\int_\Omega u_i\mathbb{L}u_i. \tag{3.26}$$

Taking $\frac{\delta}{2}\lambda_i^{\frac{1}{3}} = \frac{1}{2}$ and $\varepsilon = 5\lambda_i^{\frac{1}{3}}$ in (3.26), using Lemma 2.1, we get

$$2\int_\Omega (X_4u_i)^2 \le \frac{2}{5}\int_\Omega \mathbb{L}X_3u_iX_3u_i + 10\lambda_i. \tag{3.27}$$

And

$$\begin{aligned}\int_\Omega X_3\mathbb{L}u_iX_3u_i =& -\int_\Omega X_1\mathbb{L}u_iX_2X_3u_3 + \int_\Omega X_2\mathbb{L}u_iX_1X_3u_i\\ \le&\frac{5}{2}\int_\Omega (X_1\mathbb{L}u_i)^2 + \frac{1}{10}\int_\Omega (X_2X_3u_i)^2 + \frac{5}{2}\int_\Omega (X_2\mathbb{L}u_i)^2 + \frac{1}{10}\int_\Omega (X_1X_3u_i)^2\\ =&\frac{1}{10}\int_\Omega \mathbb{L}X_3u_iX_3u_i + \frac{5}{2}\lambda_i.\end{aligned} \tag{3.28}$$

It is from (3.24), (3.27) and (3.28) that

$$\int_\Omega \mathbb{L}X_3u_iX_3u_i \le 25\lambda_i. \tag{3.29}$$

Then, combining (3.29) with (3.27), we derive

$$\int_\Omega (X_4u_i)^2 \le 10\lambda_i. \tag{3.30}$$

Substituting (3.29) and (3.30) into (3.25), and using Lemma 2.1, we have

$$\begin{aligned}\int_\Omega (\mathbb{L}X_1u_iX_1u_i + \mathbb{L}X_2u_iX_2u_i) &\le \int_\Omega u_i\mathbb{L}^2u_i + \frac{3}{2}\int_\Omega u_i\mathbb{L}u_i + 35\lambda_i\\ &\le \lambda_i^{\frac{2}{3}} + \frac{3}{2}\lambda_i^{\frac{1}{3}} + 35\lambda_i.\end{aligned} \tag{3.31}$$

Notice that

$$\sum_{j=1}^{2}\int_\Omega (X_ju_i)^2 = \int_\Omega u_i\mathbb{L}u_i \le \lambda_i^{\frac{1}{3}}. \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.19), and using Lemma 2.1, we obtain

$$2\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2 \leq \delta\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2\left(140\lambda_i+18\lambda_i^{\frac{2}{3}}+8\lambda_i^{\frac{1}{3}}\right)+\frac{1}{\delta}\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)\lambda_i^{\frac{1}{3}}. \tag{3.33}$$

Taking

$$\delta=\frac{\left[\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)\lambda_i^{\frac{1}{3}}\right]^{\frac{1}{2}}}{\left[\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2\left(140\lambda_i+18\lambda_i^{\frac{2}{3}}+8\lambda_i^{\frac{1}{3}}\right)\right]^{\frac{1}{2}}}.$$

in (3.33), we derive this which completes the proof of Theorem 1.1.

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Similar to the proof of Theorem 1.1, take the trial functions $\varphi_{ix_j}=x_ju_i-\sum\limits_{l=1}^{m}a_{ilx_j}u_l$, where $a_{ilx_j}=\int_\Omega x_ju_iu_l$, $i=1,\ldots,m$ and $j=1,2$. According to the Rayleigh-Ritz principle, we deduce

$$\begin{aligned}\lambda_{m+1}\int_\Omega\varphi_{ix_j}^2 &\leq\int_\Omega\varphi_{ix_j}(x_j\mathbb{L}^3u_i-2\mathbb{L}^2X_ju_i-2\mathbb{L}X_j\mathbb{L}u_i-2X_j\mathbb{L}^2u_i)-\lambda_i\sum_{l=1}^{m}a_{ilx_j}\int_\Omega u_l\varphi_{ix_j}\\ &=\lambda_i\int_\Omega\varphi_{ix_j}^2-2\int_\Omega\varphi_{ix_j}(\mathbb{L}^2X_j+\mathbb{L}X_j\mathbb{L}+X_j\mathbb{L}^2)u_i.\end{aligned} \tag{3.34}$$

Then it implies

$$(\lambda_{m+1}-\lambda_m)\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega\varphi_{ix_j}^2\leq-2\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega\varphi_{ix_j}(\mathbb{L}^2X_j+\mathbb{L}X_j\mathbb{L}+X_j\mathbb{L}^2)u_i. \tag{3.35}$$

Substituting

$$\sum_{i,l=1}^{m}a_{ilx_j}\int_\Omega u_l(\mathbb{L}^2X_j+\mathbb{L}X_j\mathbb{L}+X_j\mathbb{L}^2)u_i=0 \tag{3.36}$$

to (3.36), we derive

$$\begin{aligned}(\lambda_{m+1}-\lambda_m)\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega\varphi_{ix_j}^2\leq&-2\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega x_ju_i(\mathbb{L}^2X_j+\mathbb{L}X_j\mathbb{L}+X_j\mathbb{L}^2)u_i\\ =&-2\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega(x_j\mathbb{L}^2u_iX_ju_i+x_j\mathbb{L}u_iX_j\mathbb{L}u_i+x_ju_iX_j\mathbb{L}^2u_i)\\ &-8\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega\mathbb{L}u_iX_j^2u_i+4\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega\mathbb{L}X_ju_iX_ju_i.\end{aligned} \tag{3.37}$$

Therefore, similar to the proof of Theorem 1.1, we get

$$
\begin{aligned}
(\lambda_{m+1}-\lambda_m)\sum_{i=1}^{m}\sum_{j=1}^{2}\int_\Omega \varphi_{ix_j}^2 &\le \sum_{i=1}^{m}\left(6\int_\Omega u_i\mathbb{L}^2u_i+8\int_\Omega u_i\mathbb{L}^2u_i+4\lambda_i^{\frac{2}{3}}+8\lambda_i^{\frac{1}{3}}+140\lambda_i\right)\\
&\le \sum_{i=1}^{m}\left(140\lambda_i+18\lambda_i^{\frac{2}{3}}+8\lambda_i^{\frac{1}{3}}\right).
\end{aligned} \tag{3.38}
$$

Since $a_{ilx_j}=a_{lix_j}$ and $\int_\Omega u_iX_ju_l=-\int_\Omega u_lX_ju_i$, one can easily verify

$$
\sum_{i,l=1}^{m}a_{ilx_j}\int_\Omega u_lX_ju_i=\sum_{i,l=1}^{m}a_{lix_j}\int_\Omega u_iX_ju_l=-\sum_{i,l=1}^{m}a_{ilx_j}\int_\Omega u_lX_ju_i.
$$

It implies

$$
\sum_{i,l=1}^{m}a_{ilx_j}\int_\Omega u_lX_ju_i=0. \tag{3.39}
$$

Then it is from (3.39) that

$$
\sum_{i=1}^{m}\int_\Omega \varphi_{ix_j}X_ju_i=-\sum_{i=1}^{m}\int_\Omega u_i^2-\sum_{i=1}^{m}\int_\Omega x_ju_iX_ju_i=-m-\sum_{i=1}^{m}\int_\Omega \varphi_{ix_j}X_ju_i. \tag{3.40}
$$

Thus it holds

$$
\sum_{i=1}^{m}\int_\Omega \varphi_{ix_j}X_ju_i=-\frac{m}{2}. \tag{3.41}
$$

Using (3.41) and Holder's inequality

$$
\begin{aligned}
m&=-\sum_{i=1}^{m}\int_\Omega(\varphi_{ix_1}X_1u_i+\varphi_{ix_2}X_2u_i)\\
&\le\sum_{i=1}^{m}\int_\Omega(\varphi_{ix_1}^2+\varphi_{ix_2}^2)^{\frac{1}{2}}\left[(X_1u_i)^2+(X_2u_i)^2\right]^{\frac{1}{2}}\\
&\le\left[\sum_{i=1}^{m}\int_\Omega(\varphi_{ix_1}^2+\varphi_{ix_2}^2)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{m}\int_\Omega u_i\mathbb{L}u_i\right)^{\frac{1}{2}},
\end{aligned}
$$

and using Lemma 2.2, we obtain $m\le\left[\sum\limits_{i=1}^{m}\int_\Omega(\varphi_{ix_1}^2+\varphi_{ix_2}^2)\right]^{\frac{1}{2}}\left(\sum\limits_{i=1}^{m}\lambda_i^{\frac{1}{3}}\right)^{\frac{1}{2}}$. Hence it yields

$$
\sum_{i=1}^{m}\int_\Omega(\varphi_{ix_1}^2+\varphi_{ix_2}^2)\ge\frac{m^2}{\sum_{i=1}^{m}\lambda_i^{\frac{1}{3}}}. \tag{3.42}
$$

Substituting (3.42) into (3.38), we get

$$
\lambda_{m+1}-\lambda_m\le\frac{1}{m^2}\sum_{i=1}^{m}\left(140\lambda_i+18\lambda_i^{\frac{2}{3}}+8\lambda_i^{\frac{1}{3}}\right)\sum_{i=1}^{m}\lambda_i^{\frac{1}{3}}. \tag{3.43}
$$

This finishes the proof of Theorem 1.2.

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Engle 群上次 Laplace 算子的特征值不等式

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摘要: Engel 群是次黎曼几何中的一类重要的单连通幂零李群. 本文研究了 Engel 群 $\mathbb{E}=(\mathbb{R}^4,\circ,\{\delta_\lambda\})$ 的有界区域 Ω 上次 Laplace 算子 $\Delta_{\mathbb{E}}$ 的狄利克雷特征值问题

$$\begin{cases}(-\Delta_{\mathbb{E}})^3u=\lambda u, & \text{in } \Omega,\\ u=\dfrac{\partial u}{\partial \nu}=\dfrac{\partial^2 u}{\partial \nu^2}=0, & \text{on } \partial\Omega,\end{cases}$$

其中 ν 是边界 $\partial\Omega$ 的单位外法向量场. 我们建立了该问题的一些万有特征值不等式.

关键词: 特征值; 不等式; Engel 群; 次拉普拉斯算子

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