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TWO-DIMENSIONAL MAXIMAL OPERATOR OF VILENKIN-LIKE SYSTEM ON HARDY SPACES

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Abstract: In this paper, we research the boundedness of two-dimensional maximal operator of Vilenkin-like system on Hardy spaces. By means of atomic decomposition, the two-dimensional maximal operator $T_\alpha f := \sup_{2^{-\alpha}\leqslant \frac{n}{m}\leqslant 2^\alpha} |f * P_{n,m}|$ is bounded from H^p to L^p, where $0 < p < \frac{1}{2}$ and $\alpha \geq 0$. As an application, we prove the boundedness of two-dimensional operator $\tilde{\sigma}^* f = \sup_{2^{-\alpha}\leqslant \frac{n}{m}\leqslant 2^\alpha} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/p-2}}$. By a counterexample, we also prove that two dimensional maximal operator $\hat{\sigma}^* f = \sup\limits_{n,m\in\mathbf{N}} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/2p-1}}$ is not bounded from H^p to L^p, where $0 < p < \frac{1}{2}$. The results as above generalize the known conclusions in Walsh system or in Vilenkin system.

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1 Introduction

The weak type inequality for maximal operator of Fejér means for trigonometric system can be found in Zygmund [1], in Schipp [2] for Walsh system and in Pál, Simon [3] for bounded Vilenkin system. Later, Schipp [2] showed that maximal operator $\sigma^* f := \sup\limits_n |\sigma_n f|$ is of weak type (1,1), from which the a.e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L^p \to L^p (1 < p \leqslant \infty)$. This fails to hold for $p = 1$, but Fujii [4] proved that σ^* is bounded from the dyadic Hardy space H^1 to L^1 (see also Simon [5]). Fujii's results were extended by Wesiz [6],[7] to H^p spaces for $1/2 < p \leqslant 1$, in the two-dimensional case, too. Simon [8] gave a counterexample, which shows that boundedness of σ^* does not hold for $0 < p < 1/2$. The counterexample for σ^* when $p = 1/2$ is due to Goginava [9]. Goginava [10] proved that the maximal operator $\tilde{\sigma}^*$ defined by

$$\tilde{\sigma}^* f = \sup_{n\in\mathbf{N}} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H^{1/2}$ to the space $L^{1/2}$ for Walsh system. He also proved,

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that for any nondecreasing function $\varphi : \mathbf{N} \to [1, \infty)$, satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\varphi(n)} = +\infty,$$

the maximal operator

$$\sup_{n\in\mathbf{N}} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H^{1/2}$ to the space $L^{1/2}$. Tephnadze [11] generalized this result and proved the boundedness of

$$\sup_{n\in\mathbf{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2}}$$

is bounded from the martingale Hardy space H^p to the space L^p, where $\sigma_n f$ is n-th Fejér mean with respect to bounded Vilenkin system for $0 < p < 1/2$.

In this paper the two-dimensional case will be investigated with respect to Vilenkin-like system. We show that the boundedness of some maximal operators. Throughout this paper, we denote the set of integers and the set of non-negative integers by $\mathbf{Z}$ and $\mathbf{N}$, respectively. We use c, c_p, C_p to denote constants and may denote different constants at different occurrences.

2 Definitions and Notations

Let $m := (m_0, m_1, \cdots, m_k, \cdots)$ be sequence of natural numbers such that $m_k \geq 2 (k \in \mathbf{N})$. For all $k \in \mathbf{N}$ we denote by Z_{m_k} the m_k-th discrete cyclic group. Let Z_{m_k} be represented by $\{0, 1, \cdots, m_k - 1\}$. Suppose that each (coordinate) set has the discrete topology and the measure μ_k which maps every singleton of Z_{m_k} to $1/m_k$ ($u_k(Z_{m_k}) = 1$) for $k \in \mathbf{N}$. Let G_m denote the complete direct product of Z_{m_k}'s equipped with product topology and product measure μ, then G_m forms a compact Abelian group with Haar measure 1. The elements of G_m are sequences of the form $(x_0, x_1, \cdots, x_k, \cdots)$, where $x_k \in Z_{m_k}$ for every $k \in \mathbf{N}$ and the topology of the group G_m is completely determined by the sets

$$I_n(0) := \{(x_0, x_1, \cdots, x_k, \cdots) \in G_m : x_k = 0 \ (k = 0, \cdots, n-1)\}$$

($I_0(0) := G_m$). The Vilenkin space G_m is said to be bounded if the generating system m is bounded. We assume $q = \sup_i\{m_i\} < \infty$.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$ for $k \in \mathbf{N}$, it is so-called the generalized powers. Then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum\limits_{k=0}^{\infty} n_k M_k, 0 \leq n_k < m_k, n_k \in \mathbf{N}$. The sequence $(n_0, n_1, \cdots)$ is called the expansion of n with respect to m. We often use the following notations: $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum\limits_{j=k}^{\infty} n_j M_j$.

For $k \in \mathbf{N}$ and $x \in G_m$ denote r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\iota \frac{x_k}{m_k}) \quad (x \in G_m, \iota := \sqrt{-1}, k \in \mathbf{N}).$$

It is known that for $x \in G_m, n \in \mathbf{N}$

$$\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & \text{if } x_n \neq 0, \\ m_n & \text{if } x_n = 0. \end{cases} \tag{2.1}$$

Now we define the ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k} (n \in \mathbf{N}).$$

Then $\{\psi_n : n \in \mathbf{N}\}$ is a complete orthonormal system with respect to μ.

We introduce the so-called Vilenkin-like (or $\psi\alpha$) system (see [12]). Let functions $\alpha_n, \alpha_j^k :$ $G_m \to \mathcal{C}(n, j, k \in \mathbf{N})$ satisfy for all $x, y \in G_m$:

(1) α_j^k is measurable with respect to Σ_j and $\alpha_j^k(x+y) = \alpha_j^k(x)\alpha_j^k(y)$;

(2) $|\alpha_j^k| = \alpha_j^k(0) = \alpha_0^k = \alpha_j^0 = 1 \quad (j, k \in \mathbf{N})$;

(3) $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{n^{(j)}} \quad (n \in \mathbf{N})$.

Let $\chi_n := \psi_n \alpha_n$ $(n \in \mathbf{N})$. The system $\chi := \{\chi_n : n \in \mathbf{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system.

Define Dirichlet kernels and Fejér kernels with respect to Vilenkin-like system and Vilenkin system as follows.

$$D_n(y,x) = \sum_{k=0}^{n-1} \chi_k(y)\overline{\chi}_k(x), \quad D_n(x) = \sum_{k=0}^{n-1} \psi_k(x),$$

$$K_n(y,x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(y,x), \quad K_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

It's well known that

$$D_{M_n}(y,x) = D_{M_n}(y-x) = \begin{cases} M_n & \text{if } y - x \in I_n, \\ 0 & \text{if } y - x \in G_m \backslash I_n. \end{cases} \tag{2.2}$$

Moreover for $y, x \in G_m$,

$$D_n(y,x) = \alpha_n(y)\bar{\alpha}_n(x)D_n(y-x) = \chi_n(y)\bar{\chi}_n(x)(\sum_{j=0}^{\infty} D_{M_j}(y-x) \sum_{k=m_j-n_j}^{m_j-1} r_j^k(y-x)). \tag{2.3}$$

Since $\alpha_j^k(x+y) = \alpha_j^k(x)\alpha_j^k(y)$ and $r_j(x+y) = r_j(x)r_j(y)$, we have

$$\begin{aligned} \chi_n(y)\bar{\chi}_n(x) &= \chi_n(y-x+x)\bar{\chi}_n(x) = \chi_n(y-x)\chi_n(x)\bar{\chi}_n(x) \\ &= \chi_n(y-x)|\chi_n(x)|^2 = \chi_n(y-x)\bar{\chi}_n(0). \end{aligned} \tag{2.4}$$

Thus we obtain

$$D_n(y,x) = D_n(y-x,0) \quad \text{and} \quad K_n(y,x) = K_n(y-x,0). \tag{2.5}$$

Now we define $\chi_{n,m}(x,y) := \chi_n(x)\chi_m(y), (x,y \in G_m)$. If $f \in L^1$ then the number $\hat{f}(n,m) := E(f\chi_{n,m})$ is said to be the (n,m)-th coefficient of f with respect to system χ. Denote by $S_{n,m}f$ the (n,m)-th partial sum of the Fourier series of a martingale f with respect to character system χ, namely,

$$S_{n,m}f := \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \hat{f}(k,l)\chi_{k,l}.$$

It is easy to see that

$$S_{M_n,M_m}f = f_{n,m}.$$

Let $\mathcal{F}_{n,m}(n,m \in \mathbf{N})$ be the σ-algebra generated by the rectangles $I_{n,m}(x,y) := I_n(x) \times I_m(y)$,$(x,y \in G_m)$. A sequence of integrable functions $f = (f_{n,m}; n,m \in \mathbf{N})$ is said to be a martingale if $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable for all $n,m \in \mathbf{N}$ and $S_{M_n,M_m}f_{k,l} = f_{n,m}$ for all $n,m,k,l \in \mathbf{N}$ such that $n \leqslant k$ and $m \leqslant l$.

We say that a martingale $f = (f_{n,m}; n,m \in \mathbf{N})$ is L^p-bounded if $\|f\|_p := \sup_{n,m} \| f_{n,m}\|_p < \infty$. The set of the L^p-bounded martingales will be denoted by $L^p(G_m^2)$.

The diagonal maximal function of a martingale $f = (f_{n,m}; n,m \in \mathbf{N})$ is defined by

$$f^* := \sup_{n\in\mathbf{N}} |f_{n,n}|.$$

It is easy to see that in case when f is an integrable real valued function given on G_m^2, the above maximal functions can be computed for all $x,y \in G_m$ by

$$f^*(x,y) = \sup_{n\in\mathbf{N}} \frac{1}{|I_{n,n}(x,y)|}\left|\int_{I_{n,n}(x,y)} f\right|.$$

Define the spaces $H^p(G_m^2)$ of Hardy type as the set of martingales f such that

$$\|f\|_{H^p(G_m^2)} := \|f^*\|_p < \infty.$$

The martingale Hardy spaces $H^p(G_m^2)(0 < p \leqslant 1)$ have atomic characterizations. A bounded measurable function a defined on G_m^2 is a p-atom if $a \equiv 1$ or there exists a dyadic square I such that

$$\operatorname{supp} a \subset I, \|a\|_\infty \leqslant |I|^{-1/p}, \int\int a \equiv 0.$$

We shall say also that a is supported on I. Then a martingale $f = (f_{n,m}; n,m \in \mathbf{N})$ is in $H^p(G_m^2)$ if there exists a sequence $(a_k, k \in \mathbf{N})$ of p-atoms and a sequence $(\lambda_k, k \in \mathbf{N})$ of real numbers such that $\sum\limits_{k=0}^{\infty} |\lambda_k|^p < \infty$ and

$$\sum_{k=0}^{\infty} \lambda_k S_{M_n,M_n} a_k = f_{n,n} \quad (n \in \mathbf{N}). \tag{2.6}$$

Moreover, $c_p \inf(\sum\limits_{k=0}^{\infty} |\lambda_k|^p)^{1/p} \leqslant \|f\|_{H^p} \leqslant C_p \inf(\sum\limits_{k=0}^{\infty} |\lambda_k|^p)^{1/p}$, where the infimum is taken over all decompositions of f of the form (2.6).

Next we will consider the boudedness of operator $\tilde{\sigma}^* f$ and $\hat{\sigma}^* f$ in the two-dimensional Vilenkin-like system, where $\tilde{\sigma}^* f = \sup\limits_{2^{-\alpha} \leqslant \frac{n}{m} \leqslant 2^{\alpha}} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/p-2}}$, $\hat{\sigma}^* f = \sup\limits_{n,m \in \mathbf{N}} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/2p-1}}$.

3 Some Lemmas

Lemma 3.1 ([13]) Suppose that the operator T is sublinear and for $0 < p \leq 1$, there exists a constant $C_p > 0$ such that

$$\int_{G_m \setminus I} |Ta|^p \leqslant C_p, \tag{3.1}$$

for every p-atom $a \in H^p$ supported on the dyadic interval I. If T is bounded from L^s into L^s for some $1 \leqslant s \leqslant \infty$, then

$$\|Tf\|_p \leq C_p \|f\|_{H^p} \quad (f \in H^p \cap L^1).$$

If (3.1) is true, T is called p-quasi-local.

Lemma 3.2 ([13]) Let $0 < p < 1, 1 < s \leq \infty$ and assume that the sublinear operator T is p-quasi-local and (L^s, L^s)-bounded. Then $T : H^{u,v} \to L^{u,v}$ is bounded for all $p < u < s$ and $0 < v \leqslant \infty$. Especially, T is of weak type (1,1).

Further we assume that for all $n \in \mathbf{N}$ the kernel $P_n \in L^\infty$ is given such that $\sup\limits_n \|P_n\|_1 < \infty$. If we consider the maximal operator

$$Tf := \sup_n |f * P_n| \quad (f \in L^1),$$

then $T : L^\infty \to L^\infty$ is evidently bounded. Therefore, if T is p-quasi-local for some $0 < p < 1$, then Lemma 3.2 can be applied to T.

Lemma 3.3 If P_n is a *summation kernel*, i.e. with suitable real coefficients $\lambda_{n,k}(n, k \in \mathbf{N})$

$$P_n(x, 0) = \sum_{k=0}^{n} \lambda_{n,k} \chi_k(x, 0) \quad (n \in \mathbf{N}),$$

then the assumption

$$\int_{G_m \setminus I_N} (\sup_{n \geqslant M_N} \int_{I_N} |P_n(x - t, 0)| dt)^p dx \leqslant C_p \frac{1}{M_N} \quad (n \in \mathbf{N}) \tag{3.2}$$

implies the p-quasi-locality of T.

Proof Indeed, to prove (3.1) let a be a p-atom supported on the interval I. Without loss of generality we can assume that $I = I_N$ for some $N \in \mathbf{N}$. Then $a * P_n = 0$ holds for all

$n = 0, \ldots, M_N - 1$, since the functions $\chi_k(k = 0, \ldots, M_N - 1)$ are constant on I. Therefore, $Ta = \sup_{n \geqslant M_N} |a * P_n|$ and thus

$$\begin{aligned} \int_{G_m \setminus I_N} (Ta(x))^p dx &= \int_{G_m \setminus I_N} (\sup_{n \geqslant M_N} |\int_{I_N} a(t) P_n(x - t, 0) dt|)^p dx \\ &\leqslant \|a\|_\infty^p \int_{G_m \setminus I_N} (\sup_{n \geqslant M_N} \int_{I_N} |P_n(x - t, 0)| dt)^p dx \\ &\leqslant M_N \int_{G_m \setminus I_N} (\sup_{n \geqslant M_N} \int_{I_N} |P_n(x - t, 0)| dt)^p dx. \end{aligned} \tag{3.3}$$

Hence, (3.1) follows from (3.2) and (3.3).

Lemma 3.4 ([14]) Let $z \in I_N^{k,l}, k = 0, \cdots, N-2, l = k+1, \cdots, N-1$ and $n \geqslant M_N$. Then

$$\int_{I_N} |K_n(z - t, 0)| d\mu(t) \leqslant \frac{cM_l M_k}{nM_N}. \tag{3.4}$$

Let $z \in I_N^{k,N}, k = 0, \cdots, N-1$ and $n \geq M_N$. Then

$$\int_{I_N} |K_n(z - t, 0)| d\mu(t) \leqslant \frac{cM_k}{M_N}, \tag{3.5}$$

where c is an absolute constant and

$$I_N^{k,l} = \begin{cases} I_N(0, \cdots, 0, x_k \neq 0, 0 \cdots, 0, x_l \neq 0, x_{l+1}, \cdots x_{N-1}, \cdots) & \text{if } k < l < N, \\ I_N(0, \cdots, 0, x_k \neq 0, x_{k+1} = 0, \cdots, x_{N-1} = 0, x_N \cdots) & \text{if } l = N. \end{cases}$$

.

Lemma 3.5 ([14]) Let $2 < A \in \mathbb{N}_+, k \leq s < A$, $n_A^* := M_{2A} + M_{2A-2} + \cdots + M_2 + M_0$. Then we have

$$n_{A-1}^* |K_{n_{A-1}^*}(z, 0)| \geq \frac{M_{2k} M_{2s}}{4},$$

for $z \in I_{2A}^{2k,2s}, k = 0, 1, \cdots, A-3, s = k+2, k+3, \cdots, A-1$.

If $I := I \times J$ is a dyadic square and let $I^r := I^r \times J^r$. Then it is not hard to see that the definition of the p-quasi-locality of T can be modified as follows: there exists $r = 0, 1 \ldots$ such that

$$\int_{G_m^2 \setminus I^r} |Ta|^p \leqslant C_p \tag{3.6}$$

holds for every p-atom a supported on the dyadic square I.

Let $P_{n,m}(n, m \in \mathbf{N})$ be the Kronecker product of P_n and P_m, i.e. $P_{n,m}(x_1, 0, x_2, 0) := P_n(x_1, 0) P_m(x_2, 0)$ and for a fixed $\alpha \geqslant 0$ define T_α by

$$T_\alpha f := \sup_{2^{-\alpha} \leqslant \frac{n}{m} \leqslant 2^\alpha} |f * P_{n,m}|.$$

4 Formulations of Main Results

Theorem 4.1 Assume (3.2) for a given $0 < p \leqslant 1$. Then T_α is p-quasi-local.

Proof It is enough to prove (3.6) with a suitable $r \in \mathbf{N}$. To this end let $a \in L^\infty(G_m^2)$ be a p-atom. We can assume that a is supported on the dyadic square $I_N \times I_N$ for some $N \in \mathbf{N}$. Furthermore, it follows that $a * P_{n,m} = 0$ when $n, m < M_N$. Therefore, to compute $T_\alpha a = \sup\limits_{2^{-\alpha} \leqslant \frac{n}{m} \leqslant 2^\alpha} |a * P_{n,m}|$ it can be assumed $n \geqslant M_N$ or $m \geqslant M_N$. In the first case $m \geqslant M_{N-r}$, while in the second case $n \geqslant M_{N-r}$ follows. In other words, we get the estimate

$$T_\alpha a \leqslant \sup_{n,m \geqslant M_{N-r}} |a * P_{n,m}|,$$

where $r \in \mathbf{N}$ is determined by $r - 1 \leqslant \alpha < r$. Here, $\|a\|_\infty \leqslant M_N^{\frac{2}{p}}$ implies

$$\begin{aligned} T_\alpha a(x,y) &\leqslant \sup_{n,m \geqslant M_{N-r}} \left|\int_{I_N} \int_{I_N} a(u,v) P_n(u-x,0) P_m(v-y,0) du dv\right| \\ &\leqslant M_N^{\frac{2}{p}} \sup_{n,m \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du \int_{I_N} |P_m(v-y,0)| dv. \end{aligned} \tag{4.1}$$

Therefore, to verity (3.6) it is enough to show that

$$\int_{G_m^2 \setminus (I_{N-r} \times I_{N-r})} \Big(\sup_{n,m \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du \int_{I_N} |P_m(v-y,0)| dv\Big)^p dx dy \leqslant \frac{C_p}{M_N^2}. \tag{4.2}$$

To this end let us decompose the double integral in question as follows:

$$\begin{aligned} & \int_{G_m^2 \setminus (I_{N-r} \times I_{N-r})} \Big(\sup_{n,m \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du \int_{I_N} |P_m(v-y,0)| dv\Big)^p dx dy \\ = & \int_{G_m \setminus I_{N-r}} \int_{I_{N-r}} \Big(\sup_{n,m \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du \int_{I_N} |P_m(v-y,0)| dv\Big)^p dx dy \\ & + \int_{I_{N-r}} \int_{G_m \setminus I_{N-r}} \Big(\sup_{n,m \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du \int_{I_N} |P_m(v-y,0)| dv\Big)^p dx dy \\ & + \int_{G_m \setminus I_{N-r}} \int_{G_m \setminus I_{N-r}} \Big(\sup_{n,m \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du \int_{I_N} |P_m(v-y,0)| dv\Big)^p dx dy \\ =: & A_1 + A_2 + A_3. \end{aligned} \tag{4.3}$$

Here A_1 can be estimated in the following way:

$$\begin{aligned} A_1 &\leqslant \int_{G_m \setminus I_{N-r}} \Big(\sup_{n \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du\Big)^p dx \int_{I_N} \Big(\sup_m \int_G |P_m(v-y,0)| dv\Big)^p dy \\ &\leqslant \int_{G_m \setminus I_{N-r}} \Big(\sup_{n \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)| du\Big)^p dx |I_N| \Big(\sup_m \|P_m\|_1\Big)^p \\ &\leqslant C_p \frac{1}{M_N} \int_{G_m \setminus I_{N-r}} \Big(\sup_{n \geqslant M_{N-r}} \int_{I_{N-r}} |P_n(u-x,0)| du\Big)^p dx. \end{aligned} \tag{4.4}$$

Thus we get

$$A_1 \leqslant C_p \frac{1}{M_N} \frac{1}{M_{N-r}} \leqslant \frac{C_p}{M_N^2}.$$

The estimate $A_2 \leqslant \frac{C_p}{M_N^2}$ can be derived similarly. Finally, applying (3.2) twice the estimation

$$A_3 \leqslant (\int_{G_m \setminus I_{N-r}} (\sup_{k \geqslant M_{N-r}} \int_{I_N} |P_n(u-x,0)|du)^p dx)^2 \leqslant C_p \frac{1}{M_N^2} \tag{4.5}$$

follows, which proves Theorem 4.1.

Theorem 4.2 Let $\tilde{\sigma}^* f = \sup_{2^{-\alpha} \leqslant \frac{n}{m} \leqslant 2^\alpha} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/p-2}}$. Then for all $0 < p < 1/2$ we have

$$\|\tilde{\sigma}^* f\|_p \leqslant C_p \|f\|_p \quad (f \in L^p(G_m^2)).$$

Proof Let $P_n(x,0) = \sum_{k=0}^{n} \frac{1}{(n+1)^{1/p-2}} K_n(x,0)$. By Theorem 4.1, it is enough to prove (3.2) for $P_n(x,0)$. Let $z \in I_N^{k,l}$, $0 \leqslant k < l \leqslant N$. From Lemma 3.4 and $1/p - 2 > 0$ we get

$$\sup_{n \geqslant M_N} \frac{1}{(n+1)^{1/p-2}} \int_I |K_n(z-t,0)|dt \leq c \frac{1}{M_N^{1/p-2}} \frac{M_l M_k}{n M_N} \leq c \frac{M_l M_k}{M_N^{1/p}}. \tag{4.6}$$

Thus we obtain

$$\begin{aligned}
& \int_{G_m \setminus I_N} (\sup_{n \geqslant M_N} \int_{I_N} |P_n(x-t,0)|dt)^p dx \\
= & \int_{G_m \setminus I_N} (\sup_{n \geqslant M_N} \frac{1}{(n+1)^{1/p-2}} \int_{I_N} |K_n(x-t,0)|dt)^p dx \\
= & \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j \in \{l+1,\cdots,N-1\}}^{m_j-1} \int_{I_N^{k,l}} (\sup_{n \geqslant M_N} \frac{1}{(n+1)^{1/p-2}} \int_{I_N} |K_n(x-t,0)|dt)^p d\mu(z) \\
& + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} (\sup_{n \geqslant M_N} \frac{1}{(n+1)^{1/p-2}} \int_{I_N} |K_n(x-t,0)|dt)^p d\mu(z) \\
\leq & c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_l \cdots m_N - 1}{M_N} (\frac{M_l M_k}{M_N^{1/p}})^p + \sum_{k=0}^{N-1} \frac{1}{M_N} (\frac{M_N M_k}{M_n^{1/p}})^p \\
\leq & c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^p}{M_l M_N} + \sum_{k=0}^{N-1} \frac{1}{M_N^2} (M_N M_k)^p \\
= & c \frac{1}{M_N} (\sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-2p}} \frac{(M_l M_k)^p}{M_l^{2p}} + \sum_{k=0}^{N-1} \frac{1}{M_N^{1-2p}} \frac{(M_N M_k)^p}{M_N^{2p}}) \\
= & c \frac{1}{M_N} (\sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{2^{(1-2p)l}} + \sum_{k=0}^{N-1} \frac{1}{2^{N(1-2p)}}) \\
= & c \frac{1}{M_N} (\sum_{k=0}^{N-2} \frac{1}{2^{(1-2p)k}} + \frac{N}{2^{N(1-2p)}}) \\
\leq & \frac{c}{M_N},
\end{aligned} \tag{4.7}$$

which complete the proof of Theorem 4.2.

By Lemma 3.2 and Theorem 4.2, we easily get Theorem 4.3, we omit the proof.

Theorem 4.3 Let $0 < p < 1/2$. Then $\tilde{\sigma}^* : H^{u,v}(G_m^2) \to L^{u,v}(G_m^2)$ is bounded for all $p < u < \infty$ and $0 < v \leqslant \infty$. Especially, $\tilde{\sigma}^*$ is of weak type (1,1).

Theorem 4.4 Let $0 < p < 1/2$. Then the two dimensional maximal operator $\hat{\sigma}^*$ defined by $\hat{\sigma}^* f = \sup\limits_{n,m\in\mathbf{N}} \frac{|\sigma_{n,m}f|}{[(n+1)(m+1)]^{1/2p-1}}$ is not bounded from $H^p(G_m^2)$ to $L^p(G_m^2)$.

Proof Let $A \in \mathbb{N}$ and

$$f_A(x,y) := (D_{M_{2A+1}}(x,0) - D_{M_{2A}}(x,0))(D_{M_{2A+1}}(y,0) - D_{M_{2A}}(y,0)).$$

It is simple to calculate

$$\hat{f}_A(i,j) = \begin{cases} 1, & \text{if } i,j = M_{2A}, M_{2A}+1, \cdots, M_{2A+1}-1 \\ 0, & \text{otherwise} \end{cases} \tag{4.8}$$

and

$$S_{k,l}(f_A;x,y)) = \begin{cases} (D_k(x,0) & -D_{M_{2A}}(x,0))(D_l(y,0) - D_{M_{2A}}(y,0)), \\ & \text{if } k,l = M_{2A}, M_{2A}+1, \cdots, M_{2A+1}-1 \\ f_A(x,y), & \text{if } k,l \geq M_{2A+1} \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

We have

$$\begin{aligned} f_A^* &= \sup_k |S_{M_k,M_k}(f_A;x,y)| = |f_A(x,y)|, \\ \|f_A\|_{H^p} &= \|f_A^*\|_p = \|f_A\|_p \\ &= \left(\int_{G_m} \big(D_{M_{2A+1}}(x,0) - D_{M_{2A}}(x,0)\big)^p dx \int_{G_m} \big(D_{M_{2A+1}}(y,0) - D_{M_{2A}}(y,0)\big)^p dy\right)^{1/p} \\ &= \left(\int_{G_m} \big(D_{M_{2A+1}}(x) - D_{M_{2A}}(x)\big)^p dx\right)^{2/p} \\ &= \left(\int_{I_{2A+1}} \big(D_{M_{A+1}}(x) - D_{M_{2A}}(x)\big)^p dx + \int_{I_{2A}\setminus I_{2A+1}} \big(D_{M_{2A+1}}(x) - D_{M_{2A}}(x)\big)^p dx\right)^{2/p} \\ &\leq [\frac{m_{2A-1}}{M_{2A+1}} M_{2A}^p + \frac{(m_{2A}-1)^p M_{2A}^p}{M_{2A+1}}]^{2/p} \\ &\leq cM_{2A}^{2(1-1/p)}. \end{aligned}$$

Since

$$D_{i+M_A}(x,0) - D_{M_A}(x,0) = \chi_{M_A}(x) D_k(x,0) \tag{4.10}$$

we have

$$\begin{aligned}
\hat{\sigma}^* f &= \sup_{n,m\in\mathbf{N}} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/2p-1}} \geq |f_A * \frac{K_{n_A^*,n_A^*}}{(n_A^*+1)^{(1/p-2)}}| \\
&= \frac{1}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} |\sum_{i=0}^{n_A^*-1}\sum_{j=0}^{n_A^*-1} S_{i,j} f_A| \\
&= \frac{1}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} |\sum_{i=M_{2A}+1}^{n_A^*-1}\sum_{j=M_{2A}+1}^{n_A^*-1} S_{i,j} f_A| \\
&= \frac{1}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} |\sum_{i=M_{2A}+1}^{M_{2A+1}-1}\sum_{j=M_{2A}+1}^{M_{2A+1}-1} (D_i(x,0)-D_{M_A}(x,0))(D_j(y,0)-D_{M_A}(y,0))| \\
&= \frac{1}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} |\sum_{i=1}^{n_{A-1}^*-1}\sum_{j=1}^{n_{A-1}^*-1} (D_{i+M_A}(x,0)-D_{M_A}(x,0))(D_{j+M_A}(y,0)-D_{M_A}(y,0))| \\
&= \frac{(n_{A-1}^*)^2}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} |K_{n_{A-1}^*}(x,0) K_{n_{A-1}^*}(y,0)|. \qquad (4.11)
\end{aligned}$$

Let $q=\sup_i\{m_i\}$. For every $l=1,\cdots,[\frac{1}{4}\log_q(\sqrt{A^{1/2p}})]-1$ (A is supposed to be large enough) let k_l be the smallest natural numbers, for which $M_{2A}\sqrt{A^{1/2p}}\frac{1}{q^{2l/p}} \leq M_{2k_l}^2 < M_{2A}\sqrt{A^{1/2p}}\frac{1}{q^{(2l-2)/p}}$ holds.

Suppose $x,y \in I_{2A}^{k_l,k_l+1} := I_{2A}(0,\cdots,0,z_{2k_l}\neq 0, z_{2k_l+1}\neq 0, z_{2s+1},\cdots,z_{2A-1})$, then by Lemma 3.5 we have

$$\begin{aligned}
\hat{\sigma}^* f &\geq \frac{(n_{A-1}^*)^2}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} |K_{n_{A-1}^*}(x,0) K_{n_{A-1}^*}(y,0)| \\
&\geq \frac{(M_{2k_l}M_{2k_l+1})^2}{(n_A^*)^2(n_A^*+1)^{(1/p-2)}} \geq c\frac{(M_{2k_l}M_{2k_l+1})^2}{(M_{2A})^2(M_{2A})^{(1/p-2)}} \geq \frac{1}{(M_{2A})^{(1/p-2)}}\frac{A^{1/2p}}{q^{4l/p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\hat{\sigma}^* f\|_p^p &\geq (\frac{1}{(M_{2A})^{(1-2p)}}\frac{\sqrt{A}}{q^{4l}})(\sum_{l=1}^{[\frac{1}{4}\log_q(\sqrt{A^{1/2p}})]-1}\sum_{x_{2k_l+3}=0}^{m_{2k_l+3}-1}\cdots\sum_{x_{2A-1}=0}^{m_{2A-1}-1}|I_{2A}^{k_l,k_l+1}|)^2 \\
&\geq (\frac{1}{(M_{2A})^{(1-2p)}}\frac{\sqrt{A}}{q^{4l}})(\sum_{l=1}^{[\frac{1}{4}\log_q(\sqrt{A^{1/2p}})]-1}\frac{m_{2k_l+3}\cdots m_{2A-1}}{M_{2A}})^2 \\
&\geq \frac{1}{(M_{2A})^{(1-2p)}}\frac{A}{q^{4l}}(\sum_{l=1}^{[\frac{1}{4}\log_q(\sqrt{A^{1/2p}})]-1}\frac{1}{M_{2k_l}})^2 \\
&\geq \frac{1}{(M_{2A})^{(1-2p)}}(\frac{\log_q A}{\sqrt{M_{2A}}})^2 = \frac{1}{M_{2A}^{2-2p}}(\log_q A)^2. \qquad (4.12)
\end{aligned}$$

Then

$$\frac{\|\hat{\sigma}^* f\|_p^p}{\|f_A\|_{H^p}^p} \geq \frac{\frac{1}{M_{2A}^{2-2p}}(\log_q A)^2}{cM_{2A}^{2p-2}} = (\log_q A)^2 \to \infty. \qquad (4.13)$$

Thus the proof of Theorem 4.4 is complete.

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二维 Hardy 空间维林肯型系统的极大算子

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摘要: 本文研究二维 Hardy 空间维林肯型系统的极大算子的有界性. 利用原子分解方法, 我们证明二维极大算子 $T_\alpha f := \sup_{2^{-\alpha}\leqslant \frac{n}{m}\leqslant 2^\alpha} |f * P_{n,m}|$ 是从鞅 Hardy 空间 H^p 到 L^p 有界的，其中 $0 < p < 1/2$, $\alpha \geq 0$. 作为应用, 我们得到二维极大算子 $\tilde{\sigma}^* f = \sup_{2^{-\alpha}\leqslant \frac{n}{m}\leqslant 2^\alpha} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/p-2}}$ 的有界性证明. 通过构造反例，我们证明二维极大算子 $\hat{\sigma}^* f = \sup\limits_{n,m\in\mathbf{N}} \frac{|\sigma_{n,m} f|}{[(n+1)(m+1)]^{1/2p-1}}$ 不是从鞅 Hardy 空间 H^p 到 L^p 有界的, 其中 $0 < p < \frac{1}{2}$. 上述结果推广了沃尔什系统、维林肯系统下的已知结论.

关键词: 维林肯型系统; 极大算子; Dirichlet 核; Fejér 核

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