Vol. 43 (2023)
No. 2

数 学 杂 志
J. of Math. (PRC)

WELL-POSEDNESS FOR THE COMPRESSIBLE ACTIVE LIQUID CRYSTAL MODEL

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Abstract: We consider the hydrodynamics of compressible flow of active liquid crystals model in the Q-tensor framework. The existence of local-in-time classical solution with large initial data in the whole space or torus is established. Furthermore, with an assumption on the coefficients, we also prove the global-in-time existence of classical solutions near a constant state with small initial data on the torus.

Keywords: Compressible active liquid crystal model; classical solution; near equilibrium; global in time

2010 MR Subject Classification: 35D35; 35Q35; 76A15

Document code: A **Article ID:** 0255-7797(2023)02-095-31

1 Introduction

Liquid crystals are a state of matter which may flow like a liquid but the molecules in which may be oriented in a preferred direction. They have properties between those of liquids and solid, and in general their phases can be divided into thermotropic, lyotropic and metallotropic phases, based on the differences of their optical properties. One of the most common liquid crystals phases is nematic phase for which the elongated molecules have long-range directional order with their long axes roughly parallel (see [1, 2]).

A number of mathematical theories have been made to study the hydrodynamic of nematic liquid crystals in the literature. The earliest so-called Oseen-Frank theory seems to be established by Oseen [3] in 1933, later then be revised by Frank [4] in 1958. Then Onsager [5] in 1949 and Doi [6] in 1986 proposed the Doi-Onsager theory. During the period of 1958 through 1968, Ericksen [7] and Leslie [8] established the Ericksen-Leslie theory to describe the dynamic behavior of the liquid crystals. In 1995, the Landau-de Gennes theory proposed by Gennes in [1]. In fact, under some assumptions, the above theories can be derived or related to each other, we refer the readers to [9–14] and the references therein.

In the reality, we call active particles the particles which have the ability to move autonomously in a surrounding fluid by converting energy into directed motion. So the

* **Received date:** 2021-03-19 **Accepted date:** 2021-06-21

Foundation item: Supported by National Natural Science Foundation of China(11971360).

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active particles are always taking the action to constantly maintain the motion out of the equilibrium by internal energy source, rather than by the external force applied to the system. Active hydrodynamics are often used to describe the fluids with active particles, which are fundamentally different from the fluid case constructed by the passive particles.

Generally speaking, we have two different frameworks to describe the motion of liquid crystal molecules in the fluid. For the dynamic behavior of liquid crystal molecules, one of them is based on the equation describing the orientation of the molecules. This feature is also presented in the Ericksen-Leslie model [7, 8]. Another one is the Q-tensor framework (see for an introduction to the Q-tensor framework [15], [16]), in which the local orientation of the molecules is described by a function Q taking value from $\Omega \subset \mathbb{R}^3$ in the set of the so-called 3-dimensional Q-tensors. More precisely speaking, it is a 3×3 symmetric and traceless matrix for the 3-dimensional case, namely

$$S_0^{(3)} := \{Q \in \mathbb{R}^{3\times 3} : Q_{ij} = Q_{ji},\, \mathrm{tr}(Q) = 0,\, i, j = 1, 2, 3\}\,.$$

While, the evolution of the Q's can be obtained by the free energy of the molecules, combining with the transport, distortion and alignment effects caused by the flow.

In this paper, we consider the following compressible active liquid crystal system in the Q-tensor framework (see [17]) in the domain $\mathbf{\Omega}$:

$$\begin{cases} \partial_t c + (u \cdot \nabla) c = D_0 \Delta c\,, \\ \partial_t \rho + \nabla \cdot (\rho u) = 0\,, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\nu + \mu)\nabla \mathrm{div} u = \nabla \cdot \tau + \nabla \cdot \sigma\,, \\ \partial_t Q + u \cdot \nabla Q + (Q\Omega - \Omega Q) = \Gamma H[Q, c]\,. \end{cases} \tag{1.1}$$

where $c \in \mathbb{R}$ is the concentration of active particles, ρ and $u \in \mathbb{R}^3$ are the density and velocity of the fluid, respectively. Q is the nematic tensor order parameter which is a traceless and symmetric 3×3 matrix. $P = a\rho^\gamma$ denotes the pressure satisfying γ-law with $\gamma > \frac{3}{2}$ and the constant $a > 0$. $D_0 > 0$ is the diffusion constant, $\mu > 0$ and $\nu > 0$ are the viscosity coefficients. $\Omega = \frac{1}{2}(\nabla u - \nabla u^\top)$ is the antisymmetric part of the strain tensor, and $\Gamma^{-1} > 0$ is the rotational viscosity. Moreover, the tensor

$$H[Q, c] := K\Delta Q - \tfrac{\kappa}{2}(c - c_*)Q + b(Q^2 - \tfrac{\mathrm{tr}\{Q^2\}}{3}\mathbb{I}_3) - c_* Q \mathrm{tr}\{Q^2\}$$

describes the relaxational dynamics of the nematic phase, which can be derived by the Landau-de Gennes free energy, that is, $H_{ij} = \frac{\delta \mathcal{F}}{\delta Q_{ij}}$ with

$$\mathcal{F} = \int \Big(\tfrac{\kappa}{2}(c - c_*)\mathrm{tr}(Q^2) - \tfrac{b}{3}\mathrm{tr}(Q^3) + \tfrac{c_*}{4}|\mathrm{tr}(Q^2)|^2 + \tfrac{K}{2}|\nabla Q|^2\Big)\mathrm{d}A$$

where $K > 0$ is the elastic constant for the one-constant elastic energy density, $c_* > 0$ is the critical concentration for the isotropic-nematic transition, and $\kappa > 0$ and $b \in \mathbb{R}$ are material-dependent constants. The stress tensor $\sigma = (\sigma_{ij}) := \sigma^r_{ij} + \sigma^a_{ij}$ is constructed by the following two parts:

$$\sigma^r_{ij} = Q_{ik}H_{kj}[Q,c] - H_{ik}[Q,c]Q_{kj} \quad \text{and} \quad \sigma^a_{ij} = \sigma_* c^2 Q_{ij}\,,$$

where σ^r_{ij} is the stress due to the nematic elasticity, and σ^a_{ij} is the active stress tensor which describes contractile ($\sigma_* > 0$) or extensile ($\sigma_* < 0$) stresses exerted by the active particles along the director field. $\tau = (\tau_{ij})$ is the symmetric additional stress tensor with the following form:

$$\tau_{ij} = F(Q)\delta_{ij} - K(\nabla Q \odot \nabla Q)_{ij}\,,$$

where

$$F(Q) = \tfrac{K}{2}|\nabla Q|^2 + \tfrac{1}{2}\mathrm{tr}(Q^2) + \tfrac{c_*}{4}\mathrm{tr}^2(Q^2) \quad \text{and} \quad (\nabla Q \odot \nabla Q)_{ij} = \partial_i Q_{lm}\partial_j Q_{lm}\,.$$

Here and after, we use the Einstein summation convention, i.e. The repeated indices are summed over.

We then rewrite the system (1.1) as

$$\begin{cases} \partial_t c + u\cdot\nabla c = D_0\Delta c\,,\\ \partial_t\rho + \nabla\cdot(\rho u) = 0\,,\\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + a\nabla\rho^\gamma = \mu\Delta u + (\mu+\nu)\nabla\mathrm{div}u + \nabla\cdot(F(Q)\mathbb{I}_3 - K\nabla Q\odot\nabla Q)\\ \qquad\qquad\qquad\qquad\qquad\qquad +K\nabla\cdot(Q\Delta Q - \Delta Q Q) + \sigma_*\nabla\cdot(c^2 Q)\,,\\ \partial_t Q + u\cdot\nabla Q + (Q\Omega - \Omega Q) = \Gamma H[Q,c]\,, \end{cases} \tag{1.2}$$

with $\sigma_* \in \mathbb{R}$ based on the above representations. The unknown variables are subjected to the following initial conditions:

$$(c,\rho,u,Q)(t,x)|_{t=0} = (c^{in},\rho^{in},u^{in},Q^{in})(x) \in (\mathbb{R},\mathbb{R}^+,\mathbb{R}^3,S_0^{(3)})\,, \tag{1.3}$$

and $(t,x)\in(\mathbb{R}^+,\mathbf{\Omega})$.

Active hydrodynamics have been widely used in applications and attracted more attentions in recent years, especially in the theoretical physics community. In fact, a large class of active systems can be treated as active nematic liquid crystal system. We refer the readers to [6, 18–24] and the references cited therein for more applications and discussions. While there are less rigorous mathematical description and analysis about active nematic liquid crystals. Recently, Chen-Majumdar-Wang-Zhang [25] studied the existence of global weak solutions for the incompressible active liquid crystals model [21, 26] in two and three spatial dimensions. They also obtained the higher regularity of the weak solutions and the weak-strong uniqueness by using the Littlewood-Paley decomposition for the two-dimensional case in [25]. As to the compressible active liquid crystal model [21, 26], the above four authors also established the global weak solutions in three spatial dimensions in [17].

In the present paper, we study the local-in-time classical solution for system (1.1), or (1.2) with large initial data (1.3) in the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$, as well as the existence of global-in-time classical solution on the torus $\mathbb{T}^3$ by using energy method. Our study is motivated by the works [17, 25] and the methods to be used to study the well-posedness for the compressible Ericksen-Leslie model in [27]. By noticing the mass conservation law, they derive that the L^∞-bound of density can be controlled by the L^∞-bound of div$u$ under the initial density ρ^{in} with positive lower bound $\underline{\rho}$ and upper bound $\bar{\rho}$. As a result, the L^∞-estimate of ρ and the functions typed with the term $\frac{1}{\rho}$ can be dominated by $\|\mathrm{div}u\|_{L^\infty}$ in deriving the a priori estimates and the energy estimates of approximation system.

Notation and Convenient We denote the Sobolev space by $H^k(\mathbf{\Omega})$ for integer $k \geq 1$, equipped with norm $\|\cdot\|_{H_x^k}$, and $\|\cdot\|_{\dot{H}_x^k}$ is the corresponding homogeneous Sobolev space. $\langle \cdot,\cdot\rangle$ is the standard inner product with respect to the space variables x in the whole space. For the case if f and g are vectors, then $\langle f, g\rangle = \displaystyle\int_{\mathbf{\Omega}} f(x)\cdot g(x)\mathrm{d}x$; and if A, B are matrices, then $\langle A, B\rangle = \displaystyle\int_{\mathbf{\Omega}} A : B\mathrm{d}x$ with $A : B = \mathrm{tr}(AB)$. The product of $\nabla A \otimes \nabla B$ is a matrix with ij component $(\partial_i A : \partial_j B)$.

The Frobenius norm is used as the norm of a matrix, denoted by $|Q|^2 := \mathrm{tr}(Q^2) = Q_{ij}Q_{ji}$. Then, we can define the Sobolev spaces for the Q-tensor as follows:

$$H^s(\mathbf{\Omega}) := \big\{Q : \mathbf{\Omega} \to S_0^{(3)} \big| \int_{\mathbf{\Omega}} \sum_{|\alpha|\leq s} |\partial_x^\alpha Q|^2 \mathrm{d}x < \infty\big\}\,,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with α_i is an nonnegative integer for $i = 1, 2, 3$, and $\partial_x^\alpha Q = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} Q$, $\partial_i Q = \partial_{x_i} Q$. Moreover, if there is a general positive constant C such that $f(t) \leq Cg(t)$ for any $t \in \mathbb{R}^+$, we denote it $f(t) \lesssim g(t)$ for simplification.

Furthermore, in order to state our main results in a simple way, we define the following two Sobolev weighted-norm (with wight ϕ) as

$$\|f\|_{H_\phi^s} := \big(\sum_{|\alpha|=0}^{s} \int_{\mathbf{\Omega}} \phi|\partial_x^\alpha f|^2 \mathrm{d}x\big)^{\frac{1}{2}}\,, \quad \|f\|_{\dot{H}_\phi^s} := \big(\sum_{|\alpha|=1}^{s} \int_{\mathbf{\Omega}} \phi|\partial_x^\alpha f|^2 \mathrm{d}x\big)^{\frac{1}{2}}\,,$$

and denote the quantity $\mathscr{N}_s(\rho)$ as $\mathscr{N}_s(\rho) = \|\rho\|^2_{\dot{H}^s_{\frac{P'(\rho)}{\rho}}} + \frac{2a}{\gamma-1}\|\rho\|^\gamma_{L^\gamma}$. Based on the above notation, we define the free energy $\mathcal{E}(t)$ as

$$\mathcal{E}(t) = \mathscr{N}_s(\rho) + \|c\|^2_{H^s} + \|u\|^2_{H^s_\rho} + \|Q\|^2_{H^s} + K\|\nabla Q\|^2_{H^s}\,, \tag{1.4}$$

and dissipation $\mathcal{D}(t)$ as

$$\mathcal{D}(t) = D_0\|\nabla c\|^2_{H^s} + \mu\|\nabla u(\tau)\|^2_{H^s} + (\mu+\nu)\|\mathrm{div}u\|^2_{H^s} + \Gamma K\|\nabla Q\|^2_{H^s} + \Gamma K^2\|\Delta Q\|^2_{H^s}\,. \tag{1.5}$$

In particular, the initial energy is $\mathcal{E}^{in} = \mathscr{N}_s(\rho^{in}) + \|c^{in}\|^2_{H^s} + \|u^{in}\|^2_{H^s} + \|Q^{in}\|^2_{H^s} + K\|\nabla Q^{in}\|^2_{H^s}$. As the first result of this paper, we study the local-in-time solution for the system (1.2)-(1.3) with large initial data. The main theorem is stated as follows:

Theorem 1.1 (Local existence) Let $\mathbf{\Omega}$ be the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$, and the integer $s \geq 3$, if the initial energy $\mathcal{E}^{in} < +\infty$, and $\underline{\rho} < \rho^{in} < \bar{\rho}$ for some positive constants $\underline{\rho}$ and $\bar{\rho}$, then there exist $T > 0$ and $C_0 > 0$, depending on $\mathcal{E}^{in}$ and the coefficients of (1.2), such that the system (1.2)-(1.3) admits the unique solution (c, ρ, u, Q) satisfying

$$\begin{aligned} c \in& L^\infty(0,T;H^s(\mathbf{\Omega})) \cap L^2(0,T;H^{s+1}(\mathbf{\Omega}))\,, \\ \rho \in& L^\infty(0,T;\dot{H}^s_{\frac{P'(\rho)}{\rho}}(\mathbf{\Omega})) \cap L^\gamma(\mathbf{\Omega})\,, \\ u \in& L^\infty(0,T;H^s_\rho(\mathbf{\Omega})) \cap L^2(0,T;H^{s+1}(\mathbf{\Omega}))\,, \\ Q \in& L^\infty(0,T;H^{s+1}(\mathbf{\Omega})) \cap L^2(0,T;H^{s+1}(\mathbf{\Omega}))\,. \end{aligned} \tag{1.6}$$

Moreover, the energy inequality

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau)\mathrm{d}\tau \leq C_0 \tag{1.7}$$

holds for all $t \in [0,T]$.

We will prove Theorem 1.1 by using the energy methods. We now sketch the main difficulties we have met and the novelties to be used in the proof of the above theorem. The first difficult is how to control the L^∞-bound of density. To overcome this difficulty we use some techniques inspired by the studies in compressible Navier-Stokes equations. Employing the mass conservation law, we can derive the argument that the density $\rho(t,x)$ can be bounded by the L^∞-norm of $\mathrm{div}u$ (see Lemma) in the spirit of [27], precisely speaking,

$$\underline{\rho}\exp\Big\{ -\tfrac{1}{4}\int_0^t \|\mathrm{div}u\|_{L^\infty}(s)\mathrm{d}s\Big\} \leq \rho(t,x) \leq \bar{\rho}\exp\Big\{ -\tfrac{1}{4}\int_0^t \|\mathrm{div}u\|_{L^\infty}(s)\mathrm{d}s\Big\} \tag{1.8}$$

under the assumption $0 < \underline{\rho} \leq \rho^{in} \leq \bar{\rho}$ for the initial density ρ^{in}. As a result, the L^∞-bounds of the density $\rho(t,x)$ and $\frac{1}{\rho}$ can be bounded by the L^∞-norm of $\mathrm{div}u(t,x)$ in the derivation of the a priori estimates.

The second difficulty is how to deal with the terms with highest order derivative in the derivation of the a priori estimate. Here we will use the following cancellation relation

$$\begin{aligned} &\langle \partial_x^\alpha \mathrm{div}(Q\Delta Q - \Delta QQ), \partial_x^\alpha u\rangle + \langle \partial_x^\alpha(Q\Omega - \Omega Q), \partial_x^\alpha \Delta Q\rangle \\ =&\langle \partial_x^\alpha Q\Delta Q - \Delta Q\partial_x^\alpha Q, \partial_x^\alpha \Omega\rangle + \langle \partial_x^\alpha Q\Omega - \Omega\partial_x^\alpha Q, \partial_x^\alpha \Delta Q\rangle + \langle \mathcal{M}_1, \partial_x^\alpha \Omega\rangle + \langle \mathcal{M}_2, \partial_x^\alpha \Delta Q\rangle \end{aligned} \tag{1.9}$$

with the mediate terms $\mathcal{M}_1$ and $\mathcal{M}_2$ (see (2.16)), which enables us to avoid controlling the highest order derivative term $\partial_x^\alpha \mathrm{div}\Delta Q$. We also remark that the symmetry and traceless play an important role in the cancellation relation.

Therefore, in order to construct approximation solutions (c^n, ρ^n, u^n, Q^n), we carefully design an iteration scheme. The mass conservation equation of ρ^{n+1} is designed with the velocity u^n such that the norms $\|\rho^{n+1}\|_{L^\infty}$ and $\|\frac{1}{\rho^{n+1}}\|_{L^\infty}$ can be dominated by $\|\mathrm{div}u^n\|_{L^\infty}$ by using the inequality (1.8). On the other hand, the construction of the u-equations and Q-equations in approximation system should be coupled together in order to use the cancellation relation (1.9). The corresponding terms in (1.9) are designed as $\nabla \cdot (Q^n\Delta Q^{n+1} -$

$\Delta Q^{n+1}Q^n$) and $Q^n\Omega^{n+1}-\Omega^{n+1}Q^n$ (see (3.1)). Consequently, we can close the energy estimates of the approximation system. Then we get the existence of the solution sequence of the approximation system. However, the uniform existence time interval can not be obtained directly. So before getting the solution of the initial equations (1.2)–(1.3), we need to derive that the solution sequence for the approximation system has a uniform existence lower time, see Lemma 4.1. At last, based on the estimates of the approximation system, we then prove the first main theorem of this paper.

It is worth pointing out that we do not obtain global-in-time solutions on the whole space for the active liquid crystal model at present. We remark here that the difficulty is caused by the linear term c_*Q with the critical concentration c_* in $H[Q,c]$. In fact, in the process of deriving the a priori estimates, we multiply Q in the Q-equation of the system (1.2) to get the dissipation but with no L^2-dissipation of Q, so the linear term $\langle c_*Q,Q\rangle$ can not be controlled when we deal with the global-in-time classical solutions. Even though we consider the global solution near a constant state $(c_*,1,0,0)$, this difficulty still exists, which will be converted to deal with the term $\sigma_*c_*^2\langle \mathrm{div}Q,u\rangle$. However, when the domain to be considered is the torus $\mathbb{T}^3$, then there exists a constant C depending on the volume of $\mathbb{T}^3$ such that

$$\sigma_*c_*^2\langle \mathrm{div}Q,u\rangle \le |\sigma_*|c_*^2|\mathbb{T}^3|^3\|Q\|_{L^6}\|\nabla u\|_{L^2}\le \tfrac{C\sigma_*^2c_*^4}{2\mu}\|\nabla Q\|_{L^2}^2+\tfrac{\mu}{2}\|\nabla u\|_{L^2}^2 \tag{1.10}$$

by using Hölder inequality, Sobolev embedding inequality $\|Q\|_{L^6}\lesssim\|\nabla Q\|_{L^2}$, and the Cauchy inequality. Because of the dissipations $K\Gamma\|\nabla Q\|_{H^s}$ and $\mu\|\nabla u\|_{H^s}^2$ we have obtained, then under the assumption that $\frac{C\sigma_*^2c_*^4}{2\mu}\le\frac{K\Gamma}{2}$, the terms on the right-hand side of (1.10) can be absorbed by the above two dissipation terms.

Based on the above ideas, we then study the global existence of classical solutions (c,ρ,u,Q) to the system (1.2) near a constant state $(c_*,1,0,0)$ on the torus $\mathbb{T}^3$. We rewrite c and ρ as the following forms: $c(t,x)=\tilde c(t,x)+c_*$, $\rho(t,x)=\varrho(t,x)+1$. Submitting the above identities into system (2.6), we then have

$$\begin{cases}\partial_t\tilde c+(u\cdot\nabla)\tilde c=D_0\Delta\tilde c\,,\\ \partial_t\varrho+\nabla\cdot\big((1+\varrho)u\big)=0\,,\\ \partial_tu+u\cdot\nabla u+\frac{P'(1+\varrho)}{\varrho}\nabla\varrho=\frac{1}{1+\varrho}[\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u]+\frac{1}{1+\varrho}\nabla\cdot(F(Q)\mathbb{I}_3-K\nabla Q\odot\nabla Q)\\ \qquad\qquad +\frac{K}{1+\varrho}\nabla\cdot(Q\Delta Q-\Delta QQ)+\frac{\sigma_*}{1+\varrho}\nabla\cdot[(\tilde c+c_*)^2Q]\\ \partial_tQ+u\cdot\nabla Q+(Q\Omega-\Omega Q)=\Gamma H[Q,\tilde c+c_*]\,,\end{cases}\tag{1.11}$$

where the form of $H[Q,\tilde c+c_*]$ is

$$H[Q,\tilde c+c_*]=K\Delta Q-\tfrac{\kappa}{2}\tilde cQ+b(Q^2-\tfrac{\mathrm{tr}(Q^2)}{3}\mathbb{I}_3)-c_*Q\mathrm{tr}(Q^2)$$

and $(t,x)\in\mathbb{R}^+\times\mathbb{T}^3$. We impose the initial data to the system (1.11) as

$$(\tilde c,\varrho,u,Q)(t,x)|_{t=0}=(\tilde c^{in},\varrho^{in},u^{in},Q^{in})(x)\,,\tag{1.12}$$

with $\tilde{c}^{in} = c^{in} - c_*$ and $\varrho^{in} = \rho^{in} - 1$. Here the functions $(c^{in}, \rho^{in}, u^{in}, Q^{in})$ are given in (1.3).

For convenience of notation, we denote the initial energy as

$$\tilde{\mathcal{E}}^{in} = \|\tilde{c}^{in}\|^2_{H^s} + \|\varrho^{in}\|^2_{\dot{H}^s} + \|u^{in}\|^2_{H^s} + \|Q\|^2_{H^{s+1}}\,.$$

Then we state the second main theorem of this paper in the following.

Theorem 1.2 (Global existence) Let $\mathbf{\Omega}$ be the torus $\mathbb{T}^3$, and the integer $s \geq 3$, assume that $\frac{C}{2\mu}\sigma_*^2 c_*^4 \leq \frac{K\Gamma}{2}$ for some constant C depending on the volume of torus $\mathbb{T}^3$, then there exists an $\epsilon_0 > 0$, depending only on the coefficients of the system (1.11) and s, such that if $\tilde{\mathcal{E}}^{in} < \epsilon_0$, then the system (1.11)-(1.12) admits the unique global solution $(\tilde{c}, \varrho, u, Q)$ satisfying

$$\begin{aligned}
&\varrho \in L^\infty\big(\mathbb{R}^+; H^s(\mathbb{T}^3)\big)\,,\\
&\tilde{c}, u \in L^\infty\big(\mathbb{R}^+; H^s(\mathbb{T}^3)\big) \cap L^2\big(\mathbb{R}^+; H^{s+1}(\mathbb{T}^3)\big)\,,\\
&Q \in L^\infty\big(\mathbb{R}^+; H^{s+1}(\mathbb{T}^3)\big)\,.
\end{aligned}$$

Furthermore, the energy bound

$$\sup_{t\geq 0}\Big(\|\tilde{c}\|^2_{H^s} + \|\varrho\|^2_{H^s} + \|u\|^2_{H^s} + \|Q\|^2_{H^{s+1}}\Big)(t) + \int_0^\infty D_0\|\nabla\tilde{c}\|^2_{H^s} + \tfrac{\mu}{4}\|\nabla u\|^2_{H^s}(\tau)\mathrm{d}\tau \leq C_1\tilde{\mathcal{E}}^{in}$$

holds for some constant C_1, depending on the coefficients of the system (1.11) and s.

To show the existence of global-in-time classical solutions on the torus $\mathbb{T}^3$, the dissipation we have obtained in local-in-time energy estimates is not enough. Indeed, we need some dissipation of ρ. Thanks to the construction of the pressure $P(\rho)$, satisfying $P'(\rho) > 0$, the term $\nabla P(\rho)$ in the velocity equations of (1.2) will give us some dissipation on the density ρ by multiplying the u-velocity with $\nabla\rho$ (or $\nabla^k\rho$ for higher order estimates).

As we all know, there are some mathematical works on the liquid crystal model in the Q-tensor framework. In [28], Feireisl-Schimperna-Rocca-Zarnescu proved the existence of global-in-time weak solutions for the coupled incompressible Navier-Stokes system and a nonlinear convective parabolic equation describing the evolution of the Q-tensor. For another system in the Q-tensor framework with arbitrary physically relevant initial data in case of a singular bulk potential proposed in Ball-Majumdar [9], the above four authors [29] also established the global-in-time weak solutions. In [30, 31], Paicu-Zarnescu studied the coupled incompressible Navier-Stokes and Q-tensor system, proved the global-in-time existence of weak solutions for $d = 2, 3$ and the existence of global regular solutions with sufficiently regular initial data for $d = 2$.

Organization of the Paper This paper is organized as follows: in the next section, we derive the a priori estimate, which consists of the L^2-estimate and the higher order estimate of system (1.2). In Section 3, we construct the approximation system by using iteration method and obtain its energy estimates. In the forth section, we give the proof of Theorem 1.1, namely, the existence of local-in-time solution of system (1.2)-(1.3) based on the obtaining energy estimate in Section 3. In the last section, the existence of the global-in-time classical solution (c,ρ,u,Q) near the constant state $(c_*,1,0,0)$ on the torus $\mathbb{T}^3$ is proved.

2 A Priori Estimates

2.1 Basic Energy Estimate

In this subsection, we derive the L^2-estimate of the system (1.2)-(1.3) under the assumption that (c,ρ,u,Q) are sufficiently smooth, which will give us the possible structure of the higher order estimate. We state the following lemma about the basic energy bound.

Lemma 2.1 (Basic energy bound) Let (c,ρ,u,Q) be a sufficiently smooth solution to the system (1.2), then

$$\begin{aligned}
&\tfrac{1}{2}\int_{\mathbf{\Omega}}\Big(|c|^2+\rho|u|^2+\tfrac{2a}{\gamma-1}\rho^\gamma+|Q|^2+K|\nabla Q|^2+\tfrac{c_*}{2}|Q|^4\Big)\mathrm{d}x+D_0\|\nabla c\|_{L^2}^2+\mu\|\nabla u\|_{L^2}^2\\
&\quad+(\mu+\nu)\|\mathrm{div}u\|_{L^2}^2+\Gamma K\|\nabla Q\|_{L^2}^2+\Gamma K^2\|\Delta Q\|_{L^2}^2+c_*\Gamma\|Q\|_{L^4}^4+c_*^2\Gamma\|Q\|_{L^6}^6\\
\lesssim&\|u\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}^{\frac34}\|c\|_{L^2}^{\frac14}\|\nabla c\|_{L^2}^{\frac74}+\|Q\|_{L^2}^{\frac12}\|\nabla Q\|_{L^2}^{\frac32}(\|\Delta Q\|_{L^2}+\|Q\|_{L^2}+\|Q\|_{L^2}\|Q\|_{H^2}^2)\\
&\quad+(\|c\|_{L^2}\|Q\|_{L^2})^{\frac14}(\|\nabla c\|_{L^2}\|\nabla Q\|_{L^2})^{\frac34}(\|c\|_{H^2}\|\nabla u\|_{L^2}+\|\Delta Q\|_{L^2})\\
&\quad+\|c\|_{H^2}\|Q\|_{L^2}^{\frac12}\|\nabla Q\|_{L^2}^{\frac32}(1+\|Q\|_{L^4}^2)+\|Q\|_{L^2}^2+\|\nabla Q\|_{L^2}^2+\|Q\|_{L^4}^4\,.
\end{aligned}\tag{2.1}$$

Proof We multiply the first, third and the forth equations of system (1.2) with c, u and $-K\Delta Q+Q+c_*Q\mathrm{tr}(Q^2)$ respectively, and then integrate over the space to get

$$\begin{aligned}
&\tfrac{1}{2}\int_{\mathbf{\Omega}}|c|^2+\rho|u|^2+\tfrac{2a}{\gamma-1}\rho^\gamma+|Q|^2+K|\nabla Q|^2+\tfrac{c_*}{2}|Q|^4\mathrm{d}x+D_0\|\nabla c\|_{L^2}^2+\mu\|\nabla u\|_{L^2}^2\\
&\quad+(\mu+\nu)\|\mathrm{div}u\|_{L^2}^2+\Gamma K\|\nabla Q\|_{L^2}^2+\Gamma K^2\|\Delta Q\|_{L^2}^2+c_*\Gamma\|Q\|_{L^4}^4+c_*^2\Gamma\|Q\|_{L^6}^6\\
=&-\langle u\cdot\nabla c,c\rangle-K\langle\nabla\cdot(\nabla Q\odot\nabla Q),u\rangle+\langle\nabla\cdot(F(Q)\mathbb{I}_3),u\rangle+\langle u\cdot\nabla Q,K\Delta Q\rangle\\
&\quad-\langle u\cdot\nabla Q,Q+c_*Q\mathrm{tr}(Q^2)\rangle+K\langle\nabla\cdot(Q\Delta Q-\Delta QQ),u\rangle-\langle\Omega Q-Q\Omega,K\Delta Q\rangle\\
&\quad+\langle\Omega Q-Q\Omega,Q+c_*Q\mathrm{tr}(Q^2)\rangle+\sigma_*\langle\nabla\cdot(c^2Q),u\rangle+\Gamma\kappa\langle\tfrac{c-c_*}{2}Q,-K\Delta Q+Q+c_*Q\mathrm{tr}(Q^2)\rangle\\
&\quad-b\Gamma K\langle Q^2,\Delta Q\rangle+b\Gamma\langle Q^2,Q+c_*Q\mathrm{tr}(Q^2)\rangle+2c_*\Gamma K\langle Q\mathrm{tr}(Q^2),\Delta Q\rangle\\
=&\sum_{j=1}^{13}I_j\,.
\end{aligned}\tag{2.2}$$

Straightforward calculation enables us to get $I_2+I_3+I_4+I_5=0$. Let $D=\frac{\nabla u+\nabla u^\top}{2}$. Note that $\nabla u=D+\Omega$, then using the symmetric of D and Q, skew-symmetric of Ω, one has

$$I_6+I_7=K\langle Q\Delta Q-\Delta QQ,\Omega\rangle-K\langle\Omega Q-Q\Omega,\Delta Q\rangle=0\,.$$

and

$$I_8=2\langle\Omega,Q^2+c_*Q\mathrm{tr}(Q^2)\rangle=0\,.$$

We then estimate the other terms on the right-hand side of (2.2). For the term I_{13}, simple calculation tells us

$$I_{13} = -2c_*\Gamma K \int_{\mathbf{\Omega}} |Q|^2|\nabla Q|^2 \mathrm{d}x - c_*\Gamma K \int_{\mathbf{\Omega}} |\nabla(|Q|^2)|^2 \mathrm{d}x \le 0 \,.$$

Utilizing the Hölder inequalities and interpolation inequality $\|f\|_{L^4(\mathbf{\Omega})} \lesssim \|f\|^{\frac14}_{L^2(\mathbf{\Omega})}\|\nabla f\|^{\frac34}_{L^2(\mathbf{\Omega})}$ for any $f \in H^1(\mathbf{\Omega})$, we have

$$\begin{aligned} I_1 \le& \|u\|_{L^4}\|\nabla c\|_{L^2}\|c\|_{L^4} \lesssim \|u\|^{\frac14}_{L^2}\|\nabla u\|^{\frac34}_{L^2}\|c\|^{\frac14}_{L^2}\|\nabla c\|^{\frac74}_{L^2} \,, \\ I_{11} \lesssim& \|Q\|^2_{L^4}\|\Delta Q\|_{L^2} \lesssim \|Q\|^{\frac12}_{L^2}\|\nabla Q\|^{\frac32}_{L^2}\|\Delta Q\|_{L^2} \,. \end{aligned}$$

Similarly,

$$\begin{aligned} I_9 \lesssim& \|c\|_{H^2}(\|\nabla c\|_{L^2}\|Q\|_{L^2})^{\frac14}(\|\nabla c\|_{L^2}\|\nabla Q\|_{L^2})^{\frac34}\|\nabla u\|_{L^2} \,, \\ I_{12} \lesssim& \|Q\|^{\frac32}_{L^2}\|\nabla Q\|^{\frac32}_{L^2}(1 + \|Q\|^2_{H^2}) \,, \end{aligned}$$

and

$$\begin{aligned} I_{10} \lesssim& \|c\|^{\frac14}_{L^2}\|\nabla c\|^{\frac34}_{L^2}\|Q\|^{\frac14}_{L^2}\|\nabla Q\|^{\frac34}_{L^2}\|\Delta Q\|_{L^2} + \|c\|_{L^2}\|Q\|^{\frac12}_{L^2}\|\nabla Q\|^{\frac32}_{L^2} \\ &+ \|c\|_{H^2}\|Q\|^{\frac12}_{L^2}\|\nabla Q\|^{\frac32}_{L^2}\|Q\|^2_{L^4} + \|Q\|^2_{L^2} + \|\nabla Q\|^2_{L^2} + \|Q\|^4_{L^4} \,. \end{aligned}$$

Consequently, summing up with the above estimates we have the bound (2.1). This completes Lemma 2.1.

2.2 Higher Order Estimate

In this subsection, we derive the higher order estimate for the system (1.2)-(1.3). Before doing it, we need the following two lemmas. The first lemma tells us that the L^∞-bound of density in the system can be controlled by the L^∞-norm of $\mathrm{div} u$, which plays an key role in our calculation to close the energy estimate. By deriving the L^∞-bound of density ρ, we have used the mass conservation equation and the characteristic line method. Here we point out that the proof can be found in Lemma 3.1 of Section 3 in [27].

Lemma 2.2 Assume that $\underline{\rho} \le \rho^{in} \le \bar\rho$ for some positive constants $\underline{\rho}, \bar\rho$, and density ρ satisfies the second equation of (1.2), then the following inequality

$$\underline{\rho}\exp\Big\{-\tfrac14\int_0^t \|\mathrm{div} u\|_{L^\infty}(s)\mathrm{d}s\Big\} \le \rho \le \bar\rho\exp\Big\{\tfrac14\int_0^t \|\mathrm{div} u\|_{L^\infty}\mathrm{d}s\Big\} \tag{2.3}$$

holds.

The second lemma gives us the L^2-bound of the derivative for a general smooth function $f(\rho)$. Based on the following lemma we then can bound the terms such as $\partial_x^\alpha(\frac{1}{\rho})$ $(1 \le |\alpha| \le s)$ in terms of the L^∞-bound of $\mathrm{div} u$.

Lemma 2.3 Let $f(\rho)$ be a smooth function, then for any positive integer k and $\rho \in H^k(\mathbb{R}) \cup L^\infty(\mathbb{R})$, we have

$$\partial_x^k f(\rho) = \sum_{i=1}^k f^{(i)}(\rho) \sum_{\sum_{l=1}^i k_l = k, k_l \ge 1} \prod_{l=1}^i \nabla^{k_l}\rho \,. \tag{2.4}$$

In particular, when ρ satisfies the assumption stated in Lemma , and $f(\rho)=\frac{1}{\rho}$, we then have

$$
\begin{aligned}
\|\nabla^k f(\rho)\|_{L^2} \leq & \sum_{i=1}^{k} \frac{i!}{\rho^{i+1}} \exp\left\{\frac{i+1}{4}\int_0^t \|\mathrm{div}u\|_{L^\infty}\mathrm{d}s\right\} \sum_{\sum_{l=1}^{i} k_l=k, k_l\geq 1} \|\prod_{l=1}^{i}\nabla^{k_l}\rho\|_{L^2} \\
\leq & C(\underline{\rho},k)\exp\left\{\tfrac{k+1}{4}\int_0^t \|\mathrm{div}u\|_{L^\infty}\mathrm{d}s\right\}\mathscr{P}_k(\|\rho\|_{\dot{H}^k}) \\
\leq & \mathscr{Q}(u)\mathscr{P}_k(\|\rho\|_{\dot{H}^k}),
\end{aligned} \tag{2.5}
$$

where $C(\underline{\rho},k)$ is the constant depends on $\underline{\rho}$ and k, $\mathscr{Q}(u)=a_1\exp\left\{a_2\int_0^t\|\mathrm{div}u\|_{L^\infty}\mathrm{d}s\right\}$ with generic constants a_1 and a_2, and $\mathscr{P}_k(x)=\sum_{j=1}^{k}x^j$.

Proof we can prove the lemma by induction. Here we omit it for simplicity.

Based on Lemma , we have derived that ρ has a positive lower bound. Hence we can rewrite the system (1.2) as

$$
\begin{cases}
\partial_t c + u\cdot\nabla c = D_0\Delta c\,,\\
\partial_t\rho + \nabla\cdot(\rho u) = 0\,,\\
\partial_t u + u\cdot\nabla c + \frac{a}{\rho}\nabla\rho^\gamma = \frac{1}{\rho}\mu\Delta u + \frac{(\mu+\nu)}{\rho}\nabla\mathrm{div}u + \frac{1}{\rho}\nabla\cdot(F(Q)\mathbb{I}_3 - \nabla Q\odot\nabla Q)\\
\qquad\qquad + \frac{K}{\rho}\nabla\cdot(Q\Delta Q - \Delta Q Q) + \frac{\sigma_*}{\rho}\nabla\cdot(c^2 Q)\,,\\
\partial_t Q + u\cdot\nabla Q + (Q\Omega - \Omega Q) = \Gamma H[Q,c]\,.
\end{cases} \tag{2.6}
$$

We now deal with the higher order estimates of system (1.2). It is equivalent to obtain the higher order estimate of system (2.6). The lemma about the a priori estimate of system (1.2)-(1.3) is stated as:

Lemma 2.4 (The a priori estimate) Let the integer $s\geq 3$, and under the assumption that (c,ρ,u,Q) is a sufficiently smooth solution to the system (1.2)-(1.3) on the time interval $[0,T]$. Then there exists a positive constant C, depending only on the coefficients of system (1.2) and s, such that for all $t\in[0,T]$,

$$
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\mathcal{D}(t) \leq C\Big[\sum_{i=2}^{4}\mathcal{E}^{\frac{i}{2}}(t)+\mathscr{Q}(u)\big[\mathcal{E}(t)+\mathcal{E}^{\frac{3}{2}}(t)\big]+\mathcal{D}^{\frac{1}{2}}(t)\sum_{i=2}^{4}\mathcal{E}^{\frac{i}{2}}(t)+\mathscr{Q}(u)\mathcal{D}^{\frac{1}{2}}(t)\sum_{i=1}^{s+2}\mathcal{E}^{\frac{i}{2}}(t)\Big]\,. \tag{2.7}
$$

Proof Fix any integer $s\geq 3$, for any positive integer k ($k\leq s$), we take the derivative operator ∂_x^α ($|\alpha|=k$) on the equations of (2.6), and multiply the result equalities with $\partial_x^\alpha c$, $\frac{P'(\rho)}{\rho}\partial_x^\alpha\rho$, $\rho\partial_x^\alpha u$ and $-K\partial_x^\alpha\Delta Q+\partial_x^\alpha Q$, respectively, then integrate the resultants in $\mathbf{\Omega}$ with respect to x. Then we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{\Omega}}\tfrac{P'(\rho)}{\rho}|\partial_x^\alpha\rho|^2+|\partial_x^\alpha c|^2+\rho|\partial_x^\alpha u|^2+|\partial_x^\alpha Q|^2+K|\partial_x^\alpha\nabla Q|^2\mathrm{d}x\\
&+D_0\|\partial_x^\alpha\nabla c\|_{L^2}^2+\mu\|\partial_x^\alpha\nabla u\|_{L^2}^2+(\mu+\nu)\|\partial_x^\alpha\mathrm{div}u\|_{L^2}^2+\Gamma K(\|\partial_x^\alpha\nabla Q\|_{L^2}^2+K\|\partial_x^\alpha\Delta Q\|_{L^2}^2)\\
=&I^k+J^k\,,
\end{aligned} \tag{2.8}
$$

where

$$
\begin{aligned}
I^k =& \underbrace{-\langle \partial_x^\alpha (u\cdot\nabla Q), \partial_x^\alpha Q\rangle - \langle \partial_x^\alpha (u\cdot\nabla c), \partial_x^\alpha c\rangle}_{I_1^k} + \underbrace{\langle \partial_x^\alpha (\Omega Q - Q\Omega), \partial_x^\alpha Q\rangle}_{I_2^k} \\
& \underbrace{-\langle P'(\rho)\partial_x^\alpha \nabla\rho, \partial_x^\alpha u\rangle - \langle P'(\rho)\partial_x^\alpha \mathrm{div} u, \partial_x^\alpha \rho\rangle}_{I_3^k} \\
& + \underbrace{\tfrac{1}{2}\langle \partial_t(\tfrac{P'(\rho)}{\rho}), |\partial_x^\alpha \rho|^2\rangle - \langle u\cdot\nabla\partial_x^\alpha\rho, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle}_{I_4^k} + \underbrace{\sigma_*\langle \partial_x^\alpha \nabla\cdot(c^2 Q), \partial_x^\alpha u\rangle}_{I_5^k} \\
& + \underbrace{K\langle \partial_x^\alpha \nabla\cdot(Q\Delta Q - \Delta Q Q), \partial_x^\alpha u\rangle + K\langle \partial_x^\alpha (Q\Omega - \Omega Q), \partial_x^\alpha \Delta Q\rangle}_{I_6^k} \\
& + \underbrace{K\langle \partial_x^\alpha (u\cdot\nabla Q), \partial_x^\alpha \Delta Q\rangle - K\langle \partial_x^\alpha \nabla\cdot(\nabla Q\odot\nabla Q), \partial_x^\alpha u\rangle}_{I_7^k} \\
& + \underbrace{\tfrac{K\Gamma\kappa}{2}\langle \partial_x^\alpha[(c-c_*)Q], \partial_x^\alpha\Delta Q\rangle - \tfrac{\kappa\Gamma}{2}\langle \partial_x^\alpha[(c-c_*)Q], \partial_x^\alpha Q\rangle}_{I_8^k} + \underbrace{\langle \partial_x^\alpha\nabla\cdot(F(Q)\mathbb{I}_3), \partial_x^\alpha u\rangle}_{I_9^k} \\
& + \underbrace{b\Gamma\langle \partial_x^\alpha(Q^2 - \tfrac{\mathrm{tr}(Q^2)}{3}\mathbb{I}_3), -K\partial_x^\alpha\Delta Q + \partial_x^\alpha Q\rangle}_{I_{10}^k} + \underbrace{c_*\Gamma\langle \partial_x^\alpha[Q\mathrm{tr}(Q^2)], K\partial_x^\alpha\Delta Q - \partial_x^\alpha Q\rangle}_{I_{11}^k}
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
J^k =& \underbrace{-\langle [\partial_x^\alpha, \rho\mathrm{div}]u, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle - \langle [\partial_x^\alpha, u\cdot\nabla]\rho, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle}_{J_1^k} + \underbrace{\langle [\partial_x^\alpha, u\cdot\nabla]u, \rho\partial_x^\alpha u\rangle}_{J_2^k} \\
& \underbrace{-\langle [\partial_x^\alpha, \tfrac{P'(\rho)}{\rho}\nabla]\rho, \rho\partial_x^\alpha u\rangle}_{J_3^k} + \underbrace{\mu\langle [\partial_x^\alpha, \tfrac{1}{\rho}\Delta]u, \rho\partial_x^\alpha u\rangle + (\mu+\nu)\langle [\partial_x^\alpha, \tfrac{1}{\rho}\nabla\mathrm{div}]u, \rho\partial_x^\alpha u\rangle}_{J_4^k} \\
& + \underbrace{K\langle [\partial_x^\alpha, \tfrac{1}{\rho}\mathrm{div}](Q\Delta Q - \Delta Q Q), \rho\partial_x^\alpha u\rangle}_{J_5^k} + \underbrace{\sigma_*\langle [\partial_x^\alpha, \tfrac{1}{\rho}\mathrm{div}](c^2 Q), \rho\partial_x^\alpha u\rangle}_{J_6^k} \\
& + \underbrace{\langle [\partial_x^\alpha, \tfrac{1}{\rho}\mathrm{div}](F(Q)\mathbb{I}_3 - K\nabla Q\odot\nabla Q), \rho\partial_x^\alpha u\rangle}_{J_7^k}.
\end{aligned}
\tag{2.10}
$$

We then estimate the terms in I^k and J^k. First, we estimate the terms in I^k one by one. For the estimation of I_1^k, by using Hölder inequality and the interpolation inequality $\|f\|_{L^4} \lesssim \|f\|_{L^2}^{\frac{1}{4}}\|\nabla f\|_{L^2}^{\frac{3}{4}}$ for any $f\in H^1(\mathbf{\Omega})$, we can derive that

$$
\begin{aligned}
&\langle \partial_x^\alpha(u\cdot\nabla c), \partial_x^\alpha c\rangle = \langle u\cdot\nabla\partial_x^\alpha c + \partial_x^\alpha u\cdot\nabla c, \partial_x^\alpha c\rangle + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\langle \partial_x^{\alpha_1}u\partial_x^{\alpha_2}\nabla c, \partial_x^\alpha c\rangle \\
\lesssim& \|\mathrm{div}u\|_{L^\infty}\|\partial_x^\alpha c\|_{L^2}^2 + \|\partial_x^\alpha u\|_{L^4}\|\nabla c\|_{L^4}\|\partial_x^\alpha c\|_{L^2} + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}} \|\partial_x^{\alpha_1}u\|_{L^4}\|\partial_x^{\alpha_2}\nabla c\|_{L^4}\|\partial_x^\alpha c\|_{L^2} \\
\lesssim& \|u\|_{\dot{H}^s}\|\nabla c\|_{H^s}\|c\|_{\dot{H}^s} + (\|\nabla u\|_{H^s} + \|\mathrm{div}u\|_{H^s})\|c\|_{\dot{H}^s}^2 .
\end{aligned}
$$

Similarly, $\langle \partial_x^\alpha (u\cdot\nabla Q), \partial_x^\alpha Q\rangle \lesssim \|u\|_{\dot H^s}\|\nabla Q\|_{H^s}\|Q\|_{\dot H^s} + (\|\nabla u\|_{H^2}+\|\mathrm{div} u\|_{H^s})\|Q\|_{\dot H^s}^2$. Hence, one has

$$I_1^k \lesssim \|u\|_{\dot H^s}(\|\nabla c\|_{H^s}\|c\|_{\dot H^s}+\|\nabla Q\|_{H^s}\|Q\|_{\dot H^s})+(\|\nabla u\|_{H^s}+\|\mathrm{div} u\|_{H^s})(\|c\|_{\dot H^s}^2+\|Q\|_{\dot H^s}^2)\,. \tag{2.11}$$

As to the estimate of I_2^k, straightforward calculation gives us

$$\begin{aligned}\|\partial_x^\alpha(\Omega Q)\|_{L^2} \lesssim& \|\partial_x^\alpha Q\|_{L^2}\|\nabla u\|_{L^\infty} + \|Q\|_{L^\infty}\|\partial_x^\alpha \nabla u\|_{L^2} + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}} \|\partial_x^{\alpha_1} Q\|_{L^4}\|\partial_x^{\alpha_2}\nabla u\|_{L^4}\\ \lesssim& \|Q\|_{H^s}\|\nabla u\|_{H^s}\,,\end{aligned}$$

which yields

$$I_2^k \lesssim \|Q\|_{H^s}\|Q\|_{\dot H^s}\|\nabla u\|_{H^s}\,. \tag{2.12}$$

By using the integration by parts, Hölder inequality, Sobolev embedding theorem and Lemma , we have

$$\begin{aligned}I_3^k =& -\langle P''(\rho)\nabla\rho\partial_x^\alpha u, \partial_x^\alpha\rho\rangle + \langle P'(\rho)\partial_x^\alpha u, \partial_x^\alpha \nabla\rho\rangle - \langle P'(\rho)\partial_x^\alpha\nabla\rho, \partial_x^\alpha u\rangle\\ =& \langle P''(\rho)\nabla\rho\partial_x^\alpha u, \partial_x^\alpha\rho\rangle \lesssim \mathscr{Q}(u)\|\rho\|_{\dot H^s}^2\|u\|_{\dot H^s}\,.\end{aligned} \tag{2.13}$$

Based on the mass conservation equation in the system (2.6), we have

$$\begin{aligned}I_4^k =& \frac12\int_{\Omega}\Big\{\big(\tfrac{P'(\rho)}{\rho}\big)'\big(\partial_t\rho+\mathrm{div}(\rho u)\big) - \big[\big(\tfrac{P'(\rho)}{\rho}\big)'\rho\mathrm{div}u - \tfrac{P'(\rho)}{\rho}\mathrm{div}u\big]\Big\}|\partial_x^\alpha\rho|^2\mathrm{d}x\\ \lesssim& \Big[\|\tfrac{P'(\rho)}{\rho}\|_{L^\infty}\|\mathrm{div}u\|_{L^\infty} + \|\big(\tfrac{P'(\rho)}{\rho}\big)'\|_{L^\infty}\|\rho\|_{L^\infty}\|\mathrm{div}u\|_{L^\infty}\Big]\|\partial_x^\alpha\rho\|_{L^2}^2\\ \lesssim& \mathscr{Q}(u)(1+\|\rho\|_{L^\infty})\|u\|_{\dot H^s}\|\rho\|_{\dot H^s}^2\,.\end{aligned} \tag{2.14}$$

Similar estimate as I_1^k infers that

$$I_5^k \lesssim \Big(\|\nabla Q\|_{H^s}+\|Q\|_{H^s}\Big)\|c\|_{H^s}^2\|\nabla u\|_{H^s}\,. \tag{2.15}$$

The term I_6^k can be controlled as in the following calculation:

$$\begin{aligned}I_6^k =& K\langle\partial_x^\alpha Q\Delta Q - \Delta Q\partial_x^\alpha Q, \partial_x^\alpha\Omega\rangle + K\langle\partial_x^\alpha Q\Omega - \Omega\partial_x^\alpha Q, \partial_x^\alpha\Delta Q\rangle\\ &+ K\langle\mathcal{M}_1, \partial_x^\alpha\Omega\rangle + K\langle\mathcal{M}_2, \partial_x^\alpha\Delta Q\rangle\,,\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}\mathcal{M}_1 =& \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ |\alpha_1|,|\alpha_2|\ge1}} C_\alpha^{\alpha_1}(\partial_x^{\alpha_1}Q\partial_x^{\alpha_2}\Delta Q - \partial_x^{\alpha_1}\Delta Q\partial_x^{\alpha_2}Q)\,,\\ \mathcal{M}_2 =& \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ |\alpha_1|,|\alpha_2|\ge1}} C_\alpha^{\alpha_1}(\partial_x^{\alpha_1}Q\partial_x^{\alpha_2}\Omega - \partial_x^{\alpha_1}\Omega\partial_x^{\alpha_2}Q)\,.\end{aligned}$$

Here we have used the cancellation relation

$$\langle Q\partial_x^\alpha\Delta Q - \partial_x^\alpha\Delta Q Q, \partial_x^\alpha\Omega\rangle + \langle Q\partial_x^\alpha\Omega - \partial_x^\alpha\Omega Q, \partial_x^\alpha\Delta Q\rangle = 0\,. \tag{2.17}$$

The first two terms in the last equality of (2.16) can be bounded as follows:

$$K\langle\partial_x^\alpha Q\Delta Q-\Delta Q\partial_x^\alpha Q,\partial_x^\alpha\Omega\rangle\lesssim\|\partial_x^\alpha Q\|_{L^4}\|\Delta Q\|_{L^4}\|\partial_x^\alpha\nabla u\|_{L^2}\lesssim\|\nabla Q\|_{H^s}\|\Delta Q\|_{H^1}\|\nabla u\|_{H^s}\,,$$
$$K\langle\partial_x^\alpha Q\Omega-\Omega\partial_x^\alpha Q,\partial_x^\alpha\Delta Q\rangle\lesssim\|\partial_x^\alpha Q\|_{L^4}\|\nabla u\|_{L^4}\|\partial_x^\alpha\Delta Q\|_{L^2}\lesssim\|\nabla Q\|_{H^s}\|\nabla u\|_{H^1}\|\Delta Q\|_{H^s}\,.$$

Hölder inequality tells us

$$\|\mathcal{M}_1\|_{L^2}\lesssim\|\nabla Q\|_{H^s}\|\Delta Q\|_{\dot H^{s-1}}\,,\quad \|\mathcal{M}_2\|_{L^2}\lesssim\|\nabla Q\|_{H^s}\|\nabla u\|_{\dot H^{s-1}}\,.$$

Therefore,

$$I_6^k\lesssim\|\nabla Q\|_{H^s}(\|\Delta Q\|_{H^{s-1}}\|\nabla u\|_{H^s}+\|\nabla u\|_{H^{s-1}}\|\Delta Q\|_{H^s})\,. \tag{2.18}$$

For the estimate of the other terms in I^k, using Hölder inequality and Sobolev embedding theorem directly, one has

$$\begin{aligned}I_7^k&\lesssim\|\nabla Q\|_{H^s}^2\|\nabla u\|_{H^s}+\|\nabla Q\|_{H^s}\|\Delta Q\|_{H^s}\|u\|_{H^s}\,,\\ I_8^k&\lesssim\big(\|c\|_{H^s}\|Q\|_{\dot H^s}+\|c\|_{\dot H^s}\|Q\|_{H^s}\big)(\|Q\|_{\dot H^s}+\|\Delta Q\|_{H^s})+\|Q\|_{\dot H^s}^2+\|\nabla Q\|_{H^s}^2\,.\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}I_{10}^k&\lesssim\|Q\|_{H^s}\|Q\|_{\dot H^s}(\|\Delta Q\|_{H^s}+\|Q\|_{\dot H^s})\,,\\ I_{11}^k&\lesssim\|Q\|_{H^s}^2\|Q\|_{\dot H^s}(\|Q\|_{\dot H^s}+\|\Delta Q\|_{H^s})\,.\end{aligned} \tag{2.20}$$

At last, we estimate I_9^k. Based on the representation of $F(Q)$, we only need to estimate $\langle\partial_x^\alpha\nabla\cdot(|\nabla Q|^2),\partial_x^\alpha u\rangle$, and the others can be controlled by using a similar way. Noticing that

$$\begin{aligned}&\langle\partial_x^\alpha\nabla\cdot(|\nabla Q|^2\mathbb{I}_3),\partial_x^\alpha u\rangle=-\langle\partial_x^\alpha(|\nabla Q|^2\mathbb{I}_3),\partial_x^\alpha\nabla u^\top\rangle\\ \lesssim&\|\nabla Q\|_{L^\infty}\|\partial_x^\alpha Q\|_{L^2}\|\partial_x^\alpha\nabla u\|_{L^2}+\sum_{\substack{\alpha_1+\alpha_2=\alpha,\\1\le|\alpha_1|\le k-1}}\|\partial_x^{\alpha_1}\nabla Q\|_{L^4}\|\partial_x^{\alpha_2}\nabla Q\|_{L^4}\|\partial_x^\alpha u\|_{L^2}\\ \lesssim&\|\nabla Q\|_{H^s}^2\|\nabla u\|_{H^s}\,,\end{aligned}$$

and

$$\begin{aligned}\langle\partial_x^\alpha\nabla\cdot(\mathrm{tr}(Q^2)\mathbb{I}_3),\partial_x^\alpha u\rangle&\lesssim\|Q\|_{H^s}\|Q\|_{\dot H^s}\|\nabla u\|_{H^s}\,,\\ \langle\partial_x^\alpha\nabla\cdot(\mathrm{tr}^2(Q^2)\mathbb{I}_3),\partial_x^\alpha u\rangle&\lesssim\|Q\|_{H^s}^3\|Q\|_{\dot H^s}\|\nabla u\|_{H^s}\,,\end{aligned}$$

one has

$$I_9^k\lesssim\|\nabla Q\|_{H^s}^2\|\nabla u\|_{H^s}+(\|Q\|_{H^s}+\|Q\|_{H^s}^3)\|Q\|_{\dot H^s}\|\nabla u\|_{H^s}\,. \tag{2.21}$$

Combining with the above estimates, we have

$$\begin{aligned}I^k\lesssim&\|u\|_{\dot H^s}(\|\nabla c\|_{H^s}\|c\|_{\dot H^s}+\|\nabla Q\|_{H^s}\|Q\|_{\dot H^s})+(\|\nabla u\|_{H^s}+\|\mathrm{div}u\|_{H^s})(\|c\|_{\dot H^s}^2+\|Q\|_{\dot H^s}^2)\\ &+\|Q\|_{H^s}\|Q\|_{\dot H^s}\|\nabla u\|_{H^s}+\mathscr{Q}(u)\|\rho\|_{\dot H^s}^2\|u\|_{\dot H^s}+\mathscr{Q}(u)(1+\|\rho\|_{L^\infty})\|u\|_{\dot H^s}\|\rho\|_{\dot H^s}^2\\ &+\big(\|\nabla Q\|_{H^s}+\|Q\|_{H^s}\big)\|c\|_{H^s}^2\|\nabla u\|_{H^s}+(\|Q\|_{\dot H^s}\|c\|_{H^s}+\|Q\|_{H^s}\|c\|_{\dot H^s})\|\Delta Q\|_{H^s}+\|\nabla Q\|_{H^s}^2\\ &+\|\nabla Q\|_{H^s}(\|\Delta Q\|_{H^{s-1}}\|\nabla u\|_{H^s}+\|\nabla u\|_{H^{s-1}}\|\Delta Q\|_{H^s})\\ &+(\|c\|_{H^s}\|Q\|_{\dot H^s}+\|c\|_{\dot H^s}\|Q\|_{H^s}\big)(\|Q\|_{\dot H^s}+\|\Delta Q\|_{H^s})+\|Q\|_{\dot H^s}^2+\|\nabla Q\|_{H^s}^2\\ &+\|\nabla Q\|_{H^s}^2\|\nabla u\|_{H^s}+(\|Q\|_{H^s}+\|Q\|_{H^s}^3)\|Q\|_{\dot H^s}\|\nabla u\|_{H^s}\\ &+\|Q\|_{H^s}\|Q\|_{\dot H^s}(\|\Delta Q\|_{H^s}+\|Q\|_{\dot H^s})+\|Q\|_{H^s}^2\|Q\|_{\dot H^s}(\|Q\|_{\dot H^s}+\|\Delta Q\|_{H^s})\,.\end{aligned} \tag{2.22}$$

We then estimate the terms in J^k. Based on Lemma 2.3, we know that $\|\frac{P'(\rho)}{\rho}\|_{L^\infty}$ can be bounded by $\mathscr{Q}(u)$. Therefore

$$\begin{aligned}
&\langle[\partial_x^\alpha, \rho\mathrm{div}]u, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle\\
=&\langle\partial_x^\alpha\rho\mathrm{div}u, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\langle\partial_x^\alpha\rho\partial_x^\alpha\mathrm{div}u, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle\\
\lesssim&\|\mathrm{div}u\|_{L^\infty}\|\tfrac{P'(\rho)}{\rho}\|_{L^\infty}\|\partial_x^\alpha\rho\|_{L^2}^2 + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}}\|\partial_x^{\alpha_1}\mathrm{div}u\|_{L^4}\|\partial_x^{\alpha_2}\rho\|_{L^4}\|\tfrac{P'(\rho)}{\rho}\|_{L^\infty}\|\partial_x^\alpha\rho\|_{L^2}\\
\lesssim&\mathscr{Q}(u)\|\mathrm{div}u\|_{H^s}\|\rho\|_{\dot H^s}^2\,,
\end{aligned}$$

and

$$\begin{aligned}
&\langle[\partial_x^\alpha, u\cdot\nabla]\rho, \tfrac{P'(\rho)}{\rho}\partial_x^\alpha\rho\rangle\\
\lesssim&\Big(\|\partial_x^\alpha u\|_{L^4}\|\nabla\rho\|_{L^4} + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}}\|\partial_x^{\alpha_1}u\|_{L^\infty}\|\partial_x^{\alpha_2}\rho\|_{L^2}\Big)\|\tfrac{P'(\rho)}{\rho}\|_{L^\infty}\|\partial_x^\alpha\rho\|_{L^2}\\
\lesssim&\mathscr{Q}(u)\|\nabla u\|_{H^s}\|\rho\|_{\dot H^s}^2\,.
\end{aligned}$$

Hence

$$J_1^k \lesssim \mathscr{Q}(u)(\|\mathrm{div}u\|_{H^s} + \|\nabla u\|_{H^s})\|\rho\|_{\dot H^s}^2\,. \tag{2.23}$$

Similarly, J_2^k has the following estimate:

$$J_2^k \lesssim \|\rho\|_{L^\infty}\|u\|_{H^s}\|\nabla u\|_{H^s}\|u\|_{\dot H^s}\,. \tag{2.24}$$

We then turn to estimate J_3^k and J_4^k, straightforward calculation gives us the estimate of J_3^k as

$$\begin{aligned}
J_3^k =& -\langle\partial_x^\alpha(\tfrac{P'(\rho)}{\rho})\nabla\rho + \nabla(\tfrac{P'(\rho)}{\rho})\partial_x^\alpha\rho, \rho\partial_x^\alpha u\rangle + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 2\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\langle\partial_x^{\alpha_1}(\tfrac{P'(\rho)}{\rho})\partial_x^{\alpha_2}\nabla\rho, \rho\partial_x^\alpha u\rangle\\
\lesssim&(\|\partial_x^\alpha(\tfrac{P'(\rho)}{\rho})\|_{L^2}\|\nabla\rho\|_{L^\infty}\|\rho\|_{L^\infty} + \|\nabla(\tfrac{P'(\rho)}{\rho})\|_{L^\infty}\|\partial_x^\alpha\rho\|_{L^2}\|\rho\|_{L^\infty})\|\partial_x^\alpha u\|_{L^2}\\
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 2\le|\alpha_1|\le k-1}}\|\partial_x^{\alpha_1}(\tfrac{P'(\rho)}{\rho})\|_{L^4}\|\partial_x^{\alpha_2}\nabla\rho\|_{L^4}\|\rho\|_{L^\infty}\|\partial_x^\alpha u\|_{L^2}\\
\lesssim&\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\|\rho\|_{\dot H^s}\|u\|_{\dot H^s}\,.
\end{aligned} \tag{2.25}$$

Similarly,

$$J_4^k \lesssim \mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\|u\|_{\dot H^s}(\|\mathrm{div}u\|_{H^s} + \|\nabla u\|_{H^s})\,. \tag{2.26}$$

For the estimate of J_5^k, based on

$$(Q\Delta Q) =\partial_x^\alpha(\tfrac{1}{\rho})\mathrm{div}(Q\Delta Q) + \sum_{|\beta|=1}\partial_x^\beta(\tfrac{1}{\rho})\partial_x^{\alpha-\beta}\mathrm{div}(Q\Delta Q) + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 2\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}(\tfrac{1}{\rho})\partial_x^{\alpha_2}\mathrm{div}(Q\Delta Q)\,,$$

and on the other hand, it is obvious to derive that $\sum_{|\beta|=1}\|\partial_x^{\alpha-\beta}\mathrm{div}(Q\Delta Q)\|_{L^2}$ can be bounded by $\|Q\|_{H^s}\|\Delta Q\|_{H^s}$, hence by using Hölder inequality we have

$$J_5^k \lesssim \mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot{H}^s})\|Q\|_{H^s}\|\Delta Q\|_{H^s}\|u\|_{\dot{H}^s}\,. \tag{2.27}$$

The terms J_6^k can be estimated as follows. Noticing that

$$(c^2Q)] = \partial_x^\alpha(\tfrac{1}{\rho})\mathrm{div}(c^2Q) + \sum_{|\beta|=1}\partial_x^\beta(\tfrac{1}{\rho})\partial_x^{\alpha-\beta}(c^2Q) + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 2\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}(\tfrac{1}{\rho})\partial_x^{\alpha_2}(c^2Q)\,,$$

and using the estimates

$$\begin{aligned}&\|\mathrm{div}(c^2Q)\|_{L^\infty} \lesssim \|\nabla c\|_{L^\infty}\|c\|_{L^\infty}\|Q\|_{L^\infty} + \|c\|_{L^\infty}^2\|\mathrm{div}Q\|_{L^\infty} \lesssim \|c\|_{H^s}^2\|Q\|_{H^s}\,,\\ &\sum_{|\beta|=1}\|\partial_x^{\alpha-\beta}\mathrm{div}(c^2Q)\|_{L^2} \lesssim \|c\|_{H^s}^2\|Q\|_{H^s}\,,\end{aligned}$$

and

$$\|\partial_x^{\alpha_2}\mathrm{div}(c^2Q)\|_{L^4} \lesssim \|c\|_{H^s}^2\|Q\|_{H^s},$$

for any $1 \le |\alpha_2| \le k-2$, one has

$$J_6^k \lesssim \mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot{H}^s})\|c\|_{H^s}^2\|Q\|_{H^s}\|u\|_{\dot{H}^s}\,. \tag{2.28}$$

At last, we estimate J_7^k. By using Hölder inequality, we should get the L^2-norm of $[\partial_x^\alpha, \frac{1}{\rho}\mathrm{div}](\nabla Q \odot \nabla Q)$. Since

$$\begin{aligned}(\nabla Q \odot \nabla Q) =&\partial_x^\alpha(\tfrac{1}{\rho})\mathrm{div}(\nabla Q\odot\nabla Q) + \sum_{|\beta|=1}\partial_x^\beta(\tfrac{1}{\rho})\partial_x^{\alpha-\beta}\mathrm{div}(\nabla Q\odot\nabla Q)\\ &+ \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 2\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}(\tfrac{1}{\rho})\partial_x^{\alpha_2}\mathrm{div}(\nabla Q\odot\nabla Q)\,.\end{aligned}$$

Similar estimates as J_6^k yield that

$$\|[\partial_x^\alpha, \tfrac{1}{\rho}\mathrm{div}](\nabla Q\odot\nabla Q)\|_{L^2} \lesssim \mathscr{P}_s(\|\rho\|_{\dot{H}^s})(\|\nabla Q\|_{H^s} + \|\Delta Q\|_{H^s})\|\nabla Q\|_{H^s}\,.$$

Similarly,

$$\|[\partial_x^\alpha, \tfrac{1}{\rho}\mathrm{div}](F(Q)\mathbb{I}_3)\|_{L^2} \lesssim \mathscr{P}_s(\|\rho\|_{\dot{H}^s})(\|\nabla Q\|_{H^s}^2 + \|Q\|_{H^s}\|Q\|_{\dot{H}^s} + \|Q\|_{H^s}^3\|Q\|_{\dot{H}^s})\,.$$

Therefore, it holds that

$$\begin{aligned}J_7^k \lesssim&\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot{H}^s})\big[\|\nabla Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}\|\Delta Q\|_{H^s}\\ &+ \|Q\|_{H^s}\|Q\|_{\dot{H}^s} + \|Q\|_{H^s}^3\|Q\|_{\dot{H}^s}\big]\|u\|_{\dot{H}^s}\,.\end{aligned} \tag{2.29}$$

Combining with the above estimates of $J_i^k(1\le i\le 7)$, we get

$$\begin{aligned}J^k \lesssim\ & \mathscr{Q}(u)(\|\mathrm{div}u\|_{H^s}+\|\nabla u\|_{H^s})\|\rho\|^2_{\dot H^s}+\|\rho\|_{L^\infty}\|u\|_{H^s}\|\nabla u\|_{H^s}\|u\|_{\dot H^s}\\ &+\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\|\rho\|_{\dot H^s}\|u\|_{\dot H^s}+\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\|u\|_{\dot H^s}(\|\mathrm{div}u\|_{H^s}+\|\nabla u\|_{H^s})\\ &+\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\|Q\|_{H^s}\|\Delta Q\|_{H^s}\|u\|_{\dot H^s}+\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\|c\|^2_{H^s}\|Q\|_{H^s}\|u\|_{\dot H^s}\\ &+\mathscr{Q}(u)\mathscr{P}_s(\|\rho\|_{\dot H^s})\big[\|\nabla Q\|^2_{H^s}+\|\nabla Q\|_{H^s}\|\Delta Q\|_{H^s}+\|Q\|_{H^s}\|Q\|_{\dot H^s}+\|Q\|^3_{H^s}\|Q\|_{\dot H^s}\big]\|u\|_{\dot H^s}\,.\end{aligned} \tag{2.30}$$

By the definition of $\mathcal{E}(t)$ and $\mathcal{D}(t)$, then using the interpolation inequality

$$\|Q\|_{L^4(\mathbf{\Omega})}\lesssim\|Q\|^{\frac14}_{L^2(\mathbf{\Omega})}\|\nabla Q\|^{\frac34}_{L^2(\mathbf{\Omega})}\,,$$

we know that $\|Q\|_{L^4}$ can be controlled by $\mathcal{E}^{\frac12}(t)$. As a result, the terms on the right-hand side of (2.1) can be bounded by

$$\sum_{i=2}^{5}\mathcal{E}^{\frac{i}{2}}(t)+\mathscr{Q}(u)\mathcal{E}^{\frac32}(t)+\mathcal{D}^{\frac12}(t)\mathcal{E}(t)\,. \tag{2.31}$$

On the other hand, utilizing Lemma we can derive that

$$\|\rho\|^2_{\dot H^s}\le\|\tfrac{\rho}{P'(\rho)}\|_{L^\infty}\sum_{|\alpha|=1}^{s}\int_\Omega\tfrac{P'(\rho)}{\rho}|\partial_x^\alpha\rho|^2\mathrm{d}x\lesssim\mathscr{Q}(u)\mathcal{E}(t)\,,$$

$$\|u\|^2_{H^s}\lesssim\mathscr{Q}(u)\mathcal{E}(t)\,,\quad \mathscr{P}_s(\|\rho\|_{\dot H^s})\lesssim\mathscr{Q}(u)\sum_{i=1}^{s}\mathcal{E}^{\frac{i}{2}}(t)\,.$$

Then I^k in (2.22), and J^k in (2.30) can be bounded as

$$I^k\lesssim\sum_{i=2}^{4}\mathcal{E}^{\frac{i}{2}}(t)+\mathscr{Q}(u)\big[\mathcal{E}(t)+\mathcal{E}^{\frac32}(t)\big]+\mathcal{D}^{\frac12}(t)\sum_{i=2}^{4}\mathcal{E}^{\frac{i}{2}}(t)\,, \tag{2.32}$$

and

$$J^k\lesssim\mathscr{Q}(u)\mathcal{D}^{\frac12}(t)\sum_{i=1}^{s+2}\mathcal{E}^{\frac{i}{2}}(t)\,. \tag{2.33}$$

Consequently, substituting the estimates (2.32) and (2.33) into the identity (2.8), summing up for all $1\le k\le s$, and combining with (2.31) infer that there exists constant C, such that the inequality

$$\tfrac12\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\mathcal{D}(t)\le C\Big[\sum_{i=2}^{4}\mathcal{E}^{\frac{i}{2}}(t)+\mathscr{Q}(u)\big[\mathcal{E}(t)+\mathcal{E}^{\frac32}(t)\big]+\mathcal{D}^{\frac12}(t)\sum_{i=2}^{4}\mathcal{E}^{\frac{i}{2}}(t)+\mathscr{Q}(u)\mathcal{D}^{\frac12}(t)\sum_{i=1}^{s+2}\mathcal{E}^{\frac{i}{2}}(t)\Big]$$

holds. This completes proving of Lemma 2.4.

3 Approximation System

The aim of this section is to construct the approximation system for the active liquid crystal model (1.2). Here we construct the approximation system by iteration. As the method used in the a priori estimate, in which we want the L^∞-norm of ρ be bounded by the L^∞-norm of div$u$, here ρ^{n+1} should be bounded by the L^∞-norm of divu^n in the iterating approximation system. On the other hand, in order to cancel out the terms with highest order derivative, the corresponding cancellation relation similar as the L^2-estimate in the a priori estimates for the approximation system should also be used. Thus we construct the approximate system as follows:

$$\begin{cases} \partial_t c^{n+1} + u^n \cdot \nabla c^{n+1} = D_0 \Delta c^{n+1}\,, \\ \partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} + \rho^{n+1} \mathrm{div} u^n = 0\,, \\ \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + \frac{P'(\rho^n)}{\rho^n} \nabla \rho^{n+1} = \frac{1}{\rho^n} \mu \Delta u^{n+1} + \frac{1}{\rho^n}(\mu+\nu)\nabla \mathrm{div} u^{n+1} + \frac{1}{\rho^n} \nabla \cdot (F(Q^n)\mathbb{I}_3 \\ \qquad -K\nabla Q^n \odot \nabla Q^n) + \frac{K}{\rho^n} \nabla \cdot (Q^n \Delta Q^{n+1} - \Delta Q^{n+1} Q^n) + \frac{\sigma_*}{\rho^n} \nabla \cdot [(c^n)^2 Q^n]\,, \\ \partial_t Q^{n+1} + u^n \cdot \nabla Q^{n+1} + (Q^n \Omega^{n+1} - \Omega^{n+1} Q^n) = \Gamma[K \Delta Q^{n+1} + \frac{\kappa}{2}(c_* - c^n) Q^n \\ \qquad + b((Q^n)^2 - \frac{\mathrm{tr}(Q^n)^2}{3}\mathbb{I}_3) - c_* Q^n \mathrm{tr}(Q^n)^2]\,, \end{cases} \tag{3.1}$$

with the initial data

$$(c^{n+1}, \rho^{n+1}, u^{n+1}, Q^{n+1})(t,x)|_{t=0} = (c^{in}, \rho^{in}, u^{in}, Q^{in})(x)\,, \tag{3.2}$$

where

$$F(Q^n) = \tfrac{K}{2}|\nabla Q^n|^2 + \tfrac{1}{2}\mathrm{tr}(Q^n)^2 + \tfrac{c_*}{2}\mathrm{tr}^2(Q^n)^2$$

and

$$\Omega^{n+1} = \tfrac{\nabla u^{n+1} - \nabla u^{n+1\top}}{2}\,.$$

Here we start the approximation system with

$$(c^0, \rho^0, u^0, Q^0)(t,x) = (c^{in}, \rho^{in}, u^{in}, Q^{in})(x)\,. \tag{3.3}$$

As to be a linear Stokes type system, the existence of classical solutions for iteration system can be obtained directly. Here we state it in the following lemma:

Lemma 3.1 Suppose that the integer $s \geq 3$ and the initial data $(c^{in}, \rho^{in}, u^{in}, Q^{in}) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3$ satisfying

$$\rho^{in} \in \dot{H}^s_{\frac{P'(\rho^{in})}{\rho^{in}}} \cap L^\gamma\,, \quad u^{in} \in H^s_{\rho^{in}}\,, \quad c^{in} \in H^s\,, \quad Q^{in} \in H^{s+1}\,.$$

Then there exists a maximal number $T^*_{n+1} > 0$ such that the approximating system (3.1) admits a unique solution $(c^{n+1}, \rho^{n+1}, u^{n+1}, Q^{n+1})$ satisfying

$$\begin{aligned} c^{n+1} &\in L^\infty(0, T^*_{n+1}; H^s) \cap L^2(0, T^*_{n+1}; H^{s+1})\,, \\ \rho^{n+1} &\in L^\infty(0, T^*_{n+1}; L^\gamma) \cap L^2(0, T^*_{n+1}; \dot{H}^s_{\frac{P'(\rho^n)}{\rho^n}})\,, \\ u^{n+1} &\in L^\infty(0, T^*_{n+1}; H^s_{\rho^n}) \cap L^2(0, T^*_{n+1}; \dot{H}^{s+1})\,, \\ Q^{n+1} &\in L^\infty(0, T^*_{n+1}; H^{s+1}) \cap L^2(0, T^*_{n+1}; H^{s+1})\,. \end{aligned}$$

Proof For the iterating system (3.1), the unknown vectors are $(c^{n+1},\rho^{n+1},u^{n+1},Q^{n+1})$ in the $(n+1)$-th step. By the construction of the approximation system (3.1), the first equation is a linear equation about c^{n+1}, which admits a unique solution $c^{n+1}\in L^\infty(0,T^1_{n+1};H^s)\cap L^2(0,T^1_{n+1};H^{s+1})$ on the maximal time interval $[0,T^1_{n+1})$. Similarly, the second linear equation about ρ^{n+1} has a unique solution ρ^{n+1} such that $\rho^{n+1}\in L^\infty(0,T^2_{n+1};L^\gamma)\cap L^2(0,T^2_{n+1};\dot H^s_{\frac{P'(\rho^n)}{\rho^n}})$, here we denote the time existence as T^2_{n+1}. Then substitute the solution ρ^{n+1} we have obtained into the velocity equation of the iterating system (3.1), and notice that the third and fourth equations of the approximating system are linear Stokes type equations, respectively. We then derive that there exists a time $T^3_{n+1}\le T^2_{n+1}$, such that the last two equations of the approximating system admit a unique solution (u^{n+1},Q^{n+1}) satisfying

$$\begin{aligned}u^{n+1}&\in L^\infty(0,T^3_{n+1};H^s_{\rho^n})\cap L^2(0,T^3_{n+1};\dot H^{s+1})\,,\\ Q^{n+1}&\in L^\infty(0,T^3_{n+1};H^{s+1})\cap L^2(0,T^3_{n+1};H^{s+1})\,.\end{aligned}$$

We denote $T^*_{n+1}=\min\{T^1_{n+1},T^3_{n+1}\}>0$, then Lemma is finished.

To achieve the existence of local-in-time solution for system (1.2)-(1.3), the key point is to prove that existence time sequence obtained in Lemma admits a positive uniform lower bound. To do it, we should derive a uniform-in-n energy bound for the approximation system (3.1). For the convenience of representation, we denote the iteration energy functionals $\mathcal{E}_{n+1}(t)$ and $\mathcal{D}_{n+1}(t)$ as

$$\begin{aligned}\mathcal{E}_{n+1}(t):=&\|c^{n+1}\|^2_{H^s}+\tfrac{2a}{\gamma-1}\|\rho^{n+1}\|^\gamma_{L^\gamma}+\|\rho^{n+1}\|^2_{\dot H^s_{\frac{P'(\rho^n)}{\rho^n}}}\\ &+\|u^{n+1}\|^2_{H^s_{\rho^n}}+\|Q^{n+1}\|^2_{H^s}+K\|\nabla Q^{n+1}\|_{H^s}\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\mathcal{D}_{n+1}(t):=&D_0\|\nabla c^{n+1}|^2_{H^s}+\mu\|\nabla u^{n+1}\|^2_{H^s}+(\mu+\nu)\|\mathrm{div}u^{n+1}\|^2_{H^s}\\ &+K^2\Gamma\|\Delta Q^{n+1}\|^2_{H^s}+K\Gamma\|\nabla Q^{n+1}\|^2_{H^s}\,.\end{aligned}\tag{3.5}$$

Similar calculation as the a priori estimate enables us to get the following iteration energy bound:

$$\begin{aligned}&\tfrac12\tfrac{\mathrm d}{\mathrm dt}\mathcal{E}_{n+1}(t)+\mathcal{D}_{n+1}(t)\lesssim\big(\mathscr{Q}(u^{n-2})+1\big)\mathcal{E}_n^{\frac12}(t)\mathcal{E}_{n+1}^{\frac12}(t)\mathcal{D}_{n+1}^{\frac12}(t)\\ &+\mathcal{D}_n^{\frac12}(t)\mathcal{E}_{n+1}(t)+\sum_{i=1}^4\mathcal{E}_n^{\frac i2}(t)\Big(\mathcal{E}_{n+1}^{\frac12}(t)+\mathcal{D}_{n+1}^{\frac12}(t)\Big)+\mathcal{D}_n^{\frac12}(t)\mathcal{E}_n^{\frac12}(t)\mathcal{E}_{n+1}^{\frac12}(t)\\ &+\Big(\mathscr{Q}(u^{n-2})\mathcal{E}_n^{\frac12}(t)+1\Big)\Big(\mathscr{Q}(u^{n-2})\mathcal{E}_n^{\frac12}(t)+\mathscr{Q}(u^{n-3})\mathcal{E}_{n-1}^{\frac12}(t)\Big)\mathscr{Q}(u^{n-1})\mathcal{E}_{n+1}(t)\\ &+\Big[\mathscr{Q}(u^{n-2})\mathcal{E}_n^{\frac12}(t)\mathcal{E}_{n+1}^{\frac12}(t)+\mathcal{D}_{n+1}^{\frac12}(t)+\big(\mathscr{Q}(u^n)+\mathcal{E}_{n+1}^{\frac12}(t)\big)\mathcal{D}_n^{\frac12}(t)\Big]\mathscr{Q}(u^{n-1})\mathcal{E}_{n+1}^{\frac12}(t)\\ &+\Big[1+\Big(\mathcal{E}_n^{\frac12}(t)+\mathscr{Q}(u^{n-3})\mathcal{E}_{n-1}^{\frac12}(t)\Big)\mathscr{Q}(u^{n-2})\mathcal{E}_n^{\frac12}(t)+\Big(\mathcal{D}_n^{\frac12}(t)+\mathcal{D}_{n-1}^{\frac12}(t)\Big)\Big]\mathscr{Q}(u^{n-1})\mathcal{E}_{n+1}(t)\\ &+\mathscr{Q}(u^{n-1})\mathscr{Q}(u^{n-2})\Big[\sum_{i=2}^4\mathcal{E}_n^{\frac i2}(t)+\big(\mathcal{E}_n^{\frac12}(t)+1\big)\mathcal{D}_{n+1}^{\frac12}(t)\Big]\sum_{i=1}^s\mathcal{E}_n^{\frac i2}(t)\mathcal{E}_{n+1}^{\frac12}(t)\,.\end{aligned}\tag{3.6}$$

Then we prove Theorem 1.1 in the next section.

4 Local Well-Posedness With Large Initial Data

First, we can obtain the existence of the uniform time for the iteration system (3.1) in the following lemma.

Lemma 4.1 Assume that $(c^{n+1}, \rho^{n+1}, u^{n+1}, Q^{n+1})$ is the solution to the iterating equation (3.1), for any fixed positive constant M, we define

$$T_{n+1} = \sup\left\{\tau \in [0, T^*_{n+1}); \ \sup_{t\in[0,\tau]} \mathcal{E}_{n+1}(t) + \int_0^\tau \mathcal{D}_{n+1}(t)\mathrm{d}t \le M\right\},$$

where $T^*_{n+1} > 0$ is the existence time of the iterating approximating system (3.1). Then there exists a constant time $T > 0$ depending on the coefficients of the equation (1.2), M and $\mathcal{E}^{in}$, such that

$$T_{n+1} \ge T > 0\,.$$

Proof The proof is almost the same as in Lemma 5.2 of [27]. Here we omit the details for simplicity.

The proof of Theorem 1.1 For any fixed M, if $\mathcal{E}^{in} < M$, then based on Lemma 4.1 , there exists a $T > 0$ such that for all positive integer n and $t \in [0, T]$ the following inequality holds:

$$\begin{aligned} &\mathscr{N}_s(\rho^{n+1}) + \|c^{n+1}\|^2_{H^s} + \|u^{n+1}\|^2_{H^s_{\rho^n}} + \|Q^{n+1}\|^2_{H^s} + K\|\nabla Q^{n+1}\|^2_{H^s} + \int_0^T \big[D_0\|\nabla c^{n+1}\|^2_{H^s} \\ &+ \mu\|\nabla u^{n+1}\|^2_{H^s} + (\mu+\xi)\|\mathrm{div} u^{n+1}\|^2_{H^s} + \Gamma K\|\nabla Q^{n+1}\|^2_{H^s} + \Gamma K^2\|\Delta Q^{n+1}\|^2_{H^s}\big]\mathrm{d}\tau \le M\,. \end{aligned}$$

Then, by the standard compactness arguments, we obtain that the system (1.2)-(1.3) admits the unique solution $(c, \rho, u, Q) \in (\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ satisfying

$$\begin{aligned} &\mathscr{N}_s(\rho) + \|c\|^2_{H^s} + \|u\|^2_{H^s_\rho} + \|Q\|^2_{H^s} + K\|\nabla Q\|^2_{H^s} + \int_0^T \big[D_0\|\nabla c\|^2_{H^s} \\ &+ \mu\|\nabla u\|^2_{H^s} + (\mu+\xi)\|\mathrm{div} u\|^2_{H^s} + \Gamma K\|\nabla Q\|^2_{H^s} + \Gamma K^2\|\Delta Q\|^2_{H^s}\big]\mathrm{d}\tau \le M\,. \end{aligned}$$

Finally, we claim that $Q \in S_0^{(3)}$. In fact, $Q^\top$ is also the solution of the system (1.2)-(1.3), then the uniqueness of the solution tells us $Q^\top = Q$. On the other hand, taking the trace on the Q-equations of the system (1.2), and multiplying by $\mathrm{tr}Q$, then simple calculations and the traceless of the initial datum Q^{in} give us $\mathrm{tr}(Q) = 0$ by applying the Gronwall inequality. The proof can be found in the proof of Lemma 3.2 of [17].

Therefore, the proof of Theorem 1.1 is finished.

5 Global Classical Solution on $\mathbb{T}^3$ with Small Initial Data

The main goal of this section is to study the global-in-time existence of classical solutions (c, ρ, u, Q) near a constant state $(c_*, 1, 0, 0)$ on the torus $\mathbb{T}^3$ with small initial data for the system (1.2)-(1.3), namely, to study the rewritten system (1.11). Firstly, for some small positive constant number η, we define the energy functionals as follows:

$$\begin{aligned}\mathcal{E}_\eta(t) =& \int_{\mathbb{T}^3} \tfrac{P'(1+\varrho)}{1+\varrho}|\varrho|^2 \mathrm{d}x + \sum_{|\alpha|=1}^{s} \int_{\mathbb{T}^3} \big(\tfrac{P'(1+\varrho)}{1+\varrho} - \eta\big)|\partial_x^\alpha \varrho|^2 \mathrm{d}x + \sum_{|\alpha|=0}^{s-1} \int_{\mathbb{T}^3} \big((1+\varrho)-\eta\big)|\partial_x^\alpha u|^2 \mathrm{d}x \\ &+ \sum_{|\alpha|=s} \|\sqrt{1+\varrho}\partial_x^\alpha u\|_{L^2}^2 + \|\tilde{c}\|_{H^s}^2 + \|Q\|_{H^s}^2 + K\|\nabla Q\|_{H^s}^2 ,\\ \mathcal{D}_\eta(t) =& \tfrac{1}{2}\eta \sum_{|\alpha|=1}^{s} \int_{\mathbb{T}^3} \tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^\alpha \varrho|^2 \mathrm{d}x + \tfrac{\mu}{4}\|\nabla u\|_{H^s}^2 + \tfrac{\mu+\nu}{2}\|\mathrm{div} u\|_{H^s}^2 + \tfrac{\Gamma K}{4}\|\nabla Q\|_{H^s}^2 \\ &+ \Gamma K^2\|\Delta Q\|_{H^s}^2 + D_0\|\nabla \tilde{c}\|_{H^s}^2 .\end{aligned}$$

We then obtain global energy estimates for the system (1.11)–(1.12) in the following lemma.

Lemma 5.1 There exists a small constant $\eta_0 > 0$, depending only on the coefficients of the system (1.11) and $\|1+\varrho\|_{L^\infty}$, such that if $(c_* + \tilde{c}, 1+\varrho, u, Q)$ is the solution constructed in Theorem 1.1, then for all $0 < \eta < \eta_0$,

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\eta(t) + \mathcal{D}_\eta(t) \le C\mathcal{D}_\eta(t)\sum_{k=1}^{s+3} \mathcal{E}_\eta^{\frac{s+3}{2}}(t) ,$$

where the positive constant C depends only on the coefficients of the system (1.11) and s.

Proof For global energy bound, we need more dissipation comparing with the a priori estimates in Section 2. Firstly, for the L^2-estimate of system (1.11), we take the scalar product of the four equations of system (1.11) with $\tilde{c}$, $\frac{P'(1+\varrho)}{1+\varrho}\varrho$, $(1+\varrho)u$ and $-K\Delta Q + Q + c_* Q\mathrm{tr}(Q^2)$, respectively. Summing up with the resulting identities and integrating over $\mathbb{T}^3$ yields that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} |\tilde{c}|^2 + \tfrac{P'(1+\varrho)}{1+\varrho}|\varrho|^2 + (1+\varrho)|u|^2 + K|\nabla Q|^2 + |Q|^2 + \tfrac{c_*}{2}|Q|^4 \mathrm{d}x + D_0\|\nabla\tilde{c}\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2 \\ &+ (\mu+\nu)\|\mathrm{div} u\|_{L^2}^2 + \Gamma K^2\|\Delta Q\|_{L^2}^2 + \Gamma K\|\nabla Q\|_{L^2}^2 + \Gamma c_*\|Q\|_{L^4}^4 + \Gamma c_*^2\|Q\|_{L^6}^6 \\ =& -\langle u\cdot\nabla\tilde{c}, \tilde{c}\rangle - \langle \varrho \mathrm{div} u, P'(1+\varrho)\rangle - \langle P'(1+\varrho)\nabla\varrho, u\rangle + \tfrac{1}{2}\langle \partial_t\big(\tfrac{P'(1+\varrho)}{1+\varrho}\big) + \mathrm{div}\big(\tfrac{P'(1+\varrho)}{1+\varrho}u\big), |\varrho|^2\rangle \\ &- \langle F(Q)\mathbb{I}_3 - K\nabla Q\odot\nabla Q, \nabla u^\top\rangle - K\langle Q\Delta Q - \Delta Q Q, \nabla u^\top\rangle - \sigma_*\langle(\tilde{c}+c_*)^2 Q, \nabla u^\top\rangle \\ &+ \langle u\cdot\nabla u, K\Delta Q - Q - c_* Q\mathrm{tr}Q^2\rangle - \langle \Omega Q - Q\Omega, K\Delta Q\rangle + \langle \Omega Q - Q\Omega, Q + c_* Q\mathrm{tr}Q^2\rangle \\ &- \tfrac{\kappa\Gamma}{2}\langle \tilde{c}Q, -K\Delta Q + Q + c_* Q\mathrm{tr}Q^2\rangle + 2c_*\Gamma K\langle Q\mathrm{tr}Q^2, \Delta Q\rangle + b\Gamma\langle Q^2, -K\Delta Q + Q + c_* Q\mathrm{tr}Q^2\rangle \\ =& \sum_{i=1}^{13}\mathcal{I}_i .\end{aligned} \tag{5.1}$$

The same calculations as the basic energy estimates in the a priori estimates, we have

$$\mathcal{I}_5 + \mathcal{I}_8 = \mathcal{I}_6 + \mathcal{I}_9 = \mathcal{I}_{10} = 0 ,$$

and

$$\mathcal{I}_{12} = -2c_*\Gamma K \int_{\mathbb{T}^3} |Q|^2|\nabla Q|^2 \mathrm{d}x - c_*\Gamma K \int_{\mathbb{T}^3} |\nabla(|Q|^2)|^2 \mathrm{d}x \le 0\,.$$

Through direct calculation, one can bound other terms of $\mathcal{I}$ as follows:

$$\begin{aligned}
&\mathcal{I}_1 + \mathcal{I}_4 \lesssim \|\tilde{c}\|_{H^1}\|\nabla\tilde{c}\|_{L^2}\|\nabla u\|_{L^2} + \|\mathrm{div} u\|_{L^2}\|\nabla\varrho\|_{L^2}\|\varrho\|_{H^1}\,,\\
&\mathcal{I}_2 + \mathcal{I}_3 \lesssim \|\nabla\varrho\|_{L^2}\big(\|\varrho\|_{H^1}\|u\|_{H^1}\big)^{\frac12}\big(\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2}\big)^{\frac12}\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{11} \lesssim &\ \big(\|\tilde{c}\|_{H^1}\|Q\|_{H^1}\big)^{\frac12}\big(\|\nabla\tilde{c}\|_{L^2}\|\nabla Q\|_{L^2}\big)^{\frac12}\|\Delta Q\|_{L^2}\\
&\qquad + \|\nabla\tilde{c}\|_{L^2}\|\nabla Q\|_{L^2}\big(\|Q\|_{H^1} + \|Q\|_{H^2}^3\big)\,,\\
\mathcal{I}_{13} \lesssim &\ \|Q\|_{H^1}\|\nabla Q\|_{L^2}\big(\|\Delta Q\|_{L^2} + \|\nabla Q\|_{L^2} + \|\nabla Q\|_{L^2}\|Q\|_{H^2}^2\big)\,.
\end{aligned}$$

We estimate the term $\mathcal{I}_7$. Direct calculation tells us

$$\begin{aligned}
\langle \tilde{c}^2 Q, \nabla u^\top\rangle &\lesssim \|\tilde{c}\|_{L^4}^2\|Q\|_{L^\infty}\|\nabla u\|_{L^2} \lesssim \|\tilde{c}\|_{H^1}\|\nabla\tilde{c}\|_{L^2}\|Q\|_{H^2}\|\nabla u\|_{L^2}\,,\\
\langle \tilde{c} Q, \nabla u^\top\rangle &\lesssim \|\tilde{c}\|_{L^4}\|Q\|_{L^4}\|\nabla u\|_{L^2} \lesssim \big(\|\tilde{c}\|_{H^1}\|Q\|_{H^1}\big)^{\frac12}\big(\|\nabla\tilde{c}\|_{L^2}\|\nabla Q\|_{L^2}\big)^{\frac12}\|\nabla u\|_{L^2}\,,
\end{aligned}$$

and

$$\sigma_* c_*^2\langle Q, \nabla u^\top\rangle \le |\sigma_*|c_*^2|\mathbb{T}^3|^{\frac13}\|Q\|_{L^6}\|\nabla u\|_{L^2} \le \tfrac{C}{2\mu}|\sigma_*|^2 c_*^4\|\nabla Q\|_{L^2}^2 + \tfrac{\mu}{2}\|\nabla u\|_{L^2}\,,$$

where we have used Sobolev interpolation inequality $\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}$ for $f \in H^1(\mathbb{R}^3)$, and Cauchy inequality. Consequently, we derive that

$$\begin{aligned}
\mathcal{I}_7 \le &C\big[\|\tilde{c}\|_{H^1}\|\nabla\tilde{c}\|_{L^2}\|Q\|_{H^2} + \big(\|\tilde{c}\|_{H^1}\|Q\|_{H^1}\big)^{\frac12}\big(\|\nabla\tilde{c}\|_{L^2}\|\nabla Q\|_{L^2}\big)^{\frac12}\big]\|\nabla u\|_{L^2}\\
&+ \tfrac{C}{2\mu}|\sigma_*|^2c_*^4\|\nabla Q\|_{L^2}^2 + \tfrac{\mu}{2}\|\nabla u\|_{L^2}\,.
\end{aligned} \tag{5.2}$$

Consequently, putting the above estimates into (5.1), we obtain

$$\begin{aligned}
&\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}|\tilde{c}|^2 + \tfrac{P'(1+\varrho)}{1+\varrho}|\varrho|^2 + (1+\varrho)|u|^2 + K|\nabla Q|^2 + |Q|^2 + \tfrac{c_*}{2}|Q|^4\mathrm{d}x + D_0\|\nabla\tilde{c}\|_{L^2}^2 + \tfrac{\mu}{2}\|\nabla u\|_{L^2}^2\\
&\ + (\mu+\nu)\|\mathrm{div} u\|_{L^2}^2 + \Gamma K^2\|\Delta Q\|_{L^2}^2 + \big(\Gamma K - \tfrac{C}{2\mu}|\sigma_*|^2c_*^4\big)\|\nabla Q\|_{L^2}^2 + \Gamma c_*\|Q\|_{L^4}^4 + \Gamma c_*^2\|Q\|_{L^6}^6\\
&\lesssim \big(\|\varrho\|_{H^1}\|u\|_{H^1}\big)^{\frac12}\|\nabla\varrho\|_{L^2}^{\frac32}\|\nabla u\|_{L^2}^{\frac12} + \|\nabla u\|_{L^2}\|\nabla\varrho\|_{L^2}\|\varrho\|_{H^1} + \|\tilde{c}\|_{H^1}\|\nabla\tilde{c}\|_{L^2}\|Q\|_{H^2}\|\nabla u\|_{L^2}\\
&\ + \big(\|\tilde{c}\|_{H^1}\|Q\|_{H^1}\big)^{\frac12}\big(\|\nabla\tilde{c}\|_{L^2}\|\nabla Q\|_{L^2}\big)^{\frac12}\big(\|\nabla u\|_{L^2} + \|\Delta Q\|_{L^2}\big) + \|\nabla\tilde{c}\|_{L^2}\|\nabla Q\|_{L^2}\\
&\ \times\big(\|Q\|_{H^1} + \|Q\|_{H^2}^3\big) + \|Q\|_{H^1}\|\nabla Q\|_{L^2}\big(\|\Delta Q\|_{L^2} + \|\nabla Q\|_{L^2} + \|\nabla Q\|_{L^2}\|Q\|_{H^2}^2\big)\,.
\end{aligned} \tag{5.3}$$

Secondly, we concern with the higher order estimate for the system (1.11). For any positive integer $1 \le k \le s$, acting the derivative operator ∂_x^α ($|\alpha| = k$) on the four equations of the system (1.11), then taking scalar product of the resulting identities with $\partial_x^\alpha\tilde{c}$, $\frac{P'(1+\varrho)}{1+\varrho}\partial_x^\alpha\varrho$,

$(1+\varrho)\partial_x^\alpha u$ and $-K\partial_x^\alpha\Delta Q+\partial_x^\alpha Q$, respectively, over the torus $\mathbb{T}^3$. Integrating by parts gives us

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}|\partial_x^\alpha\tilde{c}|^2+\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^\alpha\varrho|^2+(1+\varrho)|\partial_x^\alpha u|^2+K|\partial_x^\alpha\nabla Q|^2+|\partial_x^\alpha Q|^2\mathrm{d}x\\&+D_0\|\partial_x^\alpha\nabla\tilde{c}\|_{L^2}+\mu\|\partial_x^\alpha\nabla u\|_{L^2}^2+(\mu+\nu)\|\partial_x^\alpha\mathrm{div}u\|_{L^2}^2+\Gamma K^2\|\partial_x^\alpha\Delta Q\|_{L^2}^2+\Gamma K\|\partial_x^\alpha\nabla Q\|_{L^2}^2\\=&\mathcal{I}^k+\mathcal{J}^k,\end{aligned}\tag{5.4}$$

where

$$\begin{aligned}\mathcal{I}^k=&\underbrace{-\langle\partial_x^\alpha(u\cdot\nabla Q),\partial_x^\alpha Q\rangle-\langle\partial_x^\alpha(u\cdot\nabla\tilde{c}),\partial_x^\alpha\tilde{c}\rangle}_{\mathcal{I}_1^k}+\underbrace{\langle\partial_x^\alpha(\Omega Q-Q\Omega),\partial_x^\alpha Q\rangle}_{\mathcal{I}_2^k}\\&\underbrace{-\langle P'(1+\varrho)\partial_x^\alpha\nabla\varrho,\partial_x^\alpha u\rangle-\langle P'(1+\varrho)\partial_x^\alpha\mathrm{div}u,\partial_x^\alpha\varrho\rangle}_{\mathcal{I}_3^k}\\&+\underbrace{\tfrac{1}{2}\langle\partial_t\big(\tfrac{P'(1+\varrho)}{\varrho}\big),|\partial_x^\alpha\varrho|^2\rangle-\langle u\cdot\nabla\partial_x^\alpha\varrho,\tfrac{P'(1+\varrho)}{1+\varrho}\partial_x^\alpha\varrho\rangle}_{\mathcal{I}_4^k}+\underbrace{\sigma_*\langle\partial_x^\alpha\nabla\cdot((\tilde{c}+c_*)^2Q),\partial_x^\alpha u\rangle}_{\mathcal{I}_5^k}\\&+\underbrace{K\langle\partial_x^\alpha\nabla\cdot(Q\Delta Q-\Delta QQ),\partial_x^\alpha u\rangle+K\langle\partial_x^\alpha(Q\Omega-\Omega Q),\partial_x^\alpha\Delta Q\rangle}_{\mathcal{I}_6^k}\\&+\underbrace{K\langle\partial_x^\alpha(u\cdot\nabla Q),\partial_x^\alpha\Delta Q\rangle-K\langle\partial_x^\alpha\nabla\cdot(\nabla Q\odot\nabla Q),\partial_x^\alpha u\rangle}_{\mathcal{I}_7^k}\\&+\underbrace{\tfrac{\Gamma\kappa}{2}\langle\partial_x^\alpha[\tilde{c}Q],K\partial_x^\alpha\Delta Q-\partial_x^\alpha Q\rangle}_{\mathcal{I}_8^k}+\underbrace{\langle\partial_x^\alpha\nabla\cdot(F(Q)\mathbb{I}_3),\partial_x^\alpha u\rangle}_{\mathcal{I}_9^k}\\&+\underbrace{b\Gamma\langle\partial_x^\alpha(Q^2-\tfrac{\mathrm{tr}Q^2}{3}\mathbb{I}_3),-K\partial_x^\alpha\Delta Q+\partial_x^\alpha Q\rangle}_{\mathcal{I}_{10}^k}+\underbrace{c_*\Gamma\langle\partial_x^\alpha[Q\mathrm{tr}Q^2],K\partial_x^\alpha\Delta Q-\partial_x^\alpha Q\rangle}_{\mathcal{I}_{11}^k}\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}^k=&\underbrace{-\langle[\partial_x^\alpha,(1+\varrho)\mathrm{div}]u,\tfrac{P'(1+\varrho)}{1+\varrho}\partial_x^\alpha\varrho\rangle-\langle[\partial_x^\alpha,u\cdot\nabla]\varrho,\tfrac{P'(1+\varrho)}{1+\varrho}\partial_x^\alpha\varrho\rangle}_{\mathcal{J}_1^k}\\&+\underbrace{\langle[\partial_x^\alpha,u\cdot\nabla]u,(1+\varrho)\partial_x^\alpha u\rangle}_{\mathcal{J}_2^k}+\underbrace{\langle[\partial_x^\alpha,\tfrac{P'(1+\varrho)}{1+\varrho}\nabla]\varrho,(1+\varrho)\partial_x^\alpha u\rangle}_{\mathcal{J}_3^k}\\&+\underbrace{\mu\langle[\partial_x^\alpha,\tfrac{1}{1+\varrho}\Delta]u,(1+\varrho)\partial_x^\alpha u\rangle+(\mu+\nu)\langle[\partial_x^\alpha,\tfrac{1}{1+\varrho}\nabla\mathrm{div}]u,(1+\varrho)\partial_x^\alpha u\rangle}_{\mathcal{J}_4^k}\\&+\underbrace{K\langle[\partial_x^\alpha,\tfrac{1}{1+\varrho}\mathrm{div}](Q\Delta Q-\Delta QQ),(1+\varrho)\partial_x^\alpha u\rangle}_{\mathcal{J}_5^k}+\underbrace{\sigma_*\langle[\partial_x^\alpha,\tfrac{1}{1+\varrho}\mathrm{div}]((\tilde{c}+c_*)^2Q),(1+\varrho)\partial_x^\alpha u\rangle}_{\mathcal{J}_6^k}\\&+\underbrace{\langle[\partial_x^\alpha,\tfrac{1}{1+\varrho}\mathrm{div}](F(Q)\mathbb{I}_3-K\nabla Q\odot\nabla Q),(1+\varrho)\partial_x^\alpha u\rangle}_{\mathcal{J}_7^k}.\end{aligned}$$

We then estimate the terms $\mathcal{I}^k$ and $\mathcal{J}^k$. We first estimate $\mathcal{I}^k$. By the estimates of I_1^k and I_2^k, we have

$$\begin{aligned}
&\mathcal{I}_1^k \lesssim \|u\|_{\dot{H}^s}\left(\|\nabla\tilde{c}\|_{H^s}\|\tilde{c}\|_{\dot{H}^s}+\|\nabla Q\|_{H^s}\|Q\|_{\dot{H}^s}\right)+\|\nabla u\|_{H^s}\left(\|\tilde{c}\|_{\dot{H}^s}^2+\|Q\|_{\dot{H}^s}^2\right),\\
&\mathcal{I}_2^k \lesssim \|Q\|_{H^s}\|Q\|_{\dot{H}^s}\|\nabla u\|_{H^s}\,.
\end{aligned}\tag{5.5}$$

Noticing the fact that ρ has uniform positive lower and upper bounds as it near 1, i.e. $1+\varrho$ is L^∞ bounded. Then utilizing the estimates of I_3^k and I_4^k, we derive that $\mathcal{I}_3^k$ and $\mathcal{I}_4^k$ are bounded by $C\|\varrho\|_{\dot{H}^s}^2\|u\|_{\dot{H}^s}$, where C is a constant.

We then turn to control the term $\mathcal{I}_5^k$ by dividing it into the following three terms:

$$\begin{aligned}
\mathcal{I}_5^k =&\sigma_*\langle\partial_x^\alpha\nabla\cdot(\tilde{c}^2Q),\partial_x^\alpha u\rangle+2\sigma_*c_*\langle\partial_x^\alpha\nabla\cdot(\tilde{c}Q),\partial_x^\alpha u\rangle+\sigma_*c_*^2\langle\partial_x^\alpha\nabla\cdot Q,\partial_x^\alpha u\rangle\\
=&\mathcal{I}_{51}^k+\mathcal{I}_{52}^k+\mathcal{I}_{53}^k\,.
\end{aligned}$$

We control $\mathcal{I}_{51}^k$ by dividing it into the following three parts:

$$\begin{aligned}
\mathcal{I}_{51}^k =&-\sigma_*\langle\tilde{c}^2\partial_x^\alpha Q,\partial_x^\alpha\nabla u^\top\rangle-\sum_{\alpha_1+\alpha_2=\alpha}\sigma_*C_\alpha^{\alpha_1}\langle\partial_x^{\alpha_1}\tilde{c}\partial_x^{\alpha_2}\tilde{c}Q,\partial_x^\alpha\nabla u^\top\rangle\\
&-\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha,\\ 1\le|\alpha_3|\le k-1}}\sigma_*C_\alpha^{\alpha_1}C_{\alpha-\alpha_1}^{\alpha_2}\langle\partial_x^{\alpha_1}\tilde{c}\partial_x^{\alpha_2}\tilde{c}\partial_x^{\alpha_3}Q,\partial_x^\alpha\nabla u^\top\rangle\\
=&\mathcal{I}_{511}^k+\mathcal{I}_{512}^k+\mathcal{I}_{513}^k\,.
\end{aligned}$$

Straightforward calculation implies that

$$\mathcal{I}_{511}^k \lesssim \|\tilde{c}\|_{L^\infty}^2\|\partial_x^\alpha Q\|_{L^2}\|\partial_x^\alpha\nabla u\|_{L^2} \lesssim \|\tilde{c}\|_{H^s}^2\|Q\|_{\dot{H}^s}\|\nabla u\|_{H^s}\,.$$

As to the estimate of $\mathcal{I}_{512}^k$, it is obvious that it can be bounded by $\|\tilde{c}\|_{H^s}\|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s}\|\nabla u\|_{H^s}$ for the case $|\alpha_1|=0$ or $|\alpha_1|=k$. For the other cases, i.e. $1\le|\alpha_1|\le k-1$, by using

$$\|\partial_x^{\alpha_1}\tilde{c}\partial_x^{\alpha_2}\tilde{c}Q\|_{L^2}\le\|\partial_x^{\alpha_1}\tilde{c}\|_{L^4}\|\partial_x^{\alpha_2}\tilde{c}\|_{L^4}\|Q\|_{L^\infty}\lesssim\|\tilde{c}\|_{\dot{H}^s}^2\|Q\|_{H^s}\,,$$

we get

$$\mathcal{I}_{512}\lesssim\|\tilde{c}\|_{\dot{H}^s}^2\|Q\|_{H^s}\|\nabla u\|_{H^s}\,.$$

We last deal with the term $\mathcal{I}_{513}^k$ for obtaining the estimate of $\mathcal{I}_{51}^k$. When $|\alpha_3|=k-1$, we have

$$\|\partial_x^{\alpha_1}\tilde{c}\partial_x^{\alpha_2}\tilde{c}\partial_x^{\alpha_3}Q\|_{L^2}\lesssim\|\partial_x^{\alpha_1}\tilde{c}\|_{L^\infty}\|\partial_x^{\alpha_2}\tilde{c}\|_{L^4}\|\partial_x^{\alpha_3}Q\|_{L^4}\lesssim\|\tilde{c}\|_{H^s}^2\|Q\|_{\dot{H}^s}\,.$$

As $1\le|\alpha_3|<k-1$, the following estimate holds:

$$\|\partial_x^{\alpha_1}\tilde{c}\partial_x^{\alpha_2}\tilde{c}\partial_x^{\alpha_3}Q\|_{L^2}\lesssim\|\partial_x^{\alpha_1}\tilde{c}\|_{L^4}\|\partial_x^{\alpha_2}\tilde{c}\|_{L^4}\|\partial_x^{\alpha_3}Q\|_{L^\infty}\lesssim\|\tilde{c}\|_{H^s}^2\|Q\|_{\dot{H}^s}\,.$$

Hence one can derive that

$$\mathcal{I}_{513}^k\lesssim\|\tilde{c}\|_{H^s}^2\|Q\|_{\dot{H}^s}\|\nabla u\|_{H^s}\,.$$

Putting the previous estimates together, we have

$$\mathcal{I}_{51} \lesssim \big(\|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s} + \|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s}\big)\|\tilde{c}\|_{H^s}\|\nabla u\|_{H^s}\,. \tag{5.6}$$

By using similar methods as the above estimates, we can derive that

$$\mathcal{I}_{52}^k \lesssim \big(\|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s} + \|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s}\big)\|\nabla u\|_{H^s}\,. \tag{5.7}$$

We last estimate the term $\mathcal{I}_{53}^k$ before completing the estimate of $\mathcal{I}_5^k$. By using Hölder inequality, we have

$$\mathcal{I}_{53}^k \leq |\sigma_*|c_*^2\|\partial_x^\alpha Q\|_{L^2}\|\partial_x^\alpha \nabla u\|_{L^2} \leq \tfrac{\sigma_*^2 c_*^4}{2\mu}\|\partial_x^\alpha Q\|_{L^2}^2 + \tfrac{\mu}{2}\|\partial_x^\alpha \nabla u\|_{L^2}^2\,. \tag{5.8}$$

As a result, there exists a positive constant C such that

$$\begin{aligned}\mathcal{I}_5^k \leq & C\big(\|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s} + \|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s}\big)\big(1+\|\tilde{c}\|_{H^s}\big)\|\nabla u\|_{H^s}\\ & + \tfrac{\sigma_*^2 c_*^4}{2\mu}\|\partial_x^\alpha Q\|_{L^2}^2 + \tfrac{\mu}{2}\|\partial_x^\alpha \nabla u\|_{L^2}^2\,.\end{aligned} \tag{5.9}$$

In order to finish the estimate of $\mathcal{I}^k$, we only need to estimate $\mathcal{I}_8^k$, since the remainder terms in $\mathcal{I}^k$ can be estimated as controlling the corresponding terms in I^k. The following calculation

$$\begin{aligned}\|\partial_x^\alpha(\tilde{c}Q)\|_{L^2} \leq & \|\tilde{c}\|_{L^\infty}\|\partial_x^\alpha Q\|_{L^2} + \|\partial_x^\alpha \tilde{c}\|_{L^2}\|Q\|_{L^\infty} + \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\leq|\alpha_1|\leq k-1}} C_\alpha^{\alpha_1}\|\partial_x^{\alpha_1}\tilde{c}\|_{L^4}\|\partial_x^{\alpha_2}Q\|_{L^4}\\ \lesssim & \|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s} + \|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s}\end{aligned}$$

implies that

$$\mathcal{I}_8^k \lesssim \big(\|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s} + \|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s}\big)\big(\|Q\|_{\dot{H}^s} + \|\Delta Q\|_{H^s}\big)\,. \tag{5.10}$$

The remainder terms in $\mathcal{I}^k$ have the following estimates:

$$\begin{aligned}\mathcal{I}_6^k \lesssim & \|\nabla Q\|_{H^s}\big(\|\Delta Q\|_{H^{s-1}}\|\nabla u\|_{H^s} + \|\nabla u\|_{H^{s-1}}\|\Delta Q\|_{H^s}\big)\,,\\ \mathcal{I}_7^k \lesssim & \|\nabla Q\|_{H^s}\big(\|\nabla Q\|_{H^s}\|\nabla u\|_{H^s} + \|\Delta Q\|_{H^s}\|u\|_{H^s}\big)\,,\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\mathcal{I}_9^k \lesssim & \|\nabla Q\|_{H^s}^2\|\nabla u\|_{H^s} + \big(\|Q\|_{H^s} + \|Q\|_{H^s}^3\big)\|Q\|_{\dot{H}^s}\|\nabla u\|_{H^s}\,,\\ \mathcal{I}_{10}^k \lesssim & \|Q\|_{H^s}\|Q\|_{\dot{H}^s}\big(\|\Delta Q\|_{H^s} + \|Q\|_{\dot{H}^s}\big)\,,\\ \mathcal{I}_{11}^k \lesssim & \|Q\|_{H^s}^2\|Q\|_{\dot{H}^s}\big(\|\Delta Q\|_{H^s} + \|Q\|_{\dot{H}^s}\big)\,.\end{aligned} \tag{5.12}$$

Bases on the above estimates of $\mathcal{I}_i^k$ $(1 \leq i \leq 11)$, one can get the estimate of $\mathcal{I}^k$ as

$$\begin{aligned}\mathcal{I}^k \leq & \tfrac{\sigma_*^2 c_*^4}{2\mu}\|\partial_x^\alpha Q\|_{L^2}^2 + \tfrac{\mu}{2}\|\partial_x^\alpha \nabla u\|_{L^2}^2 + C\Big\{\|u\|_{\dot{H}^s}\big(\|\nabla\tilde{c}\|_{H^s}\|\tilde{c}\|_{\dot{H}^s} + \|\nabla Q\|_{H^s}\|Q\|_{\dot{H}^s}\big)\\ & + \|\nabla u\|_{H^s}\big(\|\tilde{c}\|_{\dot{H}^s}^2 + \|Q\|_{\dot{H}^s}^2\big) + \|Q\|_{H^s}\|Q\|_{\dot{H}^s}\|\nabla u\|_{H^s} + \|\varrho\|_{\dot{H}^s}^2\|u\|_{\dot{H}^s}\\ & + \big(\|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s} + \|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s}\big)\big(1+\|\tilde{c}\|_{H^s}\big)\|\nabla u\|_{H^s} + \|\nabla Q\|_{H^s}\\ & \times\big(\|\Delta Q\|_{H^s} + \|\nabla u\|_{H^s}\big)\big(\|\nabla Q\|_{H^s} + \|u\|_{H^s}\big) + \big(\|Q\|_{H^s} + \|Q\|_{H^s}^3\big)\|Q\|_{\dot{H}^s}\|\nabla u\|_{H^s}\\ & + \|\nabla Q\|_{H^s}^2\|\nabla u\|_{H^s} + \big(\|\tilde{c}\|_{H^s}\|Q\|_{\dot{H}^s} + \|\tilde{c}\|_{\dot{H}^s}\|Q\|_{H^s}\big)\big(\|Q\|_{\dot{H}^s} + \|\Delta Q\|_{H^s}\big)\\ & + (1+\|Q\|_{H^s})\|Q\|_{H^s}\|Q\|_{\dot{H}^s}\big(\|\Delta Q\|_{H^s} + \|Q\|_{\dot{H}^s}\big)\Big\}\,.\end{aligned} \tag{5.13}$$

Next it turns to estimate the terms on $\mathcal{J}^k$. In fact, we only need to estimate the terms $\mathcal{J}_6^k$, since the other terms can be bounded as the corresponding terms in J^k. The difference is that we should use the bound of $1+\varrho$. Now we control the term $\mathcal{J}_6^k$. The term can be divided into the following three parts:

$$\begin{aligned}\mathcal{J}_6^k =&\sigma_*\langle[\partial_x^\alpha, \tfrac{1}{1+\varrho}\mathrm{div}](\tilde{c}^2Q), (1+\varrho)\partial_x^\alpha u\rangle - 2\sigma_* c_*\langle[\partial_x^\alpha, \tfrac{1}{1+\varrho}\mathrm{div}](\tilde{c}Q), (1+\varrho)\partial_x^\alpha u\rangle\\ &+ \sigma_* c_*^2\langle[\partial_x^\alpha, \tfrac{1}{1+\varrho}\mathrm{div}]Q, (1+\varrho)\partial_x^\alpha u\rangle\\ =&\mathcal{J}_{61}^k + \mathcal{J}_{62}^k + \mathcal{J}_{63}^k\,.\end{aligned}\tag{5.14}$$

Similar calculation as the estimate of $\mathcal{I}_5^k$ gives us

$$\begin{aligned}\mathcal{J}_{61}^k &\lesssim \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\|\tilde{c}\|_{H^s}\big(\|\tilde{c}\|_{H^s}\|\nabla Q\|_{H^s} + \|\nabla\tilde{c}\|_{H^s}\|Q\|_{H^s}\big)\|\mathrm{u}\|_{\dot{H}^s}\,,\\ \mathcal{J}_{62}^k &\lesssim \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\big(\|\tilde{c}\|_{H^s}\|\nabla Q\|_{H^s} + \|\nabla\tilde{c}\|_{H^s}\|Q\|_{H^s}\big)\|\mathrm{u}\|_{\dot{H}^s}\,,\end{aligned}$$

and

$$\mathcal{J}_{63}^k \lesssim \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\|\nabla Q\|_{H^s}\|u\|_{\dot{H}^s}\,.$$

Putting the above three estimates together, one has

$$\begin{aligned}\mathcal{J}_6^k \lesssim&\mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\big(1+\|\tilde{c}\|_{H^s}\big)\big(\|\tilde{c}\|_{H^s}\|\nabla Q\|_{H^s} + \|\nabla\tilde{c}\|_{H^s}\|Q\|_{H^s}\big)\|\mathrm{u}\|_{\dot{H}^s}\\ &+ \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\|\nabla Q\|_{H^s}\|u\|_{\dot{H}^s}\,.\end{aligned}\tag{5.15}$$

By the estimates of J^k, one can easily get the bound of the rest terms of $\mathcal{J}^k$ as follows:

$$\begin{aligned}\mathcal{J}_1^k &\lesssim\|\nabla u\|_{H^s}\|\varrho\|_{\dot{H}^s}^2\,,\\ \mathcal{J}_2^k &\lesssim\|u\|_{H^s}\|u\|_{\dot{H}^s}\|\nabla u\|_{H^s}\,,\\ \mathcal{J}_3^k &\lesssim\mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\|\varrho\|_{\dot{H}^s}\|u\|_{\dot{H}^s}\,,\end{aligned}\tag{5.16}$$

and

$$\begin{aligned}\mathcal{J}_4^k &\lesssim\mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\|u\|_{\dot{H}^s}\|\nabla u\|_{H^s}\,,\\ \mathcal{J}_5^k &\lesssim\mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\|Q\|_{H^s}\|\Delta Q\|_{H^s}\|u\|_{\dot{H}^s}\,,\\ \mathcal{J}_7^k &\lesssim\mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\big(\|\nabla Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}\|\Delta Q\|_{H^s} + \|Q\|_{H^s}\|Q\|_{\dot{H}^s} + \|Q\|_{H^s}^3\|Q\|_{\dot{H}^s}\big)\|u\|_{\dot{H}^s}\,.\end{aligned}\tag{5.17}$$

Combining with the above estimates, we then have

$$\begin{aligned}\mathcal{J}^k \lesssim&\|\nabla u\|_{H^s}\big(\|\varrho\|_{\dot{H}^s}^2 + \|u\|_{H^s}\|u\|_{\dot{H}^s}\big) + \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\big(\|\varrho\|_{\dot{H}^s} + \|\nabla u\|_{H^s} + \|\nabla Q\|_{H^s}\big)\|u\|_{\dot{H}^s}\\ &+ \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\big(1+\|\tilde{c}\|_{H^s}\big)\big(\|\tilde{c}\|_{H^s}\|\nabla Q\|_{H^s} + \|\nabla\tilde{c}\|_{H^s}\|Q\|_{H^s} + \|Q\|_{H^s}\|\Delta Q\|_{H^s}\big)\|\mathrm{u}\|_{\dot{H}^s}\\ &+ \mathscr{P}_s(\|\varrho\|_{\dot{H}^s})\big(\|\nabla Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}\|\Delta Q\|_{H^s} + \|Q\|_{H^s}\|Q\|_{\dot{H}^s} + \|Q\|_{H^s}^3\|Q\|_{\dot{H}^s}\big)\|u\|_{\dot{H}^s}\,.\end{aligned}\tag{5.18}$$

Based on the construction of terms on the right-hand side of identity (5.4) and the bounds of $\mathcal{I}^k$ and $\mathcal{J}^k$, we can find that more dissipation should be needed for the purpose of getting the global in time solution. The positive of $P'(1+\varrho)$ enables us to obtain some dissipation by acting the derivative operator ∂_x^α ($|\alpha|=k$) on the velocity equations of system (1.11) for any integer $1\le k\le s-1$ and taking scalar product of the resulting identity with $\partial_x^{\alpha+1}\varrho$ over the torus $\mathbb{T}^3$, that is

$$\begin{aligned}&\langle\partial_t\partial_x^\alpha u,\partial_x^{\alpha+1}\varrho\rangle+\langle\tfrac{P'(1+\varrho)}{1+\varrho}\partial_x^{\alpha+1}\varrho,\partial_x^{\alpha+1}\varrho\rangle\\=&\underbrace{-\langle\partial_x^\alpha(u\cdot\nabla u),\partial_x^{\alpha+1}\varrho\rangle}_{\mathscr{I}_1^k}+\underbrace{\langle[\partial_x^\alpha,\tfrac{P'(1+\varrho)}{1+\varrho}\nabla]\varrho,-\partial_x^{\alpha+1}\varrho\rangle}_{\mathscr{I}_2^k}+\underbrace{\langle\partial_x^\alpha\Big(\tfrac{1}{1+\varrho}\big(\mu\Delta u+(\mu+\nu)\nabla\mathrm{div}u\big)\Big),\partial_x^\alpha\varrho\rangle}_{\mathscr{I}_3^k}\\&+\underbrace{\langle\partial_x^\alpha\Big(\tfrac{1}{1+\varrho}\nabla\cdot\big(F(Q)\mathbb{I}_3-K\nabla Q\odot\nabla Q\big)\Big),\partial_x^{\alpha+1}\varrho\rangle}_{\mathscr{I}_4^k}+\underbrace{K\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\nabla\cdot(Q\Delta Q-\Delta QQ)\big),\partial_x^\alpha\varrho\rangle}_{\mathscr{I}_5^k}\\&+\underbrace{\sigma_*\langle\partial_x^\alpha\Big(\tfrac{1}{1+\varrho}\nabla\cdot\big((\tilde c+c_*)^2Q\big)\Big),\partial_x^{\alpha+1}\varrho\rangle}_{\mathscr{I}_6^k}.\end{aligned}\tag{5.19}$$

We now deal with the terms on identity (5.19). The first term on the left-hand side of (5.19) can be rewritten as $\langle\partial_t\partial_x^\alpha u,\partial_x^{\alpha+1}\varrho\rangle=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_x^\alpha u+\partial_x^{\alpha+1}\varrho\|_{L^2}^2-\|\partial_x^\alpha u\|_{L^2}^2-\|\partial_x^{\alpha+1}\varrho\|_{L^2}^2\big)+\mathcal{R}_1+\mathcal{R}_2$ by using the mass conservation equation in system (1.11) with

$$\mathcal{R}_1=-\langle\partial_x^\alpha\mathrm{div}u,\partial_x^\alpha(u\cdot\nabla\varrho)\rangle,\quad \mathcal{R}_2=-\langle\partial_x^\alpha\mathrm{div}u,\partial_x^\alpha\big((1+\varrho)\mathrm{div}u\big)\rangle.$$

Straightforward calculation gives us

$$\begin{aligned}\mathcal{R}_1\lesssim&\|\nabla u\|_{H^s}\|u\|_{H^s}\|\nabla\varrho\|_{H^{s-1}},\\ \mathcal{R}_2\lesssim&\|1+\varrho\|_{L^\infty}\|\partial_x^\alpha\mathrm{div}u\|_{L^2}^2+C\|u\|_{\dot H^s}\|\nabla u\|_{H^s}\|\varrho\|_{\dot H^s}\end{aligned}\tag{5.20}$$

by using the Hölder inequality, Sobolev embedding theory.

We then estimate the terms on the right-hand side of the identity (5.19). One can calculate directly to get

$$\mathscr{I}_1^k\lesssim\|u\|_{H^s}\|\nabla u\|_{H^s}\|\varrho\|_{\dot H^s},\quad \mathscr{I}_2^k\lesssim\mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\varrho\|_{\dot H^s}^2.\tag{5.21}$$

For the estimate of $\mathscr{I}_3^k$, by using the Hölder inequality and Lemma 2.3, and noticing that $|\alpha|=k<s-1$, one has

$$\begin{aligned}&\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\Delta u\big),\partial_x^{\alpha+1}\varrho\rangle\\=&\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\big)\Delta u+\tfrac{1}{1+\varrho}\partial_x^\alpha\Delta u,\partial_x^{\alpha+1}\varrho\rangle+\sum_{\substack{\alpha_1+\alpha_2=\alpha,\\1\le|\alpha_1|\le k-1}}C_\alpha^{\alpha_1}\langle\partial_x^{\alpha_1}\big(\tfrac{1}{1+\varrho}\big)\partial_x^{\alpha_2}\Delta u,\partial_x^{\alpha+1}\varrho\rangle\\\le&C\Big(\|\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\big)\|_{L^4}\|\Delta u\|_{L^4}+\sum_{\substack{\alpha_1+\alpha_2=\alpha,\\1\le|\alpha_1|\le k-1}}\|\partial_x^{\alpha_1}\big(\tfrac{1}{1+\varrho}\big)\|_{L^4}\|\partial_x^{\alpha_2}\Delta u\|_{L^4}\Big)\|\partial_x^{\alpha+1}\varrho\|_{L^2}\\&+\|\tfrac{1}{1+\varrho}\|_{L^\infty}\|\partial_x^\alpha\Delta u\|_{L^2}\|\partial_x^{\alpha+1}\varrho\|_{L^2}\\\lesssim&C\mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\nabla u\|_{H^s}\|\varrho\|_{\dot H^s}+\|\tfrac{1}{1+\varrho}\|_{L^\infty}\|\partial_x^\alpha\Delta u\|_{L^2}\|\partial_x^{\alpha+1}\varrho\|_{L^2}.\end{aligned}$$

On the other hand, Cauchy inequality infers that

$$\Big(\mu\|\partial_x^\alpha\Delta u\|_{L^2}+(\mu+\nu)\|\partial_x^\alpha\nabla\mathrm{div}u\|_{L^2}\Big)\|\tfrac{1}{1+\varrho}\|_{L^\infty}\|\partial_x^{\alpha+1}\varrho\|_{L^2}$$
$$\leq\tfrac14\int_{\mathbb{T}^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^{\alpha+1}\varrho|^2\mathrm{d}x+2(\mu^2+\nu^2)\|\tfrac{1}{1+\varrho}\|_{L^\infty}^2\|\tfrac{1+\varrho}{P'(1+\varrho)}\|_{L^\infty}\Big(\|\partial_x^\alpha\Delta u\|_{L^2}^2+\|\partial_x^\alpha\nabla\mathrm{div}u\|_{L^2}^2\Big)\,.$$

Hence we can bound $\mathscr{I}_3^k$ as

$$\begin{aligned}\mathscr{I}_3^k\leq&C\mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\nabla u\|_{H^s}\|\varrho\|_{\dot H^s}+\tfrac14\int_{\mathbb{T}^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^{\alpha+1}\varrho|^2\mathrm{d}x\\&+2(\mu^2+\nu^2)\|\tfrac{1}{1+\varrho}\|_{L^\infty}^2\|\tfrac{1+\varrho}{P'(1+\varrho)}\|_{L^\infty}\Big(\|\partial_x^\alpha\Delta u\|_{L^2}^2+\|\partial_x^\alpha\nabla\mathrm{div}u\|_{L^2}^2\Big)\end{aligned}\tag{5.22}$$

for some constant C.

To estimate $\mathscr{I}_4^k$, we can use the similar controllation as I_9^k to derive that

$$\mathscr{I}_4^k\lesssim\big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\big)\big(\|Q\|_{H^s}^3+\|Q\|_{H^s}+\|\nabla Q\|_{H^s}+\|\Delta Q\|_{H^s}\big)\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s}\,.\tag{5.23}$$

As to the estimate of $\mathscr{I}_5^k$, which can be divided into the following three parts:

$$\begin{aligned}\mathscr{I}_5^k=&K\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\big)\mathrm{div}(Q\Delta Q-\Delta QQ),\partial_x^{\alpha+1}\varrho\rangle+K\langle\tfrac{1}{1+\varrho}\partial_x^\alpha\mathrm{div}(Q\Delta Q-\Delta QQ),\partial_x^{\alpha+1}\varrho\rangle\\&+\sum_{\substack{\alpha_1+\alpha_2=\alpha,\\1\leq|\alpha_1|\leq k-1}}C_\alpha^{\alpha_1}K\langle\partial_x^{\alpha_1}\big(\tfrac{1}{1+\varrho}\big)\partial_x^{\alpha_2}\mathrm{div}(Q\Delta Q-\Delta QQ),\partial_x^{\alpha+1}\varrho\rangle\,.\end{aligned}$$

Noting the fact that $1\leq|\alpha|\leq s-1$, then one has

$$\begin{aligned}\|\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\big)\mathrm{div}(Q\Delta Q-\Delta QQ)\|_{L^2}\lesssim&\|\partial_x^\alpha\tfrac{1}{1+\varrho}\|_{L^2}\big(\|\nabla Q\|_{L^\infty}\|\Delta Q\|_{L^\infty}+\|\Delta Q\nabla Q\|_{L^\infty}\|Q\|_{L^\infty}\big)\\\lesssim&\mathscr{P}_s(\|\varrho\|_(\dot H^s))\|Q\|_{H^s}\|\Delta Q\|_{H^s}\,,\\\|\partial_x^\alpha\mathrm{div}(\Delta QQ-\Delta Q)\|_{L^2}\lesssim&(\|Q\|_{H^s}+\|\nabla Q\|_{H^s})\|\Delta Q\|_{H^s}\,,\end{aligned}$$

and

$$\|\partial_x^{\alpha_1}\big(\tfrac{1}{1+\varrho}\big)\partial_x^{\alpha_2}\mathrm{div}(\nabla Q\Delta Q-\Delta QQ)\|_{L^2}\lesssim\mathscr{P}_s(\|\varrho\|_{\dot H^s})(\|Q\|_{H^s}+\|\nabla Q\|_{H^s})\|\Delta Q\|_{H^s}$$

for $1\leq|\alpha_1|\leq k-1$. Hence the following bound holds:

$$\mathscr{I}_5^k\lesssim\big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\big)(\|Q\|_{H^s}+\|\nabla Q\|_{H^s})\|\Delta Q\|_{H^s}\|\varrho\|_{\dot H^s}\,.\tag{5.24}$$

It remains to estimate the last term $\mathscr{I}_6^k$. Since

$$\begin{aligned}\mathscr{I}_6^k=&\sigma_*\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\nabla\cdot(\tilde c^2Q)\big),\partial_x^{\alpha+1}\varrho\rangle+2\sigma_*c_*\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\nabla\cdot(\tilde cQ)\big),\partial_x^{\alpha+1}\varrho\rangle\\&+\sigma_*c_*^2\langle\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\nabla\cdot(Q)\big),\partial_x^{\alpha+1}\varrho\rangle\,.\end{aligned}$$

Similar as the estimates of J_6^k, we deduce that

$$\begin{aligned}\|\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\nabla\cdot(c^2Q)\big)\|_{L^2}\lesssim&\Big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\Big)\Big(\|\tilde c\|_{H^s}\|\nabla\tilde c\|_{H^s}\|Q\|_{H^s}+\|\tilde c\|_{H^s}^2\|\nabla Q\|_{H^s}\Big)\,,\\\|\partial_x^\alpha\big(\tfrac{1}{1+\varrho}\nabla\cdot(cQ)\big)\|_{L^2}\lesssim&\Big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\Big)\Big(\|\nabla\tilde c\|_{H^s}\|Q\|_{H^s}+\|\tilde c\|_{H^s}^2\|\nabla Q\|_{H^s}\Big)\,.\end{aligned}$$

As to the estimates of the last part of $\mathscr{I}_6^k$, by using the Hölder inequality, Sobolev embedding theory and Cauchy inequality, we have

$$\begin{aligned}
&\sigma_* c_*^2 \langle \partial_x^\alpha \big(\tfrac{1}{1+\varrho}\nabla\cdot Q\big), \partial_x^{\alpha+1}\varrho\rangle \\
=&\sigma_* c_*^2 \langle \partial_x^\alpha \big(\tfrac{1}{1+\varrho}\big)\nabla\cdot Q + \tfrac{1}{1+\varrho}\partial_x^\alpha\nabla\cdot Q, \partial_x^{\alpha+1}\varrho\rangle + \sigma_* c_*^2 \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ 1\le|\alpha_1|\le k-1}} C_\alpha^{\alpha_1}\langle \partial_x^{\alpha_1}\big(\tfrac{1}{1+\varrho}\big)\partial_x^{\alpha_2}\nabla\cdot Q, \partial_x^{\alpha+1}\varrho\rangle \\
\le& C\mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s} + \sigma_* c_*^2\|\tfrac{1}{1+\varrho}\|_{L^\infty}\|\partial_x^\alpha\nabla\cdot Q\|_{L^2}\|\partial_x^{\alpha+1}\varrho\|_{L^2} \\
\le& \tfrac14\int_{\mathbb{T}^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^{\alpha+1}\varrho|^2 \mathrm{d}x + \sigma_*^2 c_*^4\|\tfrac{1}{1+\varrho}\|_{L^\infty}^2\|\tfrac{1+\varrho}{P'(1+\varrho)}\|_{L^\infty}\|\partial_x^\alpha\nabla\cdot Q\|_{L^2}^2 \\
&+ C\mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s}\,.
\end{aligned}$$

Gathering the above estimates, we then obtain

$$\begin{aligned}
\mathscr{I}_6^k \le& C\Big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\Big)\Big(\|\nabla\tilde c\|_{H^s}\|Q\|_{H^s} + \|\tilde c\|_{H^s}\|\nabla Q\|_{H^s}\Big)\big(1+\|\tilde c\|_{H^s}\big)\|\varrho\|_{\dot H^s} \\
&+ C\mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s} + \tfrac14\int_{\mathbb{T}^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^{\alpha+1}\varrho|^2\mathrm{d}x \\
&+ \sigma_*^2 c_*^4\|\tfrac{1}{1+\varrho}\|_{L^\infty}^2\|\tfrac{1+\varrho}{P'(1+\varrho)}\|_{L^\infty}\|\partial_x^\alpha\nabla\cdot Q\|_{L^2}^2\,.
\end{aligned}\tag{5.25}$$

On the other hand, for the case $k=0$, straightforward calculation enables us to get

$$\begin{aligned}
&\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|u+\nabla\varrho\|_{L^2}^2 - \|u\|_{L^2}^2 - \|\nabla\varrho\|_{L^2}^2\Big) + \tfrac12\int_{\mathbb{T}^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\nabla\varrho|^2\mathrm{d}x - \|1+\varrho\|_{L^\infty}\|\mathrm{div}u\|_{L^2}^2 \\
&-\|\tfrac{1}{1+\varrho}\|_{L^\infty}^2\|\tfrac{1+\varrho}{P'(1+\varrho)}\|_{L^\infty}\Big[2(\mu^2+\nu^2)\Big(\|\Delta u\|_{L^2}^2 + \|\nabla\mathrm{div}u\|_{L^2}^2\Big) + \sigma_*^2 c_*^4\|\nabla\cdot Q\|_{L^2}^2\Big] \\
\lesssim& \|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\varrho\|_{L^2} + \Big(\|Q\|_{H^s}^3 + \|Q\|_{H^s} + \|\nabla Q\|_{H^s} + \|\Delta Q\|_{H^{s-1}}\Big)\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s} \\
&+\Big(\|Q\|_{H^s} + \|\nabla Q\|_{H^s}\Big)\|\Delta Q\|_{H^s}\|\varrho\|_{\dot H^s} + \Big(\|Q\|_{H^2}\|\nabla\tilde c\|_{L^2} + \|\tilde c\|_{H^2}\|\nabla Q\|_{L^2}\Big) \\
&\times\big(1+\|\tilde c\|_{H^2}\big)\|\nabla\varrho\|_{L^2}\,.
\end{aligned}\tag{5.26}$$

Therefore, combining with the estimates of $\mathscr{I}_i$ $(1\le i\le 6)$ and $\mathcal{R}_1$, $\mathcal{R}_2$, then summing up with $1\le k\le s-1$ for identity (5.19), and combining with (5.26), we deduce that

$$\begin{aligned}
&\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|u+\nabla\varrho\|_{H^{s-1}}^2 - \|u\|_{H^{s-1}}^2 - \|\varrho\|_{\dot H^s}^2\Big) + \sum_{|\alpha|=1}^{s}\tfrac12\int_{\mathbb{T}^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^\alpha\varrho|^2\mathrm{d}x \\
&-\|1+\varrho\|_{L^\infty}\|\mathrm{div}u\|_{H^{s-1}}^2 - C(\mu,\nu)\Big(\|\Delta u\|_{H^{s-1}}^2 + \|\nabla\mathrm{div}u\|_{H^{s-1}}^2\Big) - C(\sigma_*,c_*)\|\nabla\cdot Q\|_{H^{s-1}}^2 \\
\lesssim& \|\nabla u\|_{H^s}\|u\|_{\dot H^s}\|\varrho\|_{\dot H^s} + \mathscr{P}_s(\|\varrho\|_{\dot H^s})\big(\|\varrho\|_{\dot H^s} + \|\nabla u\|_{H^s}\big)\|\varrho\|_{\dot H^s} \\
&+\Big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\Big)\Big(\|Q\|_{H^s}^3 + \|Q\|_{H^s} + \|\nabla Q\|_{H^s} + \|\Delta Q\|_{H^{s-1}}\Big)\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s} \\
&+\Big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\Big)\Big(\|Q\|_{H^s} + \|\nabla Q\|_{H^s}\Big)\|\Delta Q\|_{H^s}\|\varrho\|_{\dot H^s} + \mathscr{P}_s(\|\varrho\|_{\dot H^s})\|\nabla Q\|_{H^s}\|\varrho\|_{\dot H^s} \\
&+\Big(1+\mathscr{P}_s(\|\varrho\|_{\dot H^s})\Big)\Big(\|\nabla\tilde c\|_{H^s}\|Q\|_{H^s} + \|\tilde c\|_{H^s}\|\nabla Q\|_{H^s}\Big)\big(1+\|\tilde c\|_{H^s}\big)\|\varrho\|_{\dot H^s}\,.
\end{aligned}\tag{5.27}$$

Here we denote $C(\mu,\nu)=2(\mu^2+\nu^2)\big\|\frac{1}{1+\varrho}\big\|^2_{L^\infty}\big\|\frac{1+\varrho}{P'(1+\varrho)}\big\|_{L^\infty}$, $C(\sigma_*,c_*)=\sigma_*^2c_*^4\big\|\frac{1}{1+\varrho}\big\|^2_{L^\infty}\big\|\frac{1+\varrho}{P'(1+\varrho)}\big\|_{L^\infty}$.

Under the assumption $\frac{C}{2\mu}|\sigma_*|^2c_*^4\le\frac{\Gamma K}{2}$, we choose

$$\eta_0=\tfrac12\min\Big\{\frac{1}{\|\frac{1+\varrho}{P'(1+\varrho)}\|_{L^\infty}},\frac{1}{\|\frac{1}{1+\varrho}\|_{L^\infty}},\frac{\mu}{2C(\mu,\nu)},\frac{\mu+\nu}{\|1+\varrho\|_{L^\infty}+C(\mu,\nu)},\frac{\Gamma K}{2C(\sigma_*,c_*)},1\Big\}$$

and let $\eta\in(0,\eta_0]$, then multiply (5.27) by η and add the results to the inequality (5.4) to get

$$\begin{aligned}&\tfrac12\tfrac{\mathrm d}{\mathrm dt}\Big\{\int_{\mathbb T^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\varrho|^2\mathrm dx+\sum_{|\alpha|=1}^{s}\int_{\mathbb T^3}\big(\tfrac{P'(1+\varrho)}{1+\varrho}-\eta\big)|\partial_x^\alpha\varrho|^2\mathrm dx+\sum_{|\alpha|=0}^{s-1}\int_{\mathbb T^3}\big((1+\varrho)-\eta\big)|\partial_x^\alpha u|^2\mathrm dx\\&+\sum_{|\alpha|=s}\|\sqrt{1+\varrho}\partial_x^\alpha u\|^2_{L^2}+\|\tilde c\|^2_{H^s}+\|Q\|^2_{H^s}+K\|\nabla Q\|^2_{H^s}\Big\}+\tfrac12\eta\sum_{|\alpha|=1}^{s}\int_{\mathbb T^3}\tfrac{P'(1+\varrho)}{1+\varrho}|\partial_x^\alpha\varrho|^2\mathrm dx\\&+\big(\tfrac{\mu}{2}-\eta C(\mu,\nu)\big)\|\nabla u\|^2_{H^s}+\big((\mu+\nu)-\eta(\|1+\varrho\|_{L^\infty}+C(\mu,\nu))\big)\|\mathrm{div}u\|^2_{H^s}\\&+\big(\tfrac{\Gamma K}{2}-\eta C(\sigma_*,c_*)\big)\|\nabla Q\|^2_{H^s}+\Gamma K^2\|\Delta Q\|^2_{H^s}+D_0\|\nabla\tilde c\|^2_{H^s}\\\lesssim&\mathcal D_\eta(t)\sum_{k=1}^{s+3}\mathcal E_\eta^{\frac k2}(t)\end{aligned}\tag{5.28}$$

by using the definition of $\mathcal E_\eta(t)$ and $\mathcal D_\eta(t)$. The above estimates (5.28) is also can be written as

$$\tfrac12\tfrac{\mathrm d}{\mathrm dt}\mathcal E_\eta(t)+\mathcal D_\eta(t)\le C\mathcal D_\eta(t)\sum_{k=1}^{s+3}\mathcal E_\eta^{\frac k2}(t)\tag{5.29}$$

with the constant C depending on the the coefficients of the original system (1.11) and s. This completes the proof of Lemma 5.1.

5.1 The Proof of Global Well-Posedness

In this section, we concern with the proof of existence of global-in-time solution near the constant state $(c_*,1,0,0)$ for the original system (1.2). Here we will use the continuum arguments.

Firstly, we define the following energy: $\tilde{\mathcal E}(t)=\|\tilde c\|^2_{H^s}+\|\varrho\|^2_{H^s}+\|u\|^2_{H^s}+\|Q\|^2_{H^{s+1}}$. Obviously, by the definition of $\mathcal E_\eta(t)$, there exist constants $\widetilde C$ and $\widehat C$,

$$\begin{aligned}\widetilde C=&\min\Big\{\inf\Big|\tfrac{P'(1+\varrho)}{1+\varrho}\Big|-\eta_0,\inf|1+\varrho|-\eta_0,1-\eta_0\Big\},\\\widehat C=&\big\|\tfrac{P'(1+\varrho)}{1+\varrho}\big\|_{L^\infty}+2\|1+\varrho\|_{L^\infty}+2+K,\end{aligned}$$

such that $\widetilde C\tilde{\mathcal E}(t)\le\mathcal E_\eta(t)\le\widehat C\tilde{\mathcal E}(t)$. Consequently, we have $\widetilde C\tilde{\mathcal E}^{in}\le\mathcal E_\eta(0)\le\widehat C\tilde{\mathcal E}^{in}$.

Next we make a definition as $T^*=\sup\Big\{\tau>0;\sup_{t\in[0,\tau]}C\sum\limits_{k=1}^{s+3}\mathcal E_\eta^{\frac k2}(t)\le\frac12\Big\}$ with the positive constant C to be mentioned in Lemma 5.1. Let $\epsilon_0=\frac{1}{\widehat C}\min\Big\{1,\frac{1}{16(s+3)^2C^2}\Big\}$, if the initial energy $\tilde{\mathcal E}^{in}\le\epsilon_0$, we can deduce that $C\sum\limits_{k=1}^{s+3}\mathcal E_\eta^{\frac k2}(0)\le\frac14<\frac12$. By using the continuity of the energy function $\mathcal{E}_\eta(t)$, one can immediately get $T^* > 0$. Obviously, the above analysis implies that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\eta(t) + \Big(1 - C\sum\limits_{k=1}^{s+3}\mathcal{E}_\eta^{\frac{k}{2}}(t)\Big)\mathcal{D}_\eta(t) \leq 0$ holds for all $t \in [0, T^*]$. Hence for all $t \in [0, T^*]$, we have

$$\mathcal{E}_\eta(t) \leq \mathcal{E}_\eta(0) \leq \widehat{C}\tilde{\mathcal{E}}^{in}\,.$$

Consequently, $\sup\limits_{t\in[0,T^*]}\Big\{C\sum_{k=1}^{s+3}\mathcal{E}_\eta^{\frac{k}{2}}(t)\Big\} \leq \frac{1}{4}$. Based on the above analysis, we now claim that $T^* = +\infty$. Otherwise, by utilizing the continuity of energy function, there exists a sufficient small positive number $\epsilon > 0$, such that $\sup\limits_{t\in[0,T^*+\epsilon]}\Big\{C\sum_{k=1}^{s+3}\mathcal{E}_\eta^{\frac{k}{2}}(t)\Big\} \leq \frac{3}{8} \leq \frac{1}{2}$, which contradicts to the definition of T^*. As a result, there is a constant C_1, depending only on the coefficients of the system (1.2) and s, such that the following inequality

$$\sup_{t\geq 0}\big(\|\tilde{c}\|_{H^s}^2 + \|\varrho\|_{H^s}^2 + \|u\|_{H^s}^2 + \|Q\|_{H^{s+1}}^2\big)(t) + \int_0^\infty D_0\|\nabla\tilde{c}(\tau)\|_{H^s}^2 + \frac{\mu}{4}\|\nabla u(\tau)\|_{H^s}^2\mathrm{d}\tau \leq C_1\tilde{\mathcal{E}}^{in}$$

holds. This completes the proof of Theorem 1.2.

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可压缩活性液晶模型的解的存在性

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摘要: 本文研究了基于Q-张量框架的可压缩活性液晶模型的流体动力学问题. 在全空间或者圆环上, 我们证明了模型的大初值局部经典解的存在性. 并且，在一定的系数假设下, 我们给出了在常数态附近圆环上小初值全局经典解的存在性.

关键词: 可压缩活性液晶模型; 经典解; 平衡态附近; 全局时间

MR(2010)**主题分类号**: 35D35; 35Q35; 76A15 **中图分类号**: O175.4