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SOME IMPROVED INEQUALITIES FOR MATRICES

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Abstract: In this paper, matrix inequalities are studied. Using two new scalar inequalities, the weighted geometric mean inequalities and Hilbert-Schmidt norm inequalities for matrices are obtained, the results are refinements of some corresponding inequalities.

Keywords: scalar inequalities; weighted geometric mean inequalities; positive definite matrix; Hilbert-Schmidt norm

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1 Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n, if $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The $A > 0$ is used to mean that A is a positive definite matrix. The Hilbert-Schmidt norm of $A = (a_{ij}) \in M_n$ is denoted by

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Let $A, B \in M_n$ be positive definite and $0 \le v \le 1$, the weighted geometric mean of the matrices A and B is defined as follows:

$$A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

for $v = \frac{1}{2}$, we denote the geometric mean by $A\sharp B$.

Kittaneh and Manasrah [1] proved that if $A, B \in M_n$ are positive definite and $0 \le v \le 1$, then

$$\begin{aligned} &2r_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B \\ &\le A + B \\ &\le 2s_0(A + B - 2A\sharp B) + A\sharp_v B + A\sharp_{1-v} B, \end{aligned} \tag{1.1}$$

where $r_0 = \min\{v, 1 - v\}, s_0 = \max\{v, 1 - v\}$.

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In 2018, Liu and Yang [2] refined the inequalities (1.1) as follows:

$$\begin{aligned} & 2r_0(A+B-2A\sharp B)+A\sharp_v B+A\sharp_{1-v}B \\ & \leq A+B \\ & \leq \alpha(v)(A+B-2A\sharp B)+A\sharp_v B+A\sharp_{1-v}B \\ & \leq 2s_0(A+B-2A\sharp B)+A\sharp_v B+A\sharp_{1-v}B, \end{aligned} \tag{1.2}$$

where $\alpha(v)=\frac{3}{2}-2(v-v^2)$.

Let $A, B, X \in M_n$ such that A and B are positive definite. Bhatia and Davis [3] proved that if $0 \leq v \leq 1$, then

$$2||A^{\frac{1}{2}}XB^{\frac{1}{2}}|| \leq ||A^vXB^{1-v}+A^{1-v}XB^v|| \leq ||AX+XB||,$$

where the second inequality is known as Heinz inequality.

He and Zou [4] showed if $0 \leq v \leq 1$, then

$$||AX+XB||_2^2 \leq ||A^vXB^{1-v}+A^{1-v}XB^v||_2^2+2s_0||AX-XB||_2^2, \tag{1.3}$$

where $s_0=\max\{v, 1-v\}$.

Kittaneh and Manasrah [5] showed if $0 \leq v \leq 1$, then

$$||A^vXB^{1-v}+A^{1-v}XB^v||_2^2+2r_0||AX-XB||_2^2 \leq ||AX+XB||_2^2, \tag{1.4}$$

where $r_0=\min\{v, 1-v\}$, inequality (1.4) is the inverse of inequality (1.3).

In 2018, Liu and Yang [2] refined inequality (1.3) as follows:

$$||AX+XB||_2^2 \leq ||A^vXB^{1-v}+A^{1-v}XB^v||_2^2+\alpha(v)||AX-XB||_2^2, \tag{1.5}$$

where $\alpha(v)=\frac{3}{2}-2(v-v^2)$.

Recently, many interesting articles have been devoted to study the unitarily invariant norm inequalities for matrices, see [6-8] and references therein.

In this paper, we first give two scalar inequalities. By using scalar inequalities, we improve inequalities (1.2) and (1.5).

2 Main results

In the following, we give two scalar inequalities which will turn out to be useful in the proof of our results.

Theorem 2.1 Let $a, b>0$, $0 \leq v \leq 1$, then

$$a+b \leq a^vb^{1-v}+a^{1-v}b^v+\gamma(v)(\sqrt{a}-\sqrt{b})^2, \tag{2.1}$$

where $\gamma(v)=\frac{5}{4}-(v-v^2)$.

Proof To prove inequality (2.1), we only need prove that the following inequality

$$(1-\gamma(v))(a+b)+2\gamma(v)\sqrt{ab}\le a^vb^{1-v}+a^{1-v}b^v.$$

Let $a=e^x, b=e^y$, by the definition of the hyperbolic function, we have

$$(v-v^2-\frac{1}{4})cosh(\frac{x-y}{2})+(\frac{5}{4}-(v-v^2))\le cosh((1-2v)(\frac{x-y}{2})). \tag{2.2}$$

Let $z=\frac{x-y}{2}$, by the series expansion of the hyperbolic $coshz$ function, we know that inequality (2.2) is equivalent to

$$(v-v^2-\frac{1}{4})(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots)+(\frac{5}{4}-(v-v^2))\le 1+\frac{(1-2v)^2z^2}{2!}+\frac{(1-2v)^4z^4}{4!}+\cdots. \tag{2.3}$$

For $0\le v\le 1$, it is easy to know that inequality (2.3) holds.

This completes the proof.

Corollary 2.2 Let $a,b>0$, $0\le v\le 1$, then

$$(a+b)^2\le(a^vb^{1-v}+a^{1-v}b^v)^2+\gamma(v)(a-b)^2, \tag{2.4}$$

where $\gamma(v)=\frac{5}{4}-(v-v^2)$.

Proof By inequality (2.1), we have

$$\begin{aligned}(\sqrt{a}+\sqrt{b})^2-(a^{\frac{v}{2}}b^{\frac{1-v}{2}}+a^{\frac{1-v}{2}}b^{\frac{v}{2}})^2 &= a+b-(a^vb^{1-v}+a^{1-v}b^v)\\ &\le (\tfrac{5}{4}-(v-v^2))(\sqrt{a}-\sqrt{b})^2,\end{aligned}$$

hence

$$(a+b)^2\le(a^vb^{1-v}+a^{1-v}b^v)^2+(\frac{5}{4}-(v-v^2))(a-b)^2.$$

This completes the proof.

Theorem 2.3 Let $A,B\in M_n$ be positive definite. Then

$$\begin{aligned}&2r_0(A+B-2A\sharp B)+A\sharp_vB+A\sharp_{1-v}B\\ &\le A+B\\ &\le\gamma(v)(A+B-2A\sharp B)+A\sharp_vB+A\sharp_{1-v}B\\ &\le\alpha(v)(A+B-2A\sharp B)+A\sharp_vB+A\sharp_{1-v}B,\end{aligned} \tag{2.5}$$

where $v\in[0,1]$, $r_0=\min\{v,1-v\},\gamma(v)=\frac{5}{4}-(v-v^2),\alpha(v)=\frac{3}{2}-2(v-v^2)$.

Proof By inequalities (1.2), we know that the first inequality of (2.5) holds. For the second inequality of (2.5). Since $T \in M_n$ is positive definite, it follows by the spectral theorem that there exists unitary matrix $U \in M_n$ such that

$$T = UPU^*,$$

where $P = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), \lambda_j > 0, 1 \leq j \leq n$. For $a > 0, b = 1$, by inequality (2.1), we have

$$a + 1 \leq a^v + b^{1-v} + \gamma(v)(\sqrt{a} - 1)^2,$$

and so

$$P + I \leq P^v + P^{1-v} + \gamma(v)(\sqrt{P} - I)^2. \tag{2.6}$$

Multiplying the left and right sides of the inequality of (2.6) by U and U^*, we have

$$T + I \leq T^v + T^{1-v} + \gamma(v)(\sqrt{T} - I)^2,$$

let $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, the second inequality of (2.5) holds. For $0 \leq v \leq 1$, it easy to know that

$$\alpha(v) - \gamma(v) = \frac{1}{4} - (v - v^2) = (v - \frac{1}{2})^2 \geq 0.$$

Therefore, Theorem 2.3 is a refinement of the inequalities (1.2).

This completes the proof.

Theorem 2.4 Let $A, B, X \in M_n$ such that A, B are positive definite. Then

$$||AX + XB||_2^2 \leq ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 + \gamma(v)||AX - XB||_2^2, \tag{2.7}$$

where $v \in [0, 1]$, $\gamma(v) = \frac{5}{4} - (v - v^2)$.

Proof Since every positive definite matrix is unitarily diagonalizable, it follows that there exist unitary matrices $U, V \in M_n$ such that

$$A = UP_1U^*, B = VP_2V^*,$$

where $P_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), P_2 = diag(\mu_1, \mu_2, \cdots, \mu_n), \lambda_j, \mu_j > 0, 1 \leq j \leq n$.

Let $C = U^*XV = (c_{ij})$, then

$$\begin{aligned} A^vXB^{1-v} + A^{1-v}XB^v &= (UP_1U^*)^vX(VP_2V^*)^{1-v} + (UP_1U^*)^{1-v}X(VP_2V^*)^v \\ &= UP_1^v(U^*XV)P_2^{1-v}V^* + UP_1^{1-v}(U^*XV)P_2^vV^* \\ &= U(P_1^vCP_2^{1-v} + P_1^{1-v}CP_2^v)V^*, \end{aligned}$$

and

$$\begin{aligned} ||A^vXB^{1-v} + A^{1-v}XB^v||_2^2 &= ||P_1^vCP_2^{1-v} + P_1^{1-v}CP_2^v||_2^2 \\ &= \textstyle\sum_{i,j=1}^n (\lambda_i^v\mu_j^{1-v} + \lambda_i^{1-v}\mu_j^v)^2|c_{ij}|^2. \end{aligned}$$

Using the same method, we have

$$||AX+XB||_2^2=\sum_{i,j=1}^{n}(\lambda_i+\mu_j)^2|c_{ij}|^2,$$

$$||AX-XB||_2^2=\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|c_{ij}|^2.$$

By inequality (2.4), we obtain

$$\sum_{i,j=1}^{n}(\lambda_i^v\mu_j^{1-v}+\lambda_i^{1-v}\mu_j^v)^2|c_{ij}|^2+\gamma(v)\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|c_{ij}|^2\geq\sum_{i,j=1}^{n}(\lambda_i+\mu_j)^2|c_{ij}|^2.$$

Therefore, inequality (2.7) holds, it is a refinement of the inequality (1.5).

This completes the proof.

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几个改进的矩阵不等式

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摘要: 本文研究了矩阵不等式的问题. 利用两个新的标量不等式, 得到了矩阵的加权几何均值不等式和Hilbert-Schmidt范数不等式, 所得的结果改进了相应的不等式.

关键词: 标量不等式; 加权几何均值不等式; 正定矩阵; Hilbert-Schmidt范数

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