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A LICHNEROWICZ-OBATA TYPE ESTIMATE FOR L_E OPERATOR

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Abstract: In this paper, we investigate a class of eigenvalue problem of the L_E operator. By applying Bochner type formula, we obtain a Lichnerowicz-Obata type estimate for the first nonzero eigenvalue of this eigenvalue problem, which extends the results of [3] and [7] to the L_E case.

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1 Introduction

Let (M,g) be a compact Riemannian manifold, and let A be a smooth symmetric, positive definite (1,1)-tensor on M. Denote by ∇ and Δ the gradient operator and the Lapacian of M, respectively. Define the operator L_A as follows:

$$L_A = \mathrm{div} A\nabla.$$

It is easy to see that L_A is an elliptic operator, and if A is the $(1,1)$-tensor associated to the tensor g, $L_A = \Delta$.

For the eigenvalue problem

$$L_A(f) = -\lambda^A f, \text{ on } M, \tag{1.1}$$

when $L_A = \Delta$, Lichnerowicz [1] proved that if M is an n-dimensional compact Riemannian manifold with Ricci curvature bounded below by $(n-1)K, K > 0$, then the first nonzero eigenvalue λ_1 of the problem (1.1) satisfies $\lambda_1 \geq nK^2$. Moreover, one can get the fact that the above equality holds if and only if M is isometric to a sphere from the proof of Obata Theorem(see[2]). This is the so-called Lichnerowicz-Obata type estimate of Laplacian eigenvalue problem.

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In recent years, there are also many other Lichnerowicz-Obata type results about the first nonzero eigenvalue of the problem (1.1). When M is an n-dimensional compact immersed hypersurfaces of a space form and T_1 is the first Newton transformation associated to the shape operator of the immersion, [3] obtained a Lichnerowicz-Obata type estimate of the eigenvalue problem (1.1) with $A = T_1$. The operator L_{T_1} plays a key role in the study of stability for hypersurfaces with constant high order curvature (cf.[4,5,6]). Later, we [7] obtained more Lichnerowicz-Obata type estimates of the problem (1.1) with $A = T_r$(T_r is the rth Newton transformation, $2 \le r \le n-1$). In [3], they also considered the case $A = S$ (here S is the Schouten operator of an $n(n \ge 4)$-dimensional compact Riemannian manifold which has harmonic Weyl tensr), and proved a Lichnerowicz-Obata type result for L_S.

On the basis of the above researches, our aim in this paper is to establish Lichnerowicz-Obata type estimate for the eigenvalue problem (1.1) with $A = E$. E is the Einstein operator defined by $E = \frac{1}{2}RId - Ric$, where R is the scalar curvature and Ric is the linear operator associated with the Ricci tensor. This kind of L_E operator has very important application value both in general relativity and fuzzy mathematics(cf.[8]).

For this sake, we prove the following result:

Theorem 1.1 Let (M, g) be an n-dimensional compact Riemannian manifold and E be the Einstein operator on M. Suppose that Einstein operator E satisfies

$$0 < aId \le E \le bId,$$

where a, b are positive constants. Then, for the first nonzero eigenvalue λ_1^E of the problem (1.1) with $A = E$, we have

$$\lambda_1^E \ge \frac{nb}{2(nb-a)}[R_0 a - 2b^2 - \frac{1}{2}c + d]$$

where R_0 is the lower bound of the scalar curvature of M, c is the supremum of laplacian of the scalar curvature R and d is the infimum of laplacian of all eigenvalue functions of Ric.

Furthermore, the equalities hold if and only if M is a sphere.

2 Preliminaries

Let $\{\omega_1, \ldots, \omega_n\}$ be a locally orthonormal coframe field on the n-dimensional Riemannian manifold (M, g). Let $\phi = \Sigma_{i,j=1}^n \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric $(0, 2)$-tensor on M. Associated to tensor ϕ we have the $(1, 1)$-tensor, still denoted by ϕ, defined by

$$\langle \phi(X), Y \rangle = \phi(X, Y), \forall X, Y \in T(M),$$

and vice versa.

Then, we denote by Ric the Ricci tensor of M. Namely

$$Ric(X, Y) = \sum_{i=1}^n \langle Rm(X, e_i)Y, e_i \rangle,$$

where $Rm(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$ is the curvature tensor of M and $\{e_1,\ldots,e_n\}$ is an orthonormal frame. We will also denote by Ric the linear operator associated with the Ricci tensor, (i.e.,$Ric(X,Y) = \langle Ric(X), Y\rangle$), as well as its coordinates will be denoted by Ric_{ij}.

In [9], Cheng and Yau introduced an operator $\Box$ associated to ϕ by

$$\Box l = \sum_{i,j=1}^{n} \phi_{ij} l_{ij}, \tag{2.1}$$

where l is any smooth function on M.

Now, let us review two following basic properties of the operator $\Box$:

1. It follows from Cheng and Yau (Proposition 1 in [9]) that $\Box l = \mathrm{div}(\phi(\nabla l)) - \sum_{i=1}^{n}(\sum_{j=1}^{n}\phi_{ijj})l_i$.
2. One says that ϕ is divergence free if $\mathrm{div}\phi = 0$ or, equivalently, $\sum_{j=1}^{n}\phi_{ijj} = 0$, for all $1 \le i \le n$.

Remark If M is compact, it is not hard to check that $\Box$ is self-adjoint if and only if ϕ is divergence free from [9]. Of course, by the above properties, we know that $\Box f = \mathrm{div}(\phi(\nabla f))$ when ϕ is divergence free. If ϕ is symmetric and positive definite, then $\Box$ is strictly elliptic. therefore, we can assert that $\Box$ is strictly elliptic and self-adjoint when ϕ is divergence free, symmetric and positive definite. Furthermore, the spectrum of $\Box$ is discrete and it makes sense to consider the eigenvalue problem.

To prove the main theorem, we also need the following lemmas.

Lemma 2.1 Let (M,g) be a Riemannian manifold and E is the Einstein operator(Einstein tensor) on M. Then we have $\mathrm{div}E = 0$.

Proof It is well known that (see[10],P39) $\mathrm{div}Ric = \frac{1}{2}dR$. Then, we get

$$\mathrm{div}E = \mathrm{div}Ric - \frac{1}{2}\mathrm{div}(RId) = \frac{1}{2}dR - \frac{1}{2}dR = 0.$$

Lemma 2.2 (Bochner type formula[7]) Let (M,g) be an n-dimensional Riemannian manifold and $\phi = \Sigma_{i,j=1}^{n}\phi_{ij}\omega_i\otimes\omega_j$ be a divergence free, symmetric tensor defined on M. Then , for any smooth function $l : M \to \mathbb{R}$, we have,

$$\begin{aligned}\frac{1}{2}\mathrm{div}(\phi(\nabla|\nabla l|^2)) = \frac{1}{2}\Box(|\nabla l|^2) =& \langle\nabla l, \nabla(\mathrm{div}(\phi(\nabla l)))\rangle + \langle\phi(\nabla l), \nabla(\Delta l)\rangle \\ &+ 2\sum_{i,j,k=1}^{n}\phi_{ij}l_{ik}l_{kj} + 2\sum_{i,j,k,m=1}^{n} l_i l_j\phi_{im}R_{mkjk} \\ &- \sum_{i,j=1}^{n} l_i l_j\Delta\phi_{ij} + \sum_{k=1}^{n}(\sum_{i,j=1}^{n} l_i l_j(\phi_{jik} - \phi_{jki}))_k \\ &- \sum_{k=1}^{n}(\sum_{i,j=1}^{n} l_{ik}\phi_{ij}l_j)_k.\end{aligned} \tag{2.2}$$

For the proof details of lemma 2.2 , one can refer to the lemma 2.1 in [7].

Lemma 2.3 (Generalized Newton inequality[3]) Let P and Q be two $n\times n$ symmetric matrices. If Q is positive definite, then

$$tr(P^2Q) \geq \frac{[tr(PQ)]^2}{trQ} \tag{2.3}$$

and the equality holds if and only if $P=\alpha I$ for some $\alpha\in\mathbb{R}$.

Proof Let B be a positive definite matrix. By the fact $tr[(PQ)^2]\leq tr(P^2Q^2)$, which holds for symmetric matrices, and using the Cauchy-Schwarz inequality with $P\sqrt{B}$ and $(\sqrt{B})^{-1}Q$, one can obtian

$$[tr(PQ)]^2 = tr(P\sqrt{B}(\sqrt{B})^{-1}Q)^2 \leq tr(P^2B)tr(Q^2B^{-1}).$$

In particular, since Q is positive definite, we can choose $B=Q$ to obtain

$$[tr(PQ)]^2 \leq tr(P^2Q)trQ,$$

The equality holds if and only if

$$\begin{aligned} P\sqrt{Q} =& \alpha(\sqrt{Q})^{-1}Q \Leftrightarrow (P\sqrt{Q})\sqrt{Q} \\ =& \alpha(\sqrt{Q})^{-1}Q\sqrt{Q} \Leftrightarrow PQ=\alpha Q \Leftrightarrow P=\alpha I. \end{aligned}$$

3 Proof of the main Theorem

Proof of Theorem 1.1 Let f be an eigenfunction of λ_1^E, i.e. $L_Ef=-\lambda_1^Ef$. From Lemma 2.1, we know that E is divergence free. Now, by applying the Bochner type formula in Lemma 2.2 to tensor E and f, we obtain

$$\begin{aligned} \frac{1}{2}\mathrm{div}(E(\nabla|\nabla f|^2)) =& \langle\nabla f,\nabla(\mathrm{div}(E(\nabla f)))\rangle + \langle E(\nabla f),\nabla(\Delta f)\rangle \\ &+2\sum_{i,j,k=1}^{n}(E)_{ij}f_{ik}f_{kj} + 2\sum_{i,j,k,m=1}^{n}f_if_j(E)_{im}R_{mkjk} \\ &-\sum_{i,j=1}^{n}f_if_j\Delta(E)_{ij} + \sum_{k=1}^{n}(\sum_{i,j=1}^{n}f_if_j((E)_{jik}-(E)_{jki}))_k \\ &-\sum_{k=1}^{n}(\sum_{i,j=1}^{n}f_{ik}(E)_{ij}f_j)_k. \end{aligned} \tag{3.1}$$

By integrating both sides of (3.1) and using the divergence theorem, we have

$$\begin{aligned} 0 =& \int_M\langle\nabla f,\nabla(L_Ef)\rangle + \int_M\langle E(\nabla f),\nabla(\Delta f)\rangle + 2\int_M\sum_{i,j,k=1}^{n}(E)_{ij}f_{ik}f_{kj} \\ &+2\int_M\sum_{i,j,k,m=1}^{n}f_if_j(E)_{im}R_{mkjk} - \int_M\sum_{i,j=1}^{n}f_if_j\Delta E_{ij} \end{aligned} \tag{3.2}$$

Then, with the fact $L_E f = -\lambda_1^E f$, one have

$$\int_M \langle \nabla f, \nabla(L_E f)\rangle = -\lambda_1^E \int_M |\nabla f|^2. \tag{3.3}$$

We also have

$$\begin{aligned} \mathrm{div}(\Delta f\ E\nabla f)) =& \Delta f \mathrm{div}(E(\nabla f)) + \langle E(\nabla f), \nabla(\Delta f)\rangle \\ =& \Delta f \cdot L_E(\nabla f) + \langle E(\nabla f), \nabla(\Delta f)\rangle. \end{aligned} \tag{3.4}$$

Hence we get

$$\begin{aligned} \int_M \langle E(\nabla f), \nabla(\Delta f)\rangle =& -\int_M \Delta f \cdot L_E f \\ =& \lambda_1^E \int_M f\Delta f \\ =& -\lambda_1^E \int_M |\nabla f|^2. \end{aligned} \tag{3.5}$$

Then we estimate other parts in (3.2). For convenience, we choose an orthonormal frame $\{e_1, \ldots, e_n\}$ such that Ric is diagonalized in a neighborhood of any point $p \in M^n$, i.e. $Ric_{ij} = \mu_i \delta_{ij}$, where μ_i is eigenvalue of the Ricci tensor at point p.

Thus, for Einstein tensor $E = \frac{1}{2} RId - Ric$, at point p, we have

$$(E)_{ij} = \frac{1}{2} R\delta_{ij} - \mu_i \delta_{ij}. \tag{3.6}$$

From Lemma 2.3 and the fact E is positive definite, divergence free, we can obtain

$$\begin{aligned} 2\int_M \sum_{i,j,k=1}^n (E)_{ij} f_{ik} f_{kj} \geq& 2\int_M \frac{(\sum_{i,j=1}^n (E)_{ij} f_{ij})^2}{tr(E)} \\ =& 2\int_M \frac{(L_E f)^2}{tr(E)} \\ =& 2\int_M \frac{(\lambda_1^E f)^2}{tr(E)}. \end{aligned} \tag{3.7}$$

By appling divergence Theorem and the fact $L_E(f^2) = 2fL_E f + 2\langle E(\nabla f), \nabla f\rangle$, we have

$$\int_M \langle E(\nabla f), \nabla f\rangle = -\int_M f \cdot \lambda_1^E f = \lambda_1^E \int_M f^2. \tag{3.8}$$

With the condition $0 < aId \leq E \leq bId$, it is easy to check that

$$a|\nabla f|^2 \leq \langle E(\nabla f), \nabla f\rangle \leq b|\nabla f|^2. \tag{3.9}$$

Then, from (3.7),(3.8) and (3.9), we obtain

$$2\int_M \sum_{i,j,k=1}^n (E)_{ij} f_{ik} f_{kj} \geq \frac{2\lambda_1^E a}{nb} \int_M |\nabla f|^2. \tag{3.10}$$

We can also obtain

$$
\begin{aligned}
2\int_M \sum_{i,j,k,m=1}^{n} f_i f_j (E)_{im} R_{mkjk} =& 2\int_M Ric(\nabla f, E(\nabla f)) \\
=& 2\int_M \langle Ric(\nabla f), E(\nabla f)\rangle \\
=& 2\int_M \langle (\frac{1}{2} RId - E)\nabla f, E(\nabla f)\rangle \\
=& \int_M [R\langle \nabla f, E(\nabla f)\rangle - 2\langle \nabla f, E^2(\nabla f)\rangle] \\
\geq& (R_0 a - 2b^2)\int_M |\nabla f|^2,
\end{aligned}
\tag{3.11}
$$

where R_0 is the lower bound of the scalar curvature R.

For any function $h \in C^2(M)$, by the Hopf maximum principle, it is not hard to find that $(\Delta h)_{min} \leq 0$ and $(\Delta h)_{max} \geq 0$ on M. Then, let d be the infimum of laplacian of all eigenvalue functions of Ric. Hence we have $c \triangleq (\Delta R)_{max} \geq 0$ and $d \leq 0$.

Under the above frame, we have

$$
\begin{aligned}
-\sum_{i,j=1}^{n} f_i f_j \Delta E_{ij} =& -\sum_{i,j,p=1}^{n} f_i f_j E_{ijpp} \\
=& -\Delta R \sum_{i=1}^{n} f_i^2 + \sum_{i}^{n} (\Delta \mu_i) f_i^2 \\
\geq& (-c + d)|\nabla f|^2.
\end{aligned}
\tag{3.12}
$$

Hence

$$
-\int_M \sum_{i,j=1}^{n} f_i f_j \Delta E_{ij} \geq (-c + d)\int_M |\nabla f|^2. \tag{3.13}
$$

Finally, taking (3.3), (3.5), (3.10), (3.11) and (3.13) back into (3.2), we obtain

$$
0 \geq [-2\lambda_1^E + \frac{2\lambda_1^E a}{nb} + R_0 a - 2b^2 - c + d]\int_M |\nabla f|^2. \tag{3.14}
$$

Therefore

$$
\lambda_1^E \geq \frac{nb}{2(nb-a)}[R_0 a - 2b^2 - \frac{1}{2}c + d]. \tag{3.15}
$$

Now, we consider the equality case. If we suppose $M = \mathbb{S}^n(1)$, we have $R_0 = n(n-1)$, $E = \frac{(n-1)(n-2)}{2} Id$, $L_E f = \frac{(n-1)(n-2)}{2}\Delta f$ and $\lambda_1^E = \frac{n(n-1)(n-2)}{2}$. In this case, the estimate becomes euqality with the assumption that $a = b = \frac{(n-1)(n-2)}{2}$. On the other hand, if the equality holds, the equality case of Lemma 2.3, implies that $f_{ij} = pg_{ij}$, for some real constant p, and following the proof of Obata Theorem, cf [2], we can obtain that M is a sphere.

Remark There is still much to be studied about the Lichnerowicz-Obata type estimate of this kind of problem (1.1). Especially, when the ambient space of M is an Einstein manifold, the first nonzero eigenvalue λ_1^T is of great significance to the study of variational problem that characterizes hypersurfaces with constant 2-mean curvature in Einstein manifolds (cf.[11]). To the author's knowledge, the Lichnerowicz-Obata type estimate of L_{T_1} in this kind of ambient space is still wide open.

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L_E 算子的一个Lichnerowicz-Obata 型估计

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摘要: 本文研究了L_E算子的一类特征值问题. 利用Bochner 型公式, 我们得到了此类问题第一非零特征值的一个Lichnerowicz-Obata 型估计, 进而将[3]和[7]中的结果推广到了L_E算子的情形.

关键词: 第一非零特征值; Bochner 型公式; 椭圆算子; Einstein 张量

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