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EXISTENCE OF CLASSICAL SOLUTIONS FOR THE INCOMPRESSIBLE LIQUID CRYSTAL MODEL

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Abstract: In this paper we study a system of nonlinear partial differential equations modelling the electrokinetics of a nematic electrolyte material consisting of various ions species contained in a nematic liquid crystal. [The system coupling a Nernst-Planck system for the ions concentrations with a Maxwell's equation of electrostatics governing the evolution of the electrostatic potential, an incompressible Navier-Stokes equation for the velocity field and a non-smooth Allen-Cahn type equation for the nematic director field.] We use energy method to prove the local existence of classical solutions with large initial data and the global existence of classical solutions with small initial data.

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1 Introduction

In the 1960s, Ericksen[2, 3] and Leslie[8] established the kinetic theory of liquid crystal models. Lin-Liu[9] proved the global existence of the weak solutions and the classical solutions of a simplified Ericksen-Leslie liquid crystal equation, they also discussed uniqueness and some stability properties of the system. For the general Ericksen-Leslie system, Lin-Liu[10] proved the existence of classical solutions and the asymptotic stability of the solutions. Jiang-Luo[6] proved the global existence of classical solutions with small initial data for Ericksen-Leslie's hyperbolic incompressible liquid crystal model. Then Jiang-Luo-Tang[7] attained the the global existence of classical solutions with small initial data for the corresponding compressible system.

In this paper, we consider the system derived in [1] and modified by [4]. It describes the electrokinetics of a nematic electrolyte that consists of ions that diffuse and advect in a nematic liquid crystal environment, assuming certain simplifications commonly used in the mathematical literature on liquid crystals.

The system can be written in terms of the following variables:

* the vector n modelling the local orientation of the nematic liquid crystal molecules,

* the macroscopic velocity v of the liquid crystal molecules,

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* the pressure p resulting from the incompressibility constraint on the fluid,

* the electrostatic potential Φ,

* C_p and C_m denote the density of positive and negative charges.

Then the system takes the form

$$\frac{\partial C_p}{\partial t} + v \cdot \nabla C_p = \mathrm{div}((Id + \eta n \otimes n)\nabla C_p + C_p \nabla \Phi), \tag{1.1}$$

$$\frac{\partial C_m}{\partial t} + v \cdot \nabla C_m = \mathrm{div}((Id + \eta n \otimes n)\nabla C_m - C_m \nabla \Phi), \tag{1.2}$$

$$-\mathrm{div}((Id + \eta n \otimes n)\nabla \Phi) = C_p - C_m, \tag{1.3}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla p =& \alpha_4 \mathrm{div} D(v) - \mathrm{div}(\nabla n \odot \nabla n) \\ &+ \mathrm{div}((\nabla \Phi \otimes \nabla \Phi)(Id + \eta n \otimes n)) \\ &+ \mathrm{div}(\alpha_1 (D(v)n \cdot n) n \otimes n + \alpha_2 \mathring{n} \otimes n + \alpha_3 n \otimes \mathring{n}) \\ &+ \mathrm{div}(\alpha_5 D(v) n \otimes n + \alpha_6 n \otimes D(v) n),\end{aligned} \tag{1.4}$$

$$\mathrm{div} v = 0, \tag{1.5}$$

$$n_t + v \cdot \nabla n - \Omega(v) n + D(v) n = \triangle n + \eta(\nabla \Phi \otimes \nabla \Phi) n - F'(|n|^2) n, \tag{1.6}$$

$$(v, n, C_p, C_m, \Phi)|_{t=0} = (v^{in}, n^{in}, C_p^{in}, C_m^{in}, \Phi^{in}), \tag{1.7}$$

where

$$D(v) = \frac{\nabla v + \nabla v^T}{2}, \quad \Omega(v) = \frac{\nabla v - \nabla v^T}{2}, \quad \mathring{n} = n_t + v \cdot \nabla n - \Omega(v) n$$

and η is a constant that can be small enough. The coefficients satisfy the following relation

$$\begin{aligned}&\alpha_k > 0, \quad k = 1, 4, 5, 6; \\ &\alpha_6 - \alpha_5 = 1, \quad \alpha_6 + \alpha_5 \geq 1, \quad \alpha_2 = 0, \quad \alpha_3 = -1.\end{aligned} \tag{1.8}$$

According to [1], this relation is necessary to ensure the variational structure of the system of equations and thus the equivalency of the equation of balance of linear momentum to that derived via the Onsager's principle.We denote the material derivative $\partial_t + v \cdot \nabla$, and $\dot{n} = \partial_t n + v \cdot \nabla n$ represents the material derivative of n.

The Nernst-Planck type equations (1.1)-(1.2) correspond to the continuity equation for ions with the electric potential Φ satisfying the Maxwell's equation of electrostatics (1.3). The Navier-Stokes equations (1.4), with the incompressibility constraint (1.5), rule the evolution of the liquid crystal flow.

2 Main Results

In this section, we will state our main results and give the outline of this paper. We consider the case $s = 4$ for simplicity. In fact, the method works equally well for $s > 4$ cases. We introduce the following energy function

$$E(t) = \|v\|_{H^s}^2 + \|\nabla n\|_{H^s}^2 + \|C_p\|_{H^s}^2 + \|C_m\|_{H^s}^2,$$

the energy-dissipation

$$D(t) = \|\nabla C_p\|_{H^s}^2 + \|\nabla C_m\|_{H^s}^2 + \frac{\alpha_4}{2}\|\nabla v\|_{H^s}^2 + \|\dot{n}\|_{H^s}^2$$

and initial energy

$$E^{in} = \|v^{in}\|_{H^s}^2 + \|\nabla n^{in}\|_{H^s}^2 + \|C_p^{in}\|_{H^s}^2 + \|C_m^{in}\|_{H^s}^2.$$

Theorem 2.1 If v^{in}, ∇n^{in}, C_p^{in}, C_m^{in}, $\nabla\Phi^{in} \in H^s(\mathbb{T}^3)$, $|n^{in}| \le 1$, then there exists a $T > 0$, such that the Cauchy problem of the system (1.1)–(1.7) admits a solution

$$\begin{aligned} v, C_p, C_m, \nabla\Phi &\in L^\infty(0,T;H^s(\mathbb{T}^3)) \cap L^2(0,T;H^{s+1}(\mathbb{T}^3)), \\ \nabla n &\in L^\infty(0,T;H^s(\mathbb{T}^3)), \quad \dot{n} \in L^2(0,T;H^{s+1}(\mathbb{T}^3)). \end{aligned}$$

Moreover, the solution (v, n, C_p, C_m, Φ) satisfies

$$\begin{aligned} &\sup_{0\le t\le T}(\|v\|_{H^s}^2 + \|\nabla n\|_{H^s}^2 + \|C_p\|_{H^s}^2 + \|C_m\|_{H^s}^2) \\ &+ \int_0^T (\|\nabla C_p\|_{H^s}^2 + \|\nabla C_m\|_{H^s}^2 + \frac{\alpha_4}{2}\|\nabla v\|_{H^s}^2 + \|\dot{n}\|_{H^s}^2) d\tau \le M, \end{aligned}$$

where M and T depend on E^{in} and the coefficients.

Theorem 2.2 There is a constant $\epsilon_0 > 0$, if $E^{in} \le \epsilon_0$, then the system (1.1)–(1.7) exist a global solution

$$\begin{aligned} v, C_p, C_m, \nabla\Phi &\in L^\infty(0,\infty;H^s(\mathbb{T}^3)) \cap L^2(0,\infty;H^{s+1}(\mathbb{T}^3)), \\ \nabla n &\in L^\infty(0,\infty;H^s(\mathbb{T}^3)), \quad \dot{n} \in L^2(0,\infty;H^{s+1}(\mathbb{T}^3)). \end{aligned}$$

what's more, the solution (C_p, C_m, Φ, v, n) satisfies

$$\begin{aligned} &\sup_{t\ge 0}(\|v\|_{H^s}^2 + \|\nabla n\|_{H^s}^2 + \|C_p\|_{H^s}^2 + \|C_m\|_{H^s}^2) \\ &+ \int_0^\infty (\|\nabla C_p\|_{H^s}^2 + \|\nabla C_m\|_{H^s}^2 + \frac{\alpha_4}{2}\|\nabla v\|_{H^s}^2 + \|\dot{n}\|_{H^s}^2) d\tau \le CE^{in}, \end{aligned}$$

where C is independent of (C_p, C_m, Φ, v, n).

The rest of this paper can be organized as follows. In section 3, we establish a priori estimate of system (1.1)–(1.7). In section 4, we show the local existence of (C_p, C_m) for a given (v, n, Φ) and the local existence of (v, n) for a given Φ, which will be employed in the constructing the iterative approximate system of (1.1)–(1.7). In section 5, we construct the approximate system of (1.1)–(1.7) by iteration. In section 6, we prove the local well-posedness of (1.1)–(1.7) with large initial data by obtaining uniform energy bounds of the iterative system (5.1). In section 7, we globally extend the solution of (1.1)–(1.7) constructed in section 6 under the small initial energy condition with the same coefficients.

3 The a Priori Estimate

In this section we derive the a priori estimate. We assume (C_p, C_m, Φ, v, n) is a smooth solution of the system.

Lemma 3.1 Assuming (C_p, C_m, Φ, v, n) is a smooth solution to the system (1.1)–(1.7). Then there exists a constant $C > 0$ such that

$$\|\nabla\Phi\|_{H^4}^2 \le C(\|C_p\|_{H^4}^2 + \|C_m\|_{H^4}^2 + \|\nabla n\|_{H^4}^{10}) \le CE(t)^5. \tag{3.1}$$

Proof Thanks to Lemma2 of [1], we have $\Phi \in L^\infty(0, +\infty; L^\infty(\mathbb{T}^3))$, which is the key point to prove this lemma. For all $0 \le k \le s$, we act ∇^k on equation(1.3)and take L^2-inner product with $\nabla^k\Phi$.

$$\begin{aligned}
\|\nabla^k\nabla\Phi\|_{L^2}^2 =& \langle\nabla^k(C_p - C_m), \nabla^k\Phi\rangle - \eta\langle\nabla^k[(n\otimes n)\nabla\Phi], \nabla^k\nabla\Phi\rangle \\
\le& (\|\nabla^k C_p\|_{L^2} + \|\nabla^k C_m\|_{L^2})\|\nabla^k\Phi\|_{L^2} - \eta\langle n_i n_j \nabla^k\partial_j\Phi, \nabla^k\partial_i\Phi\rangle \\
& - \eta \sum_{\substack{a+b+c=k \\ c\le k}} \langle\nabla^a n_i \nabla^b n_j \nabla^c\Phi, \nabla^k\partial_i\Phi\rangle \\
\le& C(\|C_p\|_{H^4}^2 + \|C_m\|_{H^4}^2) + \frac{\|\nabla^k\nabla\Phi\|_{L^2}^2}{4} + C\eta\|\nabla\Phi\|_{H^4}\|\Phi\|_{H^4}\|n\|_{H^s}^2 \\
\le& C(\|C_p\|_{H^4}^2 + \|C_m\|_{H^4}^2) + \frac{\|\nabla^k\nabla\Phi\|_{L^2}^2}{4} + C\eta\|\nabla n\|_{H^4}^{\frac{10}{7}}\|\nabla\Phi\|_{H^4}^{\frac{12}{7}},
\end{aligned} \tag{3.2}$$

where we use the following Sobolev interpolation inequality $\|f\|_{H^4} < C\|f\|_{L^\infty}^{\frac{2}{7}}\|\nabla f\|_{H^4}^{\frac{5}{7}}$. Using Young inequality and summing up for all $0 \le k \le s$, we arrive at

$$\|\nabla\Phi\|_{H^4}^2 \le C(\|C_p\|_{H^4}^2 + \|C_m\|_{H^4}^2 + \|\nabla n\|_{H^4}^{10}).$$

Lemma 3.2 Assuming (C_p, C_m, Φ, v, n) is a smooth solution to the system. Then

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k C_p\|_{L^2}^2 + \|\nabla^k\nabla C_p\|_{L^2}^2 \\
\le& C\|v\|_{H^s}\|C_p\|_{H^s}^2 + C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s} \\
&+ C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\|\nabla\Phi\|_{H^s} + C\|\nabla C_p\|_{H^s}\|C_p\|_{H^s}\|\nabla\Phi\|_{H^s}, \\
&\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k C_m\|_{L^2}^2 + \|\nabla^k\nabla C_m\|_{L^2}^2 \\
\le& C\|v\|_{H^s}\|C_m\|_{H^s}^2 + C\eta\|n\|_{H^s}^2\|C_m\|_{H^s}\|\nabla C_m\|_{H^s} \\
&+ C\eta\|n\|_{H^s}^2\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}\|\nabla\Phi\|_{H^s} + C\|\nabla C_m\|_{H^s}\|C_m\|_{H^s}\|\nabla\Phi\|_{H^s}.
\end{aligned}$$

Proof For all $0 \le k \le s$, via acting ∇^k to equation (1.1), and taking L^2-inner product with $\nabla^k C_p$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^k C_p\|_{L^2}^2 + \|\nabla^k \nabla C_p\|_{L^2}^2 =& -\langle \nabla^k(v\cdot\nabla C_p), \nabla^k C_p\rangle - \eta\langle \nabla^k((n\otimes n)\nabla C_p), \nabla^k\nabla C_p\rangle \\ & - \eta\langle \nabla^k((n\otimes n)C_p\nabla\Phi), \nabla^k\nabla C_p\rangle - \langle \nabla^k(C_p\nabla\Phi), \nabla^k\nabla C_p\rangle \\ :=& \sum_{i=1}^4 A_i.\end{aligned}$$

Now we estimate the four terms on the right-hand side term by term for $0 \le k \le s$.

We take advantage of the Holder inequality, Sobolev embedding inequality and the fact that $\mathrm{div} v = 0$ to get that

$$\begin{aligned}A_1 = -\langle \nabla^k(v\cdot\nabla C_p), \nabla^k C_p\rangle =& -\langle \nabla^k v\cdot\nabla C_p, \nabla^k C_p\rangle - \langle v\cdot\nabla^k\nabla C_p, \nabla^k C_p\rangle \\ & - \langle \nabla v\cdot\nabla^k C_p, \nabla^k C_p\rangle - \sum_{\substack{a+b=k\\ a,b\le k-1}} \langle |\nabla^a v||\nabla^b C_p|, \nabla^k C_p\rangle \\ \le& \|v\|_{H^s}\|C_p\|_{H^s}^2 + \sum_{\substack{a+b=k\\ a,b\le k-1}} \|\nabla^a v\|_{L^4}\|\nabla^b C_p\|_{L^4}\|\nabla^k C_p\|_{L^2} \\ \le& C\|v\|_{H^s}\|C_p\|_{H^s}^2.\end{aligned}$$

For A_2, thanks to $\eta\langle((n\otimes n)\nabla^k\nabla C_p), \nabla^k\nabla C_p\rangle \ge 0$ and $s-\frac{3}{2} > 2$, we estimate that

$$\begin{aligned}A_2 =& -\eta\langle \nabla^k((n\otimes n)\nabla C_p), \nabla^k\nabla C_p\rangle \\ \le& -\eta\langle((n\otimes n)\nabla^k\nabla C_p), \nabla^k\nabla C_p\rangle + \eta\langle |\nabla^k n||n||\nabla C_p|, \nabla^k\nabla C_p\rangle \\ & + \eta\langle |\nabla^{k-1}n||\nabla n||\nabla C_p|, \nabla^k\nabla C_p\rangle + \eta\langle |\nabla^{k-1}n||n||\nabla^2 C_p|, \nabla^k\nabla C_p\rangle \\ & + \eta\langle |\nabla n||n||\nabla^k C_p|, \nabla^k\nabla C_p\rangle + \sum_{\substack{a+b+c+d=k\\ a,b,c\le k-2}} \langle |\nabla^a n||\nabla^b n||\nabla^c C_p|, \nabla^k\nabla C_p\rangle \\ \le& \eta\|\nabla n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s} + \eta \sum_{\substack{a+b+c+d=k\\ a,b,c\le k-2}} \|\nabla^a n\|_{L^6}\|\nabla^b n\|_{L^6}\|\nabla^c C_p\|_{L^6}\|\nabla^k\nabla C_p\|_{L^2} \\ \le& C\eta\|\nabla n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}.\end{aligned}$$

For A_3 and A_4, we use Hölder inequality and Sobolev embedding inequality to get

$$\begin{aligned}A_3 =& \eta\langle \nabla^k((n\otimes n)C_p\nabla\Phi), \nabla^k\nabla C_p\rangle \\ \le& -\eta\langle((n\otimes n)\nabla^k C_p\nabla\Phi), \nabla^k\nabla C_p\rangle \\ & + \eta\langle |\nabla^k n||n||C_p||\nabla\Phi|, \nabla^k\nabla C_p\rangle + \eta\langle |n||n||C_p||\nabla^k\nabla\Phi|, \nabla^k\nabla C_p\rangle \\ & + \eta\langle |\nabla^{k-1}n|[|\nabla n||C_p||\nabla\Phi| + |n||\nabla C_p||\nabla\Phi| + |n||C_p||\nabla^2\Phi|], \nabla^k\nabla C_p\rangle \\ & + \eta\langle |\nabla^{k-1}C_p|[|\nabla n||n||\nabla\Phi| + |n||n||\nabla^2\Phi|], \nabla^k\nabla C_p\rangle \\ & + \eta\langle |\nabla^{k-1}\nabla\Phi|[|\nabla n||n||C_p| + |n|||n|\nabla C_p|], \nabla^k\nabla C_p\rangle\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{a+b+c+d=k\\a,b,c,d\le k-2}}\langle|\nabla^a n||\nabla^b n||\nabla^c C_p||\nabla^d\nabla\Phi|,\nabla^k\nabla C_p\rangle\\
&\le C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\|\nabla\Phi\|_{H^s},
\end{aligned}
$$

$$
\begin{aligned}
A_4=\langle\nabla^k(C_p\nabla\Phi),\nabla^k\nabla C_p\rangle\le&\langle|\nabla^k C_p||\nabla\Phi|,\nabla^k\nabla C_p\rangle+\langle|(C_p||\nabla^k\nabla\Phi|,\nabla^k\nabla C_p\rangle\\
&+\sum_{\substack{a+b=k\\a,b\le k-1}}\langle|\nabla^a n||\nabla^b\nabla\Phi|,\nabla^k\nabla C_p\rangle\\
\le&C\|\nabla C_p\|_{H^s}\|C_p\|_{H^s}\|\nabla\Phi\|_{H^s}.
\end{aligned}
$$

Combing all these estimates, we obtain

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k C_p\|_{L^2}^2+\|\nabla^k\nabla C_p\|_{L^2}^2\le& C\|v\|_{H^s}\|C_p\|_{H^s}^2+C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\\
&+C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\|\nabla\Phi\|_{H^s}+C\|\nabla C_p\|_{H^s}\|C_p\|_{H^s}\|\nabla\Phi\|_{H^s}.
\end{aligned}
$$

Similarly, for C_m we get the following inequality

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k C_m\|_{L^2}^2+\|\nabla^k\nabla C_m\|_{L^2}^2\le C\|v\|_{H^s}\|C_m\|_{H^s}^2+C\eta\|n\|_{H^s}^2\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}\\
&+C\eta\|n\|_{H^s}^2\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}\|\nabla\Phi\|_{H^s}+C\|\nabla C_m\|_{H^s}\|C_m\|_{H^s}\|\nabla\Phi\|_{H^s}.
\end{aligned}
$$

Lemma 3.3 If (C_p,C_m,Φ,v,n) is a smooth solution to the system(1.1)–(1.7), for all $0\le k\le s$, we have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|\nabla^k v\|_{L^2}^2+\|\nabla^k\nabla n\|_{L^2}^2)+\tfrac{\alpha_4}{2}\|\nabla^k\nabla v\|_{L^2}^2+\|\nabla^k\dot{n}\|_{L^2}^2\\
\le&+C\|n\|_{H^s}^4\|v\|_{H^s}\|\nabla v\|_{H^s}+C\|v\|_{H^s}^3+C\|\nabla n\|_{H^s}^3\|\dot{n}\|_{H^s}\\
&+C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\eta\|\nabla\Phi\|_{H^s}^2(\|n\|_{H^s}^2\|\nabla v\|_{H^s}+\|n\|_{H^s}\|\dot{n}\|_{H^s})\\
&+C\|\nabla\Phi\|_{H^s}^2\|\nabla v\|_{H^s}+C\|\nabla n\|_{H^s}\|v\|_{H^s}\|\dot{n}\|_{H^s}+C\|\nabla n\|_{H^s}^2\|v\|_{H^s}\|\nabla v\|_{H^s}.
\end{aligned}
$$

Proof For all $0\le k\le s$, we act ∇^k on equation(1.4) and take L^2-inner product with $\nabla^k v$, we obtain

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k v\|_{L^2}^2+\tfrac{\alpha_4}{2}\|\nabla^k\nabla v\|_{L^2}^2=&-\langle\nabla^k(v\cdot\nabla v),\nabla^k v\rangle+\langle\nabla^k((\nabla n\odot\nabla n),\nabla^k\nabla v\rangle\\
&-\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)(Id+\eta n\otimes n)),\nabla^k\nabla v\rangle\\
&-\langle\nabla^k(\alpha_1(D(v)n\cdot n)n\otimes n+\alpha_2\mathring{n}\otimes n+\alpha_3 n\otimes\mathring{n}),\nabla^k\nabla v\rangle\\
&-\langle\nabla^k(\alpha_5 D(v)n\otimes n+\alpha_6 n\otimes D(v)n),\nabla^k\nabla v\rangle. \qquad (3.3)
\end{aligned}
$$

Again, via acting ∇^k on equation (1.6) and taking L^2-inner product with $\nabla^k\dot{n}$, we get

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k\nabla n\|_{L^2}^2+\|\nabla^k\dot{n}\|_{L^2}^2=&\langle\nabla^k(\Omega(v)n-(D(v)n)),\nabla^k\dot{n}\rangle+\langle\nabla^k\triangle n,\nabla^k(v\cdot\nabla n)\rangle\\
&+\eta\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)),\nabla^k\dot{n}\rangle-\langle\nabla^k(F'(|n|^2)n),\nabla^k\dot{n}\rangle. \qquad (3.4)
\end{aligned}
$$

We add up (3.3) and (3.4) to get

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|\nabla^k v\|_{L^2}^2+\|\nabla^k\nabla n\|_{L^2}^2)+\frac{\alpha_4}{2}\|\nabla^k\nabla v\|_{L^2}^2+\|\nabla^k\dot n\|_{L^2}^2\\
=&-\langle\nabla^k(v\cdot\nabla v),\nabla^k v\rangle-\langle\nabla^k(\alpha_1(D(v)n\cdot n)n\otimes n),\nabla^k\nabla v\rangle+\langle\nabla^k(F'(|n|^2)n),\nabla^k\dot n\rangle\\
&+\langle\nabla^k\triangle n,\nabla^k(v\cdot\nabla n)\rangle+\langle\nabla^k((\nabla n\odot\nabla n),\nabla^k\nabla v\rangle+\eta\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)n),\nabla^k\dot n\rangle\\
&-\eta\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)(n\otimes n)),\nabla^k\nabla v\rangle-\langle\nabla^k(\nabla\Phi\otimes\nabla\Phi),\nabla^k\nabla v\rangle\\
&-\langle\nabla^k(D(v)n),\nabla^k\dot n\rangle+\langle\nabla^k(\Omega(v)n),\nabla^k\dot n\rangle-\langle\nabla^k(\alpha_2\dot n\otimes n+\alpha_3 n\otimes\dot n),\nabla^k\nabla v\rangle\\
&-\langle\nabla^k(\alpha_5 D(v)n\otimes n+\alpha_6 n\otimes D(v)n),\nabla^k\nabla v\rangle+\langle\nabla^k(\alpha_2\Omega(v)n\otimes n+\alpha_3 n\otimes\Omega(v)n),\nabla^k\nabla v\rangle\\
:=&\sum_{i=1}^{5}B_i.
\end{aligned}
\tag{3.5}
$$

As what we do before, we estimate the terms on the right-hand side term by term.

Thanks to $\mathrm{div} v=0$, we have

$$
\begin{aligned}
B_1^1=&-\langle\nabla^k(v\cdot\nabla v),\nabla^k v\rangle\\
\le&-\langle\nabla^k v\cdot\nabla v,\nabla^k v\rangle-\langle v\cdot\nabla^k\nabla v,\nabla^k v\rangle-\langle\nabla v\cdot\nabla^k v,\nabla^k v\rangle+\sum_{\substack{a+b=k\\a,b\le k-1}}\langle|\nabla^a v||\nabla^b v|,|\nabla^k v|\rangle\\
\le& C\|v\|_{H^s}^3.
\end{aligned}
\tag{3.6}
$$

Since

$$
\begin{aligned}
-\alpha_1\langle\nabla^k(D(v)n\cdot n)n\otimes n,\nabla^k\nabla v\rangle=&-\alpha_1\langle\nabla^k D(v)_{pq}n_pn_qn_in_j,\nabla^k D(v)_{ij}+\nabla^k\Omega(v)_{ij}\rangle\\
=&-\alpha_1|n^T\nabla^k D(v)n|^2\le 0,
\end{aligned}
$$

we estimate that

$$
\begin{aligned}
B_1^2=&-\alpha_1\langle\nabla^k((D(v)n\cdot n)n\otimes n),\nabla^k\nabla v\rangle\\
\le&-\alpha_1\langle\nabla^k D(v)_{pq}n_pn_qn_in_j,\nabla^k D(v)_{ij}+\nabla^k\Omega(v)_{ij}\rangle+C\langle|\nabla^k v||\nabla n||n||n||n|,|\nabla^k\nabla v|\rangle\\
&+C\langle|\nabla v||\nabla^k n||n||n||n|,|\nabla^k\nabla v|\rangle+C\langle|\nabla^2 v||\nabla^{k-1}n||n||n||n|,|\nabla^k\nabla v|\rangle\\
&+C\langle|\nabla^{k-1}v||\nabla^2 n||n||n||n|,|\nabla^k\nabla v|\rangle+C\langle|\nabla^{k-1}v||\nabla n||\nabla n||n||n|,|\nabla^k\nabla v|\rangle\\
&+\sum_{\substack{a+b+c+d=k+1\\a,b,c,d\le k-2}}C\langle|\nabla^a v||\nabla^b n||\nabla^c n||\nabla^d n||\nabla^e n|,|\nabla^k\nabla v|\rangle\\
\le& C\|n\|_{H^s}^4\|v\|_{H^s}\|\nabla v\|_{H^s}\\
&+\sum_{\substack{a+b+c+d=k+1\\a,b,c,d\le k-2}}C|\nabla^a v|_{L^8}|\nabla^b n|_{L^8}|\nabla^c n|_{L^8}|\nabla^d n|_{L^8}|\nabla^e n|_{L^8}|\nabla^k\nabla v|_{L^2}\\
\le& C\|n\|_{H^s}^4\|v\|_{H^s}\|\nabla v\|_{H^s}.
\end{aligned}
\tag{3.7}
$$

Using $F(r) \in C_0^\infty(-\frac{1}{2}, \frac{3}{2})$, we obtain

$$\begin{aligned}
B_1^3 =& -\langle \nabla^k (F'(|n|^2)n), \nabla^k \dot{n}\rangle \\
\leq& C\langle |\nabla^{k-1}n||\nabla n||n|, |\nabla^k \dot{n}|\rangle + C\langle |\nabla^k n||n||n|, |\nabla^k \dot{n}|\rangle + C\sum_{\substack{a+b+c=k\\ a,b,c\leq k-2}} \langle |\nabla^a n||\nabla^b n||\nabla^c n|, |\nabla^k \dot{n}|\rangle \\
\leq& C\|\nabla n\|_{H^s}^3 \|\dot{n}\|_{H^s} + C\sum_{\substack{a+b+c=k\\ a,b,c\leq k-2}} |\nabla^a n|_{L^6}|\nabla^b n|_{L^6}|\nabla^c n|_{L^6}|\nabla^k \dot{n}|_{L^2} \\
\leq& C\|\nabla n\|_{H^s}^3 \|\dot{n}\|_{H^s}.
\end{aligned} \tag{3.8}$$

(3.6) along with (3.7) and (3.8) yield that

$$\begin{aligned}
B_1 =& -\langle \nabla^k (v\cdot\nabla v), \nabla^k v\rangle - \alpha_1\langle \nabla^k((D(v)n\cdot n)n\otimes n), \nabla^k\nabla v\rangle - \langle \nabla^k (F'(|n|^2)n), \nabla^k \dot{n}\rangle \\
\leq& C\|n\|_{H^s}^4\|v\|_{H^s}\|\nabla v\|_{H^s} + C\|v\|_{H^s}^3 + C\|\nabla n\|_{H^s}^3\|\dot{n}\|_{H^s}.
\end{aligned} \tag{3.9}$$

For B_2, thanks to $\mathrm{div} v = 0$, we have $\langle \nabla^k \partial_p n_i, v_j\partial_j\partial_p\nabla^k n_i\rangle = 0$. So we estimate that

$$\begin{aligned}
B_2 =& \langle \nabla^k((\nabla n\odot\nabla n), \nabla^k\nabla v\rangle + \langle \nabla^k \triangle n, \nabla^k(v\cdot\nabla n)\rangle \\
=& \langle \nabla^k(\partial_i n_k\partial_j n_k), \nabla^k\partial_j v_i\rangle + \langle \nabla^k\partial_p\partial_p n_i, \nabla^k(v_j\partial_j n_i)\rangle \\
=& \langle \nabla^k(\partial_i n_k\partial_j n_k), \nabla^k\partial_j v_i\rangle - \langle \nabla^k\partial_p n_i, \nabla^k(\partial_p v_j\partial_j n_i)\rangle - \langle \nabla^k\partial_p n_i, \nabla^k(v_j\partial_j\partial_p n_i)\rangle \\
\leq& C\langle |\nabla^k\nabla n||\nabla n|, |\nabla^k\nabla v|\rangle + C\sum_{\substack{a+b=k\\ 1\leq a,b\leq k-1}} \langle |\nabla^a\nabla n||\nabla^b\nabla n|, |\nabla^k\nabla v|\rangle + C\langle |\nabla^k\nabla n|, |\nabla^k\nabla v||\nabla n|\rangle \\
&+ C\langle |\nabla^k\nabla n|, |\nabla^k\nabla n||\nabla v|\rangle + C\sum_{\substack{a+b=k\\ 1\leq a,b\leq k-1}} \langle |\nabla^a\nabla n||\nabla^b\nabla v|, |\nabla^k\nabla n|\rangle \\
&+ C\langle |\nabla^k\nabla n|, |\nabla^k v||\nabla^2 n|\rangle + C\langle |\nabla^k\nabla n|, |\nabla v||\nabla^{k+1}n|\rangle + C\sum_{\substack{a+b=k\\ 2\leq a\leq k-1}} \langle |\nabla^a v||\nabla^b\nabla n|, |\nabla^k\nabla n|\rangle \\
\leq& C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s} + C\sum_{\substack{a+b=k\\ 1\leq a,b\leq k-1}} |\nabla^a\nabla n|_{L^4}|\nabla^b\nabla n|_{L^4}|\nabla^k\nabla v|_{L^2} \\
&+ C\sum_{\substack{a+b=k\\ 1\leq a,b\leq k-1}} |\nabla^a\nabla n|_{L^4}|\nabla^b\nabla v|_{L^4}|\nabla^k\nabla n|_{L^2} + C\sum_{\substack{a+b=k\\ 2\leq a\leq k-1}} |\nabla^a v|_{L^4}|\nabla^b\nabla n|_{L^4}|\nabla^k\nabla n|_{L^2} \\
\leq& C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}.
\end{aligned} \tag{3.10}$$

On account of Holder inequality and Sobolev embedding inequality, we can easily get that

$$
\begin{aligned}
B_3 =&\eta\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)n),\nabla^k\dot{n}\rangle-\eta\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)(n\otimes n)),\nabla^k\nabla v\rangle-\langle\nabla^k(\nabla\Phi\otimes\nabla\Phi),\nabla^k\nabla v\rangle\\
\le& C\eta\langle|\nabla^k\nabla\Phi||\nabla\Phi||n|,|\nabla^k\dot{n}|\rangle+C\eta\langle|\nabla\Phi||\nabla\Phi||\nabla^k n|,|\nabla^k\dot{n}|\rangle\\
&+C\eta\langle|\nabla^{k-1}\nabla\Phi||\nabla^2\Phi||n|,|\nabla^k\dot{n}|\rangle+\eta\sum_{\substack{a+b+c=k\\ a,b\le k-2\\ c\le k-1}}\langle|\nabla^a\nabla\Phi||\nabla^b\nabla\Phi||\nabla^c n|,|\nabla^k\dot{n}|\rangle\\
&+C\eta\langle|\nabla^k\nabla\Phi||\nabla\Phi||n||n|,|\nabla^k\nabla v|\rangle+C\eta\langle|\nabla\Phi||\nabla\Phi||\nabla^k n||n|,|\nabla^k\nabla v|\rangle\\
&+C\eta\langle|\nabla^{k-1}\nabla\Phi||\nabla\nabla\Phi||n||n|,|\nabla^k\nabla v|\rangle+\eta\sum_{\substack{a+b+c+d=k\\ a,b\le k-2\\ c,d\le k-1}}\langle|\nabla^a\nabla\Phi||\nabla^b\nabla\Phi||\nabla^c n||\nabla^d n|,|\nabla^k\nabla v|\rangle\\
&+C\langle|\nabla^k\nabla\Phi||\nabla\Phi|,|\nabla^k\nabla v|\rangle+\sum_{\substack{a+b=k\\ a,b\le k-1}}\langle|\nabla^a\nabla\Phi||\nabla^b\nabla\Phi|,|\nabla^k\dot{n}|\rangle\\
\le& C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}+C\|\nabla\Phi\|_{H^s}^2\|\nabla v\|_{H^s}\\
&+\eta\sum_{\substack{a+b+c=k\\ a,b\le k-2\\ c\le k-1}}|\nabla^a\nabla\Phi|_{L^6}|\nabla^b\nabla\Phi|_{L^6}|\nabla^c n|_{L^6},|\nabla^k\dot{n}|_{L^2}+\sum_{\substack{a+b=k\\ a,b\le k-1}}|\nabla^a\nabla\Phi|_{L^4}|\nabla^b\nabla\Phi|_{L^4}|\nabla^k\nabla v|_{L^2}\\
&+\eta\sum_{\substack{a+b+c+d=k\\ a,b\le k-2\\ c,d\le k-1}}|\nabla^a\nabla\Phi|_{L^8}|\nabla^b\nabla\Phi|_{L^8}|\nabla^c n|_{L^8}|\nabla^d n|_{L^8}|\nabla^k\nabla v|_{L^2}\\
\le& C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}+C\|\nabla\Phi\|_{H^s}^2\|\nabla v\|_{H^s}.
\end{aligned}\tag{3.11}
$$

For B_4, thanks to $\alpha_2=0$ and $\alpha_3=-1$, we have

$$
\begin{aligned}
\tilde{B}_4=&-\alpha_2\langle\nabla^k\dot{n}_i n_j,\nabla^k\Omega(v)_{ij}\rangle-\alpha_3\langle\nabla^k\dot{n}_j n_i,\nabla^k\Omega(v)_{ij}\rangle+\langle\nabla^k\Omega(v)_{ij}n_j,\nabla^k\dot{n}_i\rangle\\
&-\alpha_2\langle\nabla^k\dot{n}_i n_j,\nabla^k D(v)_{ij}\rangle-\alpha_3\langle\nabla^k\dot{n}_j n_i,\nabla^k D(v)_{ij}\rangle-\langle\nabla^k D(v)_{ij}n_j,\nabla^k\dot{n}_i\rangle=0.
\end{aligned}
$$

So we estimate that

$$
\begin{aligned}
B_4=&-\langle\nabla^k(D(v)n),\nabla^k\dot{n}\rangle+\langle\nabla^k(\Omega(v)n),\nabla^k\dot{n}\rangle-\langle\nabla^k(\alpha_2\dot{n}\otimes n+\alpha_3 n\otimes\dot{n}),\nabla^k\nabla v\rangle\\
=&-\alpha_2\langle\nabla^k\dot{n}_i n_j,\nabla^k\Omega(v)_{ij}\rangle-\alpha_3\langle\nabla^k\dot{n}_j n_i,\nabla^k\Omega(v)_{ij}\rangle+\langle\nabla^k\Omega(v)_{ij}n_j,\nabla^k\dot{n}_i\rangle\\
&-\alpha_2\langle\nabla^k\dot{n}_i n_j,\nabla^k D(v)_{ij}\rangle-\alpha_3\langle\nabla^k\dot{n}_j n_i,\nabla^k D(v)_{ij}\rangle-\langle\nabla^k D(v)_{ij}n_j,\nabla^k\dot{n}_i\rangle\\
&-\alpha_2\sum_{\substack{a+b=k\\ a\le k-1}}\langle\nabla^a\dot{n}_i\nabla^b n_j,\nabla^k\partial_j v_i\rangle-\alpha_3\sum_{\substack{a+b=k\\ b\le k-1}}\langle\nabla^a n_i\nabla^b\dot{n}_j,\nabla^k\partial_j v_i\rangle\\
&+\sum_{\substack{a+b=k\\ a\le k-1}}\langle\nabla^a\Omega(v)_{ij}\nabla^b n_j,\nabla^k\dot{n}_i\rangle-\sum_{\substack{a+b=k\\ a\le k-1}}\langle\nabla^a D(v)_{ij}\nabla^b n_j,\nabla^k\dot{n}_i\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\alpha_2 \sum_{\substack{a+b=k\\ a\le k-1}} \langle \nabla^a \partial_j \dot{n}_i \nabla^b n_j + \nabla^a \dot{n}_i \nabla^b \partial_j n_j, \nabla^k v_i \rangle \\
&\quad - \alpha_3 \sum_{\substack{a+b=k\\ b\le k-1}} \langle \nabla^a \partial_j n_i \nabla^b \dot{n}_j + \nabla^a n_i \nabla^b \partial_j \dot{n}_j, \nabla^k v_i \rangle \\
&\quad + \sum_{\substack{a+b=k\\ a\le k-1}} \langle \nabla^a \Omega(v)_{ij} \nabla^b n_j, \nabla^k \dot{n}_i \rangle - \sum_{\substack{a+b=k\\ a\le k-1}} \langle \nabla^a D(v)_{ij} \nabla^b n_j, \nabla^k \dot{n}_i \rangle \\
&\le C \|\nabla n\|_{H^s} \|v\|_{H^s} \|\dot{n}\|_{H^s}. \qquad (3.12)
\end{aligned}
$$

As for B_5, by using $\alpha_6 - \alpha_5 = 1$, $\alpha_6 + \alpha_5 > 1$ and $|\nabla^k D(v)n|^2 \ge |\nabla^k \Omega(v)n|^2$, we obtain

$$
-(\alpha_5+\alpha_6)|\nabla^k D(v)n|^2 + |\nabla^k \Omega(v)n|^2 + (\alpha_6 - \alpha_5 - 1)\langle \nabla^k \Omega(v)n, \nabla^k D(v)n\rangle \le 0.
$$

Thus, we have

$$
\begin{aligned}
B_5 &= -\langle \nabla^k(\alpha_5 D(v)n\otimes n + \alpha_6 n \otimes D(v)n), \nabla^k \nabla v\rangle \\
&\quad + \langle \nabla^k(\alpha_2 \Omega(v)n\otimes n + \alpha_3 n\otimes \Omega(v)n), \nabla^k\nabla v\rangle \\
&= -\langle \nabla^k(\alpha_5 D(v)n\otimes n + \alpha_6 n \otimes D(v)n), \nabla^k \nabla v\rangle + \langle \nabla^k(n\otimes \Omega(v)n), \nabla^k \nabla v\rangle \\
&= -\alpha_5 \langle \nabla^k D(v)_{ip} n_p n_j, \nabla^k \partial_j v_i\rangle - \alpha_6 \langle n_i \nabla^k D(v)_{jp} n_p, \nabla^k \partial_j v_i\rangle + \langle n_i \nabla^k \Omega(v)_{jp} n_p, \nabla^k \partial_j v_i\rangle \\
&\quad - \alpha_5 \sum_{\substack{a+b+c=k\\ a\le k-1}} \langle \nabla^a D(v)_{ip} \nabla^b n_p \nabla^c n_j, \nabla^k \partial_j v_i\rangle - \alpha_6 \sum_{\substack{a+b+c=k\\ b\le k-1}} \langle \nabla^a n_i \nabla^b D(v)_{jp} \nabla^c n_p, \nabla^k \partial_j v_i\rangle \\
&\quad + \sum_{\substack{a+b+c=k\\ b\le k-1}} \langle \nabla^a n_i \nabla^b \Omega(v)_{jp} \nabla^c n_p, \nabla^k \partial_j v_i\rangle \\
&\le -(\alpha_5+\alpha_6)|\nabla^k D(v)n|^2 + |\nabla^k \Omega(v)n|^2 + (\alpha_6-\alpha_5-1)\langle \nabla^k \Omega(v)n, \nabla^k D(v)n\rangle \\
&\quad + C\|\nabla n\|_{H^s}^2 \|v\|_{H^s} \|\nabla v\|_{H^s} \\
&\le C\|\nabla n\|_{H^s}^2 \|v\|_{H^s} \|\nabla v\|_{H^s}. \qquad (3.13)
\end{aligned}
$$

Now plugging the inequalities (3.9), (3.10), (3.11), (3.12) and (3.13) into the equation(3.5), we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|\nabla^k v\|_{L^2}^2 + \|\nabla^k \nabla n\|_{L^2}^2) + \tfrac{\alpha_4}{2}\|\nabla^k \nabla v\|_{L^2}^2 + \|\nabla^k \dot{n}\|_{L^2}^2 \\
&\le + C\|n\|_{H^s}^4 \|v\|_{H^s}\|\nabla v\|_{H^s} + C\|v\|_{H^s}^3 + C\|\nabla n\|_{H^s}^3 \|\dot{n}\|_{H^s} + C\|\nabla n\|_{H^s}^2 \|\nabla v\|_{H^s} \\
&\quad + C\eta \|\nabla \Phi\|_{H^s}^2 (\|n\|_{H^s}^2 \|\nabla v\|_{H^s} + \|n\|_{H^s}\|\dot{n}\|_{H^s}) + C\|\nabla\Phi\|_{H^s}^2 \|\nabla v\|_{H^s} \\
&\quad + C\|\nabla n\|_{H^s}\|v\|_{H^s}\|\dot{n}\|_{H^s} + C\|\nabla n\|_{H^s}^2 \|v\|_{H^s}\|\nabla v\|_{H^s}.
\end{aligned}
$$

Lemma 3.4 Let (C_p, C_m, Φ, v, n) be a sufficiently smooth solution to the system(1.1)-(1.6) complemented with the initial condition(1.7) and satisfying the coefficient relation(1.8). Then for any $M > E^{in}$, there is a $T > 0$, which depends only on E^{in} and M, such that there holds the energy inequality

$$
\sup_{t\in[0,T]} E(t) + \int_0^T D(t)dt \le M.
$$

Proof Taking Lemma 1.2 and Lemma 1.3 into consideration, summing up for all $0 \le k \le s$, we obtain the inequality that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|C_p\|_{H^s}^2 + \|C_m\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\nabla n\|_{H^s}^2) + \|\nabla C_p\|_{H^s}^2 + \|\nabla C_m\|_{H^s}^2 + \tfrac{\alpha_4}{2}\|\nabla v\|_{H^s}^2 + \|\dot{n}\|_{H^s}^2 \\
&\le C\|v\|_{H^s}\|C_p\|_{H^s}^2 + C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s} + C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\|\nabla\Phi\|_{H^s} \\
&\quad + C\|\nabla C_p\|_{H^s}\|C_p\|_{H^s}\|\nabla\Phi\|_{H^s} + C\|v\|_{H^s}\|C_m\|_{H^s}^2 + C\eta\|n\|_{H^s}^2\|C_m\|_{H^s}\|\nabla C_m\|_{H^s} \\
&\quad + C\eta\|n\|_{H^s}^2\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}\|\nabla\Phi\|_{H^s} + C\|\nabla C_m\|_{H^s}\|C_m\|_{H^s}\|\nabla\Phi\|_{H^s} \\
&\quad + C\|n\|_{H^s}^4\|v\|_{H^s}\|\nabla v\|_{H^s} + C\|v\|_{H^s}^3 + C\|\nabla n\|_{H^s}^3\|\dot{n}\|_{H^s} + C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s} \\
&\quad + C\eta\|\nabla\Phi\|_{H^s}^2(\|n\|_{H^s}^2\|\nabla v\|_{H^s} + \|n\|_{H^s}\|\dot{n}\|_{H^s}) + C\|\nabla\Phi\|_{H^s}^2\|\nabla v\|_{H^s} \\
&\quad + C\|\nabla n\|_{H^s}\|v\|_{H^s}\|\dot{n}\|_{H^s} + C\|\nabla n\|_{H^s}^2\|v\|_{H^s}\|\nabla v\|_{H^s}.
\end{aligned} \tag{3.14}$$

We can handle the terms that contain $\|\nabla\Phi\|_{H^s}$ on the right-hand side by Lemma 3.1. Therefore we have $\frac{1}{2}\frac{d}{dt}E(t) + D(t) \le CE(t)^{\frac{3}{2}} + C\sum_{1\le k\le 12} E(t)^{\frac{k}{2}}D(t)^{\frac{1}{2}} \le C(1+E(t))^{12} + \frac{1}{2}D(t)$, which follows that

$$\tfrac{d}{dt}E(t) + D(t) \le C(1+E(t))^{12}. \tag{3.15}$$

We solve the ode inequality that

$$E(t) \le [(1+E^{in})^{-13} - Ct]^{-\frac{1}{13}} - 1 = \varphi(t),$$

where the function $\varphi(t)$ is strictly increasing and continuous and $\varphi(0) = E^{in}$. So for any $M > E^{in}$, there is a $T_1 > 0$ such that $\varphi(t) \le M$ for any $t \in [0, T_1]$. We integrate on $[0,t]$ for any $t \in [0, T_1]$ to get that

$$E(t) + \int_0^t D(s)ds \le E^{in} + C\int_0^t (\varphi(s)+1)^{12}ds \le E^{in} + Ct(\varphi(t)+1)^{12} = \phi(t).$$

The same as $\varphi(t)$, for any $M > E^{in}$, there is a $T > 0$ such that $\phi(t) \le M$ for any $t \in [0, T]$. Then we have

$$\sup_{t\in[0,T]} E(t) + \int_0^T D(t)dt \le M.$$

4 Approximate Solution

Lemma 4.1 For $s - \frac{3}{2} > 2$ and $T_0 > 0$, let vector fields $(C_p^{in}, C_m^{in}, v, \nabla n, \nabla\Phi)$ satisfy $C_p^{in} \in H^s$, $C_m^{in} \in H^s$, $v \in L^\infty(0,T_0;H^s)\bigcap L^2(0,T_0;H^{s+1})$, $\nabla n \in L^\infty(0,T_0;H^s)$ and $\nabla\Phi \in L^\infty(0,T_0;H^s)$. Then there exists some $0 < T < T_0$, depending only on C_p^{in}, C_m^{in}, n, $\nabla\Phi$ and v, such that the following system

$$\begin{aligned}
&\frac{\partial C_p}{\partial t} + v\cdot\nabla C_p = \operatorname{div}((Id + \eta n\otimes n)\nabla C_p + C_p\nabla\Phi), \\
&\frac{\partial C_m}{\partial t} + v\cdot\nabla C_m = \operatorname{div}((Id + \eta n\otimes n)\nabla C_m - C_m\nabla\Phi), \\
&C_p|_{t=0} = C_p^{in}, \quad C_m|_{t=0} = C_m^{in}
\end{aligned} \tag{4.1}$$

has a solution $C_p \in L^\infty(0,T;H^s)\bigcap L^2(0,T;H^{s+1})$ and $C_m \in L^\infty(0,T;H^s)\bigcap L^2(0,T;H^{s+1})$.

Proof Let radial function $\rho \in C^\infty(\mathbb{R}^3)$ such that

$$\rho(|x|) \in C_0^\infty(\mathbb{R}^3), \quad \rho \le 0, \quad \int_{\mathbb{R}^3} \rho dx = 1,$$

and $\rho(x) = 0$ for $|x| > 1$. We then define $\rho^\epsilon(x) \in C^\infty(\mathbb{T}^3)$ by $\rho^\epsilon(x) = \frac{1}{\epsilon^3}\sum\limits_{l\in\mathbb{L}^3} \rho(\frac{x+l}{\epsilon})$ for any $\epsilon > 0$, where $\mathbb{L}^3 \subset \mathbb{R}^3$ is some 3-dimensional lattice. Then we define a mollifier J_ϵ as

$$(J_\epsilon f)(x) = \rho^\epsilon * f(x) = \int_{\mathbb{T}^3} \rho^\epsilon(x-y)f(y)dy.$$

We construct the following approximate system such that

$$\begin{aligned} \partial_t C_p^\epsilon + J_\epsilon(v\cdot\nabla J_\epsilon C_p^\epsilon) =& J_\epsilon \mathrm{div}((Id+\eta n\otimes n)(\nabla J_\epsilon C_p^\epsilon + J_\epsilon C_p^\epsilon \nabla\Phi)), \\ C_p^\epsilon|_{t=0} =& J_\epsilon C_p^{in}. \end{aligned} \tag{4.2}$$

By ODE theory, we can prove that there is a maximal time $T_\epsilon > 0$ such that the approximate system(4.2) admits a unique solution $C_p^\epsilon \in C([0,T^\epsilon);H^s)$. Acting ∇^k on the equation(3.3) and taking $L^2-inner$ product with $\nabla^k C_p^\epsilon$, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\nabla^k C_p^\epsilon|^2 + |\nabla^k\nabla J_\epsilon C_p^\epsilon|^2 =& -\langle\nabla^k(v\cdot\nabla J_\epsilon C_p^\epsilon), \nabla^k J_\epsilon C_p^\epsilon\rangle - \langle\nabla^k(J_\epsilon C_p^\epsilon\nabla\Phi), \nabla^k J_\epsilon\nabla C_p^\epsilon\rangle \\ &- \eta\langle\nabla^k(n\otimes n\nabla J_\epsilon C_p), \nabla^k J_\epsilon\nabla C_p^\epsilon\rangle - \eta\langle\nabla^k(n\otimes nJ_\epsilon C_p\nabla\Phi), \nabla^k J_\epsilon\nabla C_p^\epsilon\rangle. \end{aligned}$$

As what we do in Lemma 1.2, we get the following energy inequality that

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}|C_p^\epsilon|_{H^s}^2 + |J_\epsilon\nabla C_p^\epsilon|_{H^s}^2 \le& C|v|_{H^s}|J_\epsilon C_p^\epsilon|_{H^s}^2 + C|J_\epsilon C_p^\epsilon|_{H^s}|J_\epsilon\nabla C_p^\epsilon|_{H^s}|n|_{H^s}^2 \\ &+ C|\nabla\Phi|_{H^s}|J_\epsilon C_p^\epsilon|_{H^s}|J_\epsilon\nabla C_p^\epsilon|_{H^s} + C|J_\epsilon C_p^\epsilon|_{H^s}|J_\epsilon\nabla C_p^\epsilon|_{H^s}|n|_{H^s}^2|\nabla\Phi|_{H^s}. \end{aligned} \tag{4.3}$$

Now we let $E_{C_p}^\epsilon = |C_p^\epsilon|_{H^s}^2$ and $D_{C_p}^\epsilon = |J_\epsilon\nabla C_p^\epsilon|_{H^s}^2$. We define

$$T^\epsilon = \sup\{\tau \in [0,T_0); \sup_{t\in[0,\tau]} E_{C_p}^\epsilon(t) + \int_0^\tau D_{C_p}^\epsilon(t)dt \le 2E_{C_p}^0\}$$

According to the above energy inequality we have

$$\tfrac{1}{2}\tfrac{d}{dt}E_{C_p}^\epsilon + D_{C_p}^\epsilon \le C(E_{C_p}^\epsilon + E_{C_p}^{\epsilon\frac{1}{2}} D_{C_p}^{\epsilon\frac{1}{2}}).$$

By the Young inequality, it follows that

$$\tfrac{d}{dt}E_{C_p}^\epsilon(t) + D_{C_p}^\epsilon(t) \le C_1 E_{C_p}^\epsilon(t),$$

where C_1 is a constant that is independent of ϵ.

Solving the ODE inequality, we have

$$E_{C_p}^\epsilon(t) \le E_{C_p}^{0\epsilon} e^{C_1 t} \le E_{C_p}^0 e^{C_1 t}.$$

We can find a $T>0$ independent of ϵ such that $\int_0^T e^{C_1 t}dt \le \frac{1}{C_1}$.

For all $t\in[0,T]$, we have

$$E^\epsilon_{C_p}(t)+\int_0^t D^\epsilon_{C_p}(\tau)d\tau \le \int_0^t C_1E^\epsilon_{C_p}(\tau)d\tau + E^0_{C_p} \le \int_0^t C_1 E^0_{C_p} e^{C_1\tau}d\tau + E^0_{C_p} \le 2E^0_{C_p}.$$

Hence $T^\epsilon \ge T>0$ for all $\epsilon>0$. Therefore, for all $\epsilon>0$ and $t\in[0,T]$, we have

$$E^\epsilon_{C_p}(t)+\int_0^t D^\epsilon_{C_p}(\tau)d\tau \le 2E^0_{C_p}.$$

Namely, we obtain the following uniform energy bound $|C^\epsilon_p|^2_{H^s} \le 2E^0_{C_p}$, $\displaystyle\int_0^t |J_\epsilon \nabla C^\epsilon_p|^2_{H^s} \le 2E^0_{C_p}$, for all $\epsilon>0$ and $t\in[0,T]$. By the bounds we know that there is a

$$C_p\in L^\infty(0,T;H^s)\cap L^2(0,T;H^{s+1})$$

such that C_p obeys 4.1 after passing limits in (4.2) as $\epsilon\to 0$. Similarly, we can get a

$$C_m\in L^\infty(0,T;H^s)\cap L^2(0,T;H^{s+1})$$

that obeys 4.1.

Lemma 4.2 For $s-\frac{3}{2}>2$ and $T_1>0$, let vector fields $(v^{in}, n^{in}, \nabla\Phi)$ satisfy $v^{in}\in H^s$, $\nabla n^{in}\in H^s$ and $\nabla\Phi\in L^\infty(0,T_1;H^s)$. Then there exists some $0<T<T_1$, depending only on v^{in}, n^{in} and $\nabla\Phi$ such that

$$\begin{aligned}
\frac{\partial v}{\partial t}+v\cdot\nabla v+\nabla p =& \alpha_4 \mathrm{div} D(v) - \mathrm{div}(\nabla n\odot\nabla n)\\
&+\mathrm{div}((\nabla\Phi\otimes\nabla\Phi)(Id+\eta n\otimes n))\\
&+\mathrm{div}(\alpha_1(D(v)n\cdot n)n\otimes n+\alpha_2\mathring{n}\otimes n+\alpha_3 n\otimes\mathring{n})\\
&+\mathrm{div}(\alpha_5 D(v)n\otimes n+\alpha_6 n\otimes D(v)n), \qquad (4.4)\\
\mathrm{div} v =& 0,\\
n_t+v\cdot\nabla n-\Omega(v)n+D(v)n =& \triangle n+\eta(\nabla\Phi\otimes\nabla\Phi)n-F'(|n|^2)n,\\
(v,n)|_{t=0} =& (v^{in}, n^{in})
\end{aligned}$$

which has a unique solution $v\in L^\infty(0,T;H^s)\bigcap L^2(0,T;H^{s+1})$, $\nabla n\in L^\infty(0,T;H^s)$ and $\dot{n}\in L^2(0,T;H^s)$.

Proof As what we do in Lemma 3.1, we construct the following approximate system such that

$$\begin{aligned}
&\partial_t v^\epsilon + J_\epsilon(J_\epsilon v^\epsilon\cdot\nabla J_\epsilon v^\epsilon)+\nabla p^\epsilon\\
=&\alpha_4 J_\epsilon \mathrm{div} D(v^\epsilon) - J_\epsilon\mathrm{div}(\nabla J_\epsilon n^\epsilon\odot\nabla J_\epsilon n^\epsilon) + J_\epsilon \mathrm{div}[(\nabla\Phi\otimes\nabla\Phi)(Id+\eta J_\epsilon n^\epsilon\otimes J_\epsilon n^\epsilon)]\\
&+J_\epsilon\mathrm{div}[\alpha_1(J_\epsilon D(v^\epsilon)J_\epsilon n^\epsilon\cdot J_\epsilon n^\epsilon)J_\epsilon n^\epsilon\otimes J_\epsilon n^\epsilon+\alpha_2\mathring{n}^\epsilon\otimes J_\epsilon n^\epsilon+\alpha_3 J_\epsilon n^\epsilon\otimes\mathring{n}^\epsilon] \qquad (4.5)\\
&+J_\epsilon\mathrm{div}(\alpha_5 J_\epsilon D(v^\epsilon)J_\epsilon n^\epsilon\otimes J_\epsilon n^\epsilon+\alpha_6 J_\epsilon n^\epsilon\otimes J_\epsilon D(v^\epsilon)J_\epsilon n^\epsilon),\\
&\mathrm{div} v^\epsilon = 0,
\end{aligned}$$

$$
\begin{aligned}
&n_t^\epsilon + J_\epsilon(v^\epsilon \cdot \nabla J_\epsilon n^\epsilon) - J_\epsilon(\Omega(v^\epsilon) J_\epsilon n^\epsilon) + J_\epsilon(D(v^\epsilon)J_\epsilon n^\epsilon)\\
=&\Delta J_\epsilon^2 n^\epsilon + \eta J_\epsilon((\nabla\Phi \otimes \nabla\Phi)J_\epsilon n^\epsilon) - J_\epsilon(F'(|J_\epsilon n^\epsilon|^2)J_\epsilon n^\epsilon),\\
\mathring{n}^\epsilon =& n_t^\epsilon + J_\epsilon(v^\epsilon \cdot \nabla J_\epsilon n^\epsilon) - J_\epsilon(\Omega(v^\epsilon)J_\epsilon n^\epsilon), \quad v^\epsilon|_{t=0} = J_\epsilon v^{in}, \quad n^\epsilon|_{t=0} = J_\epsilon n^{in}.
\end{aligned}
$$

By ODE theorem, we can prove that there is a maximal time $T_\epsilon > 0$, depending only on$\nabla\Phi$, v_0, d_0 and T_0 such that the approximate system admits a unique solution $n^\epsilon \in C([0,T_\epsilon]; H^{s+1})$ and $v^\epsilon \in C([0,T_\epsilon]; H^s)$. We point out that $T_\epsilon \le T_0$ for all $\epsilon > 0$, which is determined by the regularity of $\nabla\Phi$.

For all $1 \le l \le s$, we act the l-order derivative operator ∇^l on the first equation of the approximate system and take L^2-inner product by multiplying $\nabla^l v^\epsilon$. Similarly, we act ∇^l on the third equation of the approximate system and take L^2-inner product by multiplying $\nabla^l \dot{n}^\epsilon$. Using integration by parts we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(|\nabla^l J_\epsilon v|_{L^2}^2 + |\nabla^l \nabla J_\epsilon n^\epsilon|_{L^2}^2) + |\nabla^l \dot{n}^\epsilon|_{L^2}^2 + |\nabla^l v^\epsilon|_{L^2}^2 = -\langle \nabla^l (J_\epsilon v^\epsilon \cdot \nabla J_\epsilon v^\epsilon), \nabla^l \nabla J_\epsilon v^\epsilon\rangle\\
&+ \langle \nabla^l(\nabla J_\epsilon n^\epsilon \odot \nabla J_\epsilon n^\epsilon), \nabla^l \nabla J_\epsilon v^\epsilon\rangle - \langle \nabla^l[(\nabla\Phi\otimes\nabla\Phi)(Id + \eta\nabla n \otimes \nabla n)], \nabla^l\nabla J_\epsilon v^\epsilon\rangle\\
&- \langle \nabla^l[\alpha_1(J_\epsilon D(v^\epsilon)J_\epsilon n^\epsilon \cdot J_\epsilon n^\epsilon)J_\epsilon n^\epsilon \otimes J_\epsilon n^\epsilon + \alpha_2 \mathring{n}^\epsilon \otimes J_\epsilon n^\epsilon + \alpha_3 J_\epsilon n^\epsilon \otimes \mathring{n}^\epsilon], \nabla^l \nabla J_\epsilon v^\epsilon\rangle\\
&- \langle \nabla^l(\alpha_5 J_\epsilon D(v^\epsilon)J_\epsilon n^\epsilon \otimes J_\epsilon n^\epsilon + \alpha_6 J_\epsilon n^\epsilon \otimes J_\epsilon D(v^\epsilon)J_\epsilon n^\epsilon), \nabla^l\nabla J_\epsilon v^\epsilon\rangle\\
&+ \langle \nabla^l[J_\epsilon(\Omega(v^\epsilon)J_\epsilon n^\epsilon) - J_\epsilon(D(v^\epsilon)J_\epsilon n^\epsilon)], \nabla^l \dot{n}^\epsilon\rangle + \langle \nabla^l \Delta J_\epsilon^2 n^\epsilon, \nabla^l J_\epsilon(v^\epsilon\cdot\nabla J_\epsilon n^\epsilon)\rangle\\
&+ \langle \nabla^l \eta J_\epsilon((\nabla\Phi\otimes\nabla\Phi)J_\epsilon n^\epsilon), \nabla^l \dot{n}^\epsilon\rangle - \langle \nabla^l J_\epsilon(F'(|J_\epsilon n^\epsilon|^2)J_\epsilon n^\epsilon), \nabla^l \dot{n}^\epsilon\rangle.
\end{aligned}
$$

According to Lemma 1.3, we have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}(\|\nabla^l v^\epsilon\|_{L^2}^2 + \|\nabla^l J_\epsilon \nabla n^\epsilon\|_{L^2}^2) + \tfrac{\alpha_4}{2}\|\nabla^l J_\epsilon \nabla v^\epsilon\|_{L^2}^2 + \|\nabla^l \dot{n^\epsilon}\|_{L^2}^2\\
\le& C\|J_\epsilon \nabla n^\epsilon\|_{H^s}^4 \|v^\epsilon\|_{H^s}\|\nabla v^\epsilon\|_{H^s} + C\|v^\epsilon\|_{H^s}^3 + C\|\nabla J_\epsilon n^\epsilon\|_{H^s}^3 \|\dot{n}\|_{H^s} + C\|\nabla J_\epsilon n^\epsilon\|_{H^s}^2 \|\nabla J_\epsilon v^\epsilon\|_{H^s}\\
&+ C\eta\|\nabla\Phi\|_{H^s}^2(\|\nabla J_\epsilon n^\epsilon\|_{H^s}^2 \|J_\epsilon \nabla v^\epsilon\|_{H^s} + \|\nabla J_\epsilon n^\epsilon\|_{H^s}\|\dot{n^\epsilon}\|_{H^s}) + C\|\nabla\Phi\|_{H^s}^2\|\nabla v^\epsilon\|_{H^s}\\
&+ C\|\nabla J_\epsilon n^\epsilon\|_{H^s}\|v^\epsilon\|_{H^s}\|\dot{n^\epsilon}\|_{H^s} + C\|\nabla J_\epsilon n^\epsilon\|_{H^s}^2\|v^\epsilon\|_{H^s}\|\nabla J_\epsilon v^\epsilon\|_{H^s}. \qquad (4.6)
\end{aligned}
$$

Now we let $E_{v,n}^\epsilon(t) = \|v^\epsilon\|_{H^s}^2 + \|\nabla J_\epsilon n^\epsilon\|_{H^s}^2$ and $D_{v,n}^\epsilon(t) = \frac{\alpha_4}{2}\|\nabla J_\epsilon v^\epsilon\|_{H^s}^2 + \|\dot{n^\epsilon}\|_{H^s}^2$. Then we obtain that

$$
\tfrac{1}{2}\tfrac{d}{dt}E_{v,n}^\epsilon(t) + D_{v,n}^\epsilon(t) \le CE_{v,n}^{\epsilon\frac{3}{2}}(t) + C(1 + E_{v,n}^{\epsilon\frac{1}{2}}(t))^5 D_{v,n}^{\epsilon\frac{1}{2}}(t).
$$

Thanks to Young inequality, we get that

$$
\tfrac{d}{dt}E_{v,n}^\epsilon(t) + D_{v,n}^\epsilon(t) \le C(1 + E_{v,n}^\epsilon(t))^5.
$$

We solve the ODE inequality that $E_{v,n}^\epsilon(t) \le [(1 + E_{v,n}^\epsilon(0))^{-4} - Ct]^{-\frac{1}{4}} - 1$. Therefore

$$
\begin{aligned}
E_{v,n}^\epsilon(t) + \int_0^t D_{v,n}^\epsilon(\tau)d\tau \le& C\int_0^t (1 + E_{v,n}^\epsilon(t))^{5z} d\tau + E_{v,n}(0)\\
\le& C\int_0^t [(1 + E_{v,n}(0))^{-4} - C\tau]^{-\frac{5}{4}} d\tau + E_{v,n}(0).
\end{aligned}
$$

We let $h(t) = C\int_0^t [(1+E_{v,n}(0))^{-4} - C\tau]^{-\frac{5}{4}} d\tau + E_{v,n}(0)$, which is nonnegative and $h(0) = E_{v,n}(0)$. Since $h(t)$ is continuous in t and independent of $\epsilon > 0$, there is a $T > 0$ such that $h(t) \le 2E_{v,n}(0)$ for all $t \in [0,T]$. Now we obtain the energy bound

$$E^{\epsilon}_{v,n}(t) + \int_0^t D^{\epsilon}_{v,n}(\tau)d\tau \le h(t) \le 2E_{v,n}(0),$$

for all $\epsilon > 0$ and $t \in [0,T]$. By the bound we get that there is a (v,n) satisfying $\nabla n \in L^\infty(0,T;H^s)$, $\dot{n} \in L^2(0,T;H^s)$ and $v \in L^\infty(0,T;H^s)\bigcap L^2(0,T;H^{s+1})$ such that (v,n) obeys the system (4.4) after passing limits in the approximate system as $\epsilon \to 0$. Therefore, we have completed the proof of the lemma.

5 The Iterative Approximate System

In this section, we construct the approximate system by iteration. More precisely, the iterative approximate system is constructed as follows: for all integer $k \ge 0$

$$\begin{aligned}
&\frac{\partial C_p^{k+1}}{\partial t} + v^k\cdot\nabla C_p^{k+1} = \mathrm{div}((Id+\eta n^k\otimes n^k)\nabla C_p^{k+1} + C_p^{k+1}\nabla\Phi^k),\\
&\frac{\partial C_m^{k+1}}{\partial t} + v^k\cdot\nabla C_m^{k+1} = \mathrm{div}((Id+\eta n^k\otimes n^k)\nabla C_m^{k+1} - C_m^{k+1}\nabla\Phi^k),\\
&-\mathrm{div}((Id+\eta n^k\otimes n^k)\nabla\Phi^{k+1}) = C_p^k - C_m^k,\\
&\partial_t v^{k+1} + v^{k+1}\cdot\nabla v^{k+1} + \nabla p^{k+1} = \alpha_4 \mathrm{div} D(v^{k+1}) - \mathrm{div}(\nabla n^{k+1}\odot\nabla n^{k+1})\\
&\qquad + \mathrm{div}[(\nabla\Phi^k\otimes\nabla\Phi^k)(Id+\eta n^{k+1}\otimes n^{k+1})]\\
&\qquad + \mathrm{div}[\alpha_1(D(v^{k+1})n^{k+1}\cdot n^{k+1})n^{k+1}\otimes n^{k+1}\\
&\qquad + \alpha_2 \mathring{n}^{k+1}\otimes n^{k+1} + \alpha_3 n^{k+1}\otimes\mathring{n}^{k+1}]\\
&\qquad + \mathrm{div}(\alpha_5 D(v^{k+1})n^{k+1}\otimes n^{k+1} + \alpha_6 n^{k+1}\otimes D(v^{k+1})n^{k+1}),\\
&\mathrm{div} v^{k+1} = 0,\\
&n_t^{k+1} + v^{k+1}\cdot\nabla n^{k+1} - \Omega(v^{k+1})n^{k+1} + D(v^{k+1})n^{k+1}\\
&\qquad = \Delta n^{k+1} + \eta(\nabla\Phi^k\otimes\nabla\Phi^k)n^{k+1} - F'(|n^{k+1}|^2)n^{k+1},\\
&(v^{k+1}, n^{k+1}, C_p^{k+1}, C_m^{k+1}, \Phi^{k+1})|_{t=0} = (v^{in}, n^{in}, C_p^{in}, C_m^{in}, \Phi^{in}),
\end{aligned}\tag{5.1}$$

where $D(v^{k+1}) = \frac{\nabla v^{k+1} + \nabla(v^{k+1})^T}{2}$, $\Omega(v^{k+1}) = \frac{\nabla v^{k+1} - \nabla(v^{k+1})^T}{2}$ and $\mathring{n}^{k+1} = n_t^{k+1} + v^{k+1}\cdot\nabla n^{k+1} - \Omega(v^{k+1})n^{k+1}$. The iteration starts from $k=0$, i.e.,

$$(v^0, n^0, C_p^0, C_m^0, \Phi^0) = (v^{in}, n^{in}, C_p^{in}, C_m^{in}, \Phi^{in}).$$

Now we state the existence result of the iterative approximate system (5.1) as follows.

Lemma 5.1 Suppose that $s - \frac{3}{2} > 2$ and the initial data $(v^{in}, n^{in}, C_p^{in}, C_m^{in}, \Phi^{in})$ satisfies $v^{in}, \nabla n^{in}, C_p^{in}, C_m^{in}$ and $\nabla\Phi^{in} \in H^s$. Then there is a maximal number $T^*_{k+1} > 0$ such that the system (5.1) admits a solution $(v^{k+1}, n^{k+1}, C_p^{k+1}, C_m^{k+1}, \Phi^{k+1})$ satisfying v^{k+1}, C_p^{k+1}, $C_m^{k+1} \in$

$L^\infty(0,T^*_{k+1};H^s)\bigcap L^2(0,T^*_{k+1};H^{s+1})$, $\nabla n^{k+1}\in L^\infty(0,T^*_{k+1};H^s)$, $\dot n^{k+1}\in L^2(0,T^*_{k+1};H^s)$ and $\nabla\Phi^{k+1}\in L^\infty(0,T;H^s)$.

Proof For the case $k+1$, the function $(v^k,n^k,C^k_p,C^k_m,\Phi^k)$ is given. Namely, the electrostatic potential equation of Φ^{k+1} is a divergence type elliptic equation

$$-\mathrm{div}((Id+\eta n^k\otimes n^k)\nabla\Phi^{k+1})=C^k_p-C^k_m,\quad \Phi^{k+1}|_{t=0}=\Phi^{in},$$

which admits a solution $\nabla\Phi^{k+1}\in L^\infty(0,T^\Phi_{k+1};H^s)$. Moreover, the equations of C^{k+1}_p and C^{k+1}_m are linear system with given v^k, n^k and Φ^k,

$$\begin{aligned}
\frac{\partial C^{k+1}_p}{\partial t}+v^k\cdot\nabla C^{k+1}_p=&\mathrm{div}((Id+\eta n^k\otimes n^k)\nabla C^{k+1}_p+C^{k+1}_p\nabla\Phi^k),\\
\frac{\partial C^{k+1}_m}{\partial t}+v^k\cdot\nabla C^{k+1}_m=&\mathrm{div}((Id+\eta n^k\otimes n^k)\nabla C^{k+1}_m-C^{k+1}_m\nabla\Phi^k),\\
(C^{k+1}_p,C^{k+1}_m)|_{t=0}=&(C^{in}_p,C^{in}_m),
\end{aligned}$$

which, by lemma2.1, has a solution (C^{k+1}_p,C^{k+1}_m) satisfying $C^{k+1}_p\in L^\infty(0,T^{p,m}_{k+1};H^s)\bigcap L^2(0,T^{p,m}_{k+1};H^{s+1})$ and $C^{k+1}_m\in L^\infty(0,T^{p,m}_{k+1};H^s)\bigcap L^2(0,T^{p,m}_{k+1};H^{s+1})$. Finally, for the system

$$\begin{aligned}
\partial_t v^{k+1}+v^{k+1}\cdot\nabla v^{k+1}+\nabla p^{k+1}=&\alpha_4\mathrm{div}D(v^{k+1})-\mathrm{div}(\nabla n^{k+1}\odot\nabla n^{k+1})\\
&+\mathrm{div}[(\nabla\Phi^k\otimes\nabla\Phi^k)(Id+\eta n^{k+1}\otimes n^{k+1})]\\
&+\mathrm{div}[\alpha_1(D(v^{k+1})n^{k+1}\cdot n^{k+1})n^{k+1}\otimes n^{k+1}\\
&+\alpha_2\mathring{n}^{k+1}\otimes n^{k+1}+\alpha_3 n^{k+1}\otimes\mathring{n}^{k+1}]\\
&+\mathrm{div}(\alpha_5 D(v^{k+1})n^{k+1}\otimes n^{k+1}+\alpha_6 n^{k+1}\otimes D(v^{k+1})n^{k+1}),\\
\mathrm{div}v^{k+1}=&0,\\
n^{k+1}_t+v^{k+1}\cdot\nabla n^{k+1}-\Omega(v^{k+1})n^{k+1}&+D(v^{k+1})n^{k+1}\\
=&\Delta n^{k+1}+\eta(\nabla\Phi^k\otimes\nabla\Phi^k)n^{k+1}-F'(|n^{k+1}|^2)n^{k+1},\\
(v^{k+1},n^{k+1})|_{t=0}=&(v^{in},n^{in}),
\end{aligned}$$

by Lemma 2.2, we obtain a solution (v^{k+1},n^{k+1}) satisfying $v^{k+1}\in L^\infty(0,T^{v,n}_{k+1};H^s)\bigcap L^2(0,T^{v,n}_{k+1};H^{s+1})$, $\nabla n^{k+1}\in L^\infty(0,T^{v,n}_{k+1};H^s)$ and $\dot n^{k+1}\in L^2(0,T^{v,n}_{k+1};H^s)$. We denote by

$$T^*_{k+1}=\min\{T^\Phi_{k+1},T^{p,m}_{k+1},T^{v,n}_{k+1}\},$$

and the proof of lemma 5.1 is finished.

6 Local Well-posedness with Large Initial Data

In this section, we prove the local well-posedness of system(1.1)-(1.7) with large initial data. The key point is to justify the positive lower bound of T^*_{k+1} and the uniform energy bounds of the iterative approximate system (5.1), which will be shown in Lemma 4.1. In the end, by the compactness argument, we can pass to the limits in the system(5.1) and then reach our goal, which is a standard process. We define the following energy functions

$$E_{k+1}(t)=\|v^{k+1}\|^2_{H^s}+\|\nabla n^{k+1}\|^2_{H^s}+\|C^{k+1}_p\|^2_{H^s}+\|C^{k+1}_m\|^2_{H^s}$$

and

$$D_{k+1}(t) =\|\nabla C_p^{k+1}\|_{H^s}^2 + \|\nabla C_m^{k+1}\|_{H^s}^2 + \tfrac{\alpha_4}{2}\|\nabla v^{k+1}\|_{H^s}^2 + \|\dot{n}^{k+1}\|_{H^s}^2.$$

Lemma 6.1 Assume that $(v^{k+1},\ n^{k+1},\ C_p^{k+1},\ C_m^{k+1},\ \Phi^{k+1})$ is the solution to the iterative approximate system (5.1), and we define

$$T_{k+1} = \sup\{\tau \in [0, T_{k+1}^*];\ \sup_{t\in[0,\tau]} E_{k+1}(t) + \int_0^\tau D_{k+1}(t)dt \le M\},$$

where $T_{k+1}^* > 0$ is the existence time of the iterative approximate system(5.1). Then for any fixed $M > E_0$ there is a constant $T > 0$, depending only on M and E_0, such that $T_{k+1} \ge T > 0$.

Proof By the continuity of the functionals $E_{k+1}(t)$, we know that $T_{k+1} > 0$. If the sequence $\{T_k; k = 1, 2, ...\}$ is increasing, the conclusion immediately holds. So we consider that the sequence T_k is not increasing. Now we choose a strictly increasing sequence $\{k_p\}_{p=1}^{\Lambda}$ as follows:

$$k_1 = 1, \quad k_{p+1} = \min\{k; k > k_p, T_k < T_{k_p}\}.$$

If $\Lambda < \infty$, the conclusion holds. Consequently, we consider the case $\Lambda = \infty$. By the definition of k_p, the sequence $\{k_p\}_{p=1}^{\infty}$ is strictly decreasing, so that our goal is to prove $\lim_{p\to\infty} T_{k_p} > 0$. As what we do in a priori estimate, we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}E_{k+1} + D_{k+1} \le& C\{E_k^{\frac{1}{2}}E_{k+1} + E_{k+1}^{\frac{3}{2}} + [E_{k+1} + E_{k+1}^{\frac{3}{2}} + E_{k+1}^{\frac{5}{2}} \\ &+ (E_{k+1} + E_{k+1}^{\frac{1}{2}})E_k^5 + E_k^5 + E_k^{\frac{7}{2}}E_{k+1}^{\frac{1}{2}} + E_kE_{k+1}^{\frac{1}{2}} + E_k^{\frac{5}{2}}E_{k+1}^{\frac{1}{2}}]D_{k+1}^{\frac{1}{2}}\},\end{aligned}$$

which follows that

$$\tfrac{d}{dt}E_{k+1} + D_{k+1} \le C(1 + E_k(t))^{10}(1 + E_{k+1}(t))^5. \tag{6.1}$$

Recalling the definition of the sequence $\{k_p\}$, we know that for any integer $N < k_p$, $T_N > T_{k_p}$. We take $k = k_p - 1$ in the inequality(6.1), and then by the definition of T_k we have for all $t \in [0, T_{k_p}]$

$$\tfrac{d}{dt}E_{k_p}(t) + D_{k_p}(t) \le C(1 + M)^{10}(1 + E_{k_p})^5.$$

We solve the ODE inequality that for all $t \in [0, T_{k_p}]$,

$$E_{k_p}(t) \le [(1 + E_0)^{-4} - C(1 + M)^{10}t]^{-\frac{1}{4}} - 1 \equiv x(t),$$

where the function $x(t)$ is strictly increasing and continuous and $x(0) = E_0$. Plugging the above inequality into the ODE inequality(6.1) and then integrating on $[0, t]$, for any $t \in [0, T_{k_p}]$, we estimate that

$$E_{k_p}(t) + \int_0^t D_{k_p}(\tau)d\tau \le E_0 + C(1 + M)^{10}(1 + x(t))^5t \equiv y(t),$$

where $y(t)$ is also strictly increasing and continuous and $y(0) = E_0$. Thus, by the continuity and monotonicity of the function $y(t)$, we know that for any $M > E_0$ and $p \in N^+$, there is a number $t^* > 0$, depending only on M and initial energy E_0, such that for all $t \in [0, t^*]$,

$$E_{k_p}(t) + \int_0^t D_{k_p}(\tau) d\tau \le M.$$

By the definition of T_k we derive that $T_{k_p} \ge t^* > 0$, hence $T = \lim_{p\to\infty} T_{k_p} \ge t^* > 0$. Consequently, we complete the proof of Lemma 4.1.

According to Lemma 4.1 we know that for any fixed $M > E^0$ there is a $T > 0$ such that for all integer $k \ge 0$ and $t \in [0, T]$

$$\begin{aligned}&\sup_{t\in[0,T]} \{\|v_{k+1}\|^2_{H^s} + \|\nabla n_{k+1}\|^2_{H^s} + \|C_{p_{k+1}}\|^2_{H^s} + \|C_{m_{k+1}}\|^2_{H^s}\}\\ &+ \int_0^t (\|\nabla C_{p_{k+1}}\|^2_{H^s} + \|\nabla C_{m_{k+1}}\|^2_{H^s} + \tfrac{\alpha_4}{2}\|\nabla v_{k+1}\|^2_{H^s} + \|\dot n_{k+1}\|^2_{H^s}) d\tau \le M.\end{aligned}$$

Then, by compactness arguments, we get vector (C_p, C_m, Φ, v, n) satisfying C_p, C_m and $v \in L^\infty(0, T; H^s) \bigcap L^2(0, T; H^{s+1})$, $\nabla n \in L^\infty(0, T; H^s)$ and $\dot n \in L^2(0, T; H^s)$, which solves the system(1.1)–(1.6) with the initial condition(1.7). Moreover, (C_p, C_m, Φ, v, n) satisfies the bound

$$\begin{aligned}&\sup_{t\in[0,T]} \|v\|^2_{H^s} + \|\nabla n\|^2_{H^s} + \|C_p\|^2_{H^s} + \|C_m\|^2_{H^s}\\ &+ \int_0^t (\|\nabla C_p\|^2_{H^s} + \|\nabla C_m\|^2_{H^s} + \frac{\alpha_4}{2}\|\nabla v\|^2_{H^s} + \|\dot n\|^2_{H^s}) d\tau \le M.\end{aligned}$$

Then the proof of the Theorem 2.1 is finished.

7 Global Existence with Small Initial Data

We set the following energy functionals:

$$\tilde E(t) = \|v\|^2_{H^s} + \|\nabla n\|^2_{H^s} + \|C_p\|^2_{H^s} + \|C_m\|^2_{H^s} + \|n\|^2_{H^s}$$

and

$$\tilde D(t) = \|\nabla C_p\|^2_{H^s} + \|\nabla C_m\|^2_{H^s} + \frac{\alpha_4}{2}\|\nabla v\|^2_{H^s} + \|\dot n\|^2_{H^s} + \|\nabla n\|^2_{H^s}.$$

We observe that

$$C_* \tilde E(t) \le E(t) \le \tilde E(t), \quad \text{and} \quad D(t) \le \tilde D(t),$$

where C_* is a constant which depends only on the *Poincaré* inequality in torus.

Lemma 7.1 If (v, n, C_p, C_m, Φ) is the local solution constructed in Theorem 2.1, then

$$\frac{1}{2}\frac{d}{dt}\tilde E(t) + \tilde D(t) \le \hat C \sum_{k=1}^{11} \tilde E(t)^{\frac{k}{2}} \tilde D(t),$$

where the positive constant C is independent of (v, n, C_p, C_m, Φ).

Proof Firstly, by lemma(3.2) and Poincaré inequality, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{dt}\|\nabla^k C_p\|_{L^2}^2+\|\nabla^k\nabla C_p\|_{L^2}^2\\
&\le C\|v\|_{H^s}\|C_p\|_{H^s}^2+C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\\
&\quad+C\eta\|n\|_{H^s}^2\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}\|\nabla\Phi\|_{H^s}+C\|\nabla C_p\|_{H^s}\|C_p\|_{H^s}\|\nabla\Phi\|_{H^s}\\
&\le C\|v\|_{H^s}\|\nabla C_p\|_{H^s}^2+C\eta\|n\|_{H^s}^2\|\nabla C_p\|_{H^s}^2\\
&\quad+C\eta\|n\|_{H^s}^2\|\nabla C_p\|_{H^s}^2\|\nabla\Phi\|_{H^s}+C\|\nabla C_p\|_{H^s}^2\|\nabla\Phi\|_{H^s}\\
&\le C(\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t)+\tilde{E}(t)^{\frac{7}{2}}+\tilde{E}(t)^{\frac{5}{2}})\tilde{D}(t).
\end{aligned}\tag{7.1}$$

By a similar argument, we also have

$$\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k C_m\|_{L^2}^2+\|\nabla^k\nabla C_m\|_{L^2}^2\le C(\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t)+\tilde{E}(t)^{\frac{7}{2}}+\tilde{E}(t)^{\frac{5}{2}})\tilde{D}(t).\tag{7.2}$$

Secondly, via acting ∇^k on the equation (1.6) and taking L^2-inner product with $\nabla^k n$, we get

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k n\|_{L^2}^2+\|\nabla^k n\|_{L^2}^2=&\langle\nabla^k(\Omega(v)n),\nabla^k n\rangle-\langle\nabla^k(D(v)n)),\nabla^k n\rangle\\
&+\eta\langle\nabla^k((\nabla\Phi\otimes\nabla\Phi)n),\nabla^k n\rangle-\langle\nabla^k(F'(|n|^2)n),\nabla^k n\rangle.
\end{aligned}\tag{7.3}$$

Now we estimate the terms on the right-hand side of(7.3) term by term as follows:

$$\begin{aligned}
&G_1=\langle\nabla^k[\Omega(v)n-D(v)n],\nabla^k n\rangle\le C\|n\|_{H^s}\|\nabla v\|_{H^s}\|\nabla n\|_{H^s}\le C\tilde{E}(t)^{\frac{1}{2}}\tilde{D}(t),\\
&G_2=\eta\langle\nabla^k[(\nabla\Phi\otimes\nabla\Phi)n],\nabla^k n\rangle\le C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}^2\le C\tilde{E}(t)^5\tilde{D}(t),\\
&G_3=-\langle\nabla^k[F'(|n|^2)n],\nabla^k n\rangle\le C\|n\|_{H^s}^2\|\nabla n\|_{H^s}^2\le C\tilde{E}(t)\tilde{D}(t).
\end{aligned}\tag{7.4}$$

Plugging the inequalities in (7.4) into (7.3), we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^k n\|_{L^2}^2+\|\nabla^k n\|_{L^2}^2\le C(\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t)^5+\tilde{E}(t))\tilde{D}(t).\tag{7.5}$$

Now we estimate the terms on the right-hand side of (3.5) in other way by using Poincaré inequality. Naturally, we can get the following estimates:

$$\begin{aligned}
B_1\le&C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}\le C\tilde{E}(t)^{\frac{1}{2}}\tilde{D}(t),\\
B_2\le&C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}\le C\tilde{E}(t)^{\frac{1}{2}}\tilde{D}(t),\\
B_4\le&C\|v\|_{H^s}\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}\le C\tilde{E}(t)^{\frac{1}{2}}\tilde{D}(t),\\
B_5\le&C\|\nabla n\|_{H^s}^2\|v\|_{H^s}\|\nabla v\|_{H^s}\le C\tilde{E}(t)\tilde{D}(t).
\end{aligned}\tag{7.6}$$

In the light of lemma1.1 and Poincaré inequality, we obtain

$$\begin{aligned}
\|\nabla\Phi\|_{H^s}^2\le&C(\|C_p\|_{H^s}^2+\|C_m\|_{H^s}^2+\|\nabla n\|_{H^4}^{10})\\
\le&C(\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}+\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}+\|\nabla n\|_{H^s}^{10})\\
\le&C(\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t)^{\frac{9}{2}})\tilde{D}(t)^{\frac{1}{2}},
\end{aligned}$$

which implies that

$$\begin{aligned}B_3 \le& C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}+C\|\nabla\Phi\|_{H^s}^2\|\nabla v\|_{H^s}\\ \le& C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}\\ &+C(\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}+\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}+\|\nabla n\|_{H^s}^{10})\|\nabla v\|_{H^s}\\ \le& C(\tilde{E}(t)^{\frac{11}{2}}+\tilde{E}(t)^5+\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t)^{\frac{9}{2}})\tilde{D}(t).\end{aligned} \tag{7.7}$$

We plug inequalities(7.6) and (7.7) into (3.5) to obtain that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(\|\nabla^k v\|_{L^2}^2+\|\nabla^k\nabla n\|_{L^2}^2)+\tfrac{\alpha_4}{2}\|\nabla^k\nabla v\|_{L^2}^2+\|\nabla^k\dot{n}\|_{L^2}^2\\ \le& C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}\\ &+C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}^2\|\nabla v\|_{H^s}+C\eta\|\nabla\Phi\|_{H^s}^2\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}\\ &+C(\|C_p\|_{H^s}\|\nabla C_p\|_{H^s}+\|C_m\|_{H^s}\|\nabla C_m\|_{H^s}+\|\nabla n\|_{H^s}^{10})\|\nabla v\|_{H^s}\\ &+C\|v\|_{H^s}\|\nabla n\|_{H^s}\|\dot{n}\|_{H^s}+C\|\nabla n\|_{H^s}^2\|v\|_{H^s}\|\nabla v\|_{H^s}\\ \le& C(\tilde{E}(t)^{\frac{11}{2}}+\tilde{E}(t)^5+\tilde{E}(t)^{\frac{1}{2}}+\tilde{E}(t)^{\frac{9}{2}}+\tilde{E}(t))\tilde{D}(t).\end{aligned} \tag{7.8}$$

Therefore, adding the inequalities (7.1),(7.2), (7.5) and(7.8) together and summing up for all $0 \le k \le s$, we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\tilde{E}(t)+\tilde{D}(t) \le \hat{C}\sum_{1}^{11}\tilde{E}(t)^{\frac{k}{2}}\tilde{D}(t).$$

Then we complete the proof of Lemma 7.1.

We define the following number $T^* = \sup\{\tau > 0; \sup\limits_{t\in[0,\tau]}\hat{C}\sum\limits_{1}^{11}\tilde{E}(t)^{\frac{k}{2}} \le \frac{1}{2}\} \ge 0$, where the constant $\hat{C} > 0$ is mentioned in Lemma 7.1.

Since $C_*\tilde{E}(t) \le E(t) \le \tilde{E}(t)$, there exists a positive number ϵ_0, such that

$$\hat{C}\sum_{k=1}^{11}\tilde{E}(0)^{\frac{k}{2}} \le \frac{1}{4} \le \frac{1}{2},$$

when $E(0) \le \epsilon_0$. From the continuity of the energy function $\tilde{E}(t)$, we can deduce that $T^* > 0$. Thus for all $t \in [0, T^*]$

$$\tfrac{1}{2}\tfrac{d}{dt}\tilde{E}(t)+[1-\hat{C}\sum_{k=1}^{11}\tilde{E}(t)^{\frac{k}{2}}]\tilde{D}(t) \le 0,$$

which immediately means that $\tilde{E}(t) \le \tilde{E}(0) \le \frac{1}{C_*}E^{in}$ holds for all $t \in [0, T^*]$, and consequently $\sup\limits_{t\in[0,T^*]}\hat{C}\sum\limits_{k=1}^{11}\tilde{E}(t)^{\frac{k}{2}} \le \frac{1}{4}$. We now can claim that $T^* = +\infty$. Otherwise, if $T^* < +\infty$, the continuity of the energy $\tilde{E}(t)$ implies that there is a constant $\theta > 0$ such that

$$\sup_{t\in[0,T^*+\theta]}\hat{C}\sum_{k=1}^{11}\tilde{E}(t)^{\frac{k}{2}} \le \frac{3}{8} \le \frac{1}{2},$$

which contradicts to the definition of T^*. Therefore we get

$$\begin{aligned}&\sup_{t\geq 0}(\|v\|_{H^s}^2+\|\nabla n\|_{H^s}^2+\|C_p\|_{H^s}^2+\|C_m\|_{H^s}^2)\\&+\int_0^{+\infty}(\|\nabla C_p\|_{H^s}^2+\|\nabla C_m\|_{H^s}^2+\frac{\alpha_4}{2}\|\nabla v\|_{H^s}^2+\|\dot{n}\|_{H^s}^2)d\tau\leq CE^{in},\end{aligned}$$

where C is independent of (C_p, C_m, Φ, v, n), and as a consequence, the proof of Theorem 2.2 is finished.

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不可压液晶模型经典解的存在性

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摘要: 本文研究一个描述离子在向列型液晶中输运和扩散的非线性偏微分方程模型. 该模型耦合了对应于电势满足Maxwell's方程的离子的连续性方程的Nernst-Planck系统, 控制液晶流演变的不可压Naiver-Stokes方程与关于液晶方向场的非线性Allen-Cahn型方程. 我们利用能量方法证明了该系统的大初值经典解的局部存在性和小初值经典解的整体存在性.

关键词: 液晶; 能量方法; 经典解

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