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THE DIRICHLET PROBLEM OF A SPECIAL LAGRANGIAN TYPE EQUATION WITH SUPERCRITICAL PHASE

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Abstract: In this paper, we introduce a special Lagrangian type operator, and consider the corresponding Dirichlet problem of the special Lagrangian type equation with supercritical phase. By establishing the global C^2 estimates, we obtain the existence theorem of classical solutions by the method of continuity.

Keywords: special lagrangian type equation; Dirichlet problem; supercritical phase

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1 Introduction

For a smooth function $u(x)$ in $\mathbb{R}^n$ $(n \geq 3)$, we assume $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalue of Hessian matrix $D^2u := \{\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\}_{1 \leq i,j \leq n}$. Then there is a mapping induced by D^2u as follows

$$Pe_i = \sum_{j=1}^{n} u_{ij} e_j,$$

where $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$. As in Caffarelli-Nirenberg-Spruck [1], we consider the self-adjoint mapping

$$U = \sum_{k=1}^{n-1} 1 \otimes \cdots \otimes \underset{k}{P} \otimes \cdots \otimes 1,$$

acting on the real vector space $\Lambda^{n-1}\mathbb{R}^n$, that is

$$U(e_{i_1} \wedge \cdots \wedge e_{i_{n-1}}) = \sum_{k=1}^{n-1} e_{i_1} \otimes \cdots \otimes \underset{k}{Pe_{i_k}} \otimes \cdots \otimes e_{i_{n-1}}.$$

Then the eigenvalues of U are $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ with

$$\eta_i = \sum_{k \neq i} \lambda_k, \quad \forall i = 1, 2, \cdots, n.$$

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Hence we have a special Lagrangian type operator

$$\arctan \eta =: \arctan \eta_1 + \arctan \eta_2 + \cdots + \arctan \eta_n.$$

In fact, if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalue of Hessian matrix D^2u, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ are the eigenvalue of matrix $\{\Delta u \mathrm{I}_n - D^2u\}$, and

$$\arctan \eta = \arctan\{\Delta u \mathrm{I}_n - D^2 u\}.$$

In this paper, we study the Dirichlet problems of the corresponding special Lagrangian type equation

$$\begin{cases} \arctan\{\Delta u \mathrm{I}_n - D^2u\} = \Theta(x), & \text{in} \quad \Omega \subset \mathbb{R}^n, \\ u = \varphi(x), & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

where $\Theta(x) \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$ is called the phase. In particular, $\Theta = \frac{(n-2)\pi}{2}$ is the critical phase, and if $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$, the equation (1.1) is called special Lagrangian type equation with supercritical phase.

The special Lagrangian equation

$$\arctan D^2u =: \arctan \lambda_1 + \arctan \lambda_2 + \cdots + \arctan \lambda_n = \Theta.$$

was introduced by Harvey-Lawson [2] in the study of calibrated geometries. Here Θ is a constant called the phase angle. In this case the graph $x \mapsto (x, Du(x))$ defines a calibrated, minimal submanifold of $\mathbb{R}^{2n}$. Since the work of Harvey-Lawson, special Lagrangian manifolds have gained wide interests, due in large part to their fundamental role in the Strominger-Yau-Zaslow description of mirror symmetry [3]. For the special Lagrangian equations with supercritical phase, Yuan obtained the interior C^1 estimate with Warren in [4] and the interior C^2 estimate with Wang in [5]. Recently Collins-Picard-Wu [6] obtained the existence theorem of the Dirichlet problem by adopting the classic method with some important observation about the concavity of the operator.

In fact, the Dirichlet problems of elliptic equations in $\mathbb{R}^n$ were widely studied. For the Laplace equation, the Dirichlet problem was well studied in [7, 8]. For fully nonlinear elliptic equations, the pioneering work was done by Caffarelli-Nirenberg-Spruck in [1, 9] and Ivochkina in [10]. In their papers, they solved the Dirichlet problem for Monge-Ampère equations and k-Hessian equations elegantly. Since then, many interesting fully nonlinear equations with different structure conditions have been researched, such as Hessian quotient equations, which were solved by Trudinger in [11]. For more information, we refer the citations of [9].

In this paper, we establish the following existence theorem of (1.1)

Theorem 1.1 Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 strictly convex domain, $\varphi \in C^2(\partial\Omega)$ and $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Then there exists a unique solution $u \in C^{3,\alpha}(\overline{\Omega})$ to the Dirichlet problem (1.1).

Remark 1.2 In addition, if Ω, Θ and φ are all smooth, the solution u is also smooth on $\overline{\Omega}$.

Remark 1.3 As in [6], if we assume there is a subsolution $\underline{u}$ instead of the strict convexity of Ω, Theorem 1.1 still holds.

The rest of the paper is organized as follows. In Section 2, we give some properties and establish the C^0 estimates. In Section 3 and 4, we establish the C^1 and C^2 estimates for the Dirichlet problem (1.1). And Theorem 1.1 is proved in the Section 5.

2 Some Properties and a Priori Estimates

In this section, we give some properties and establish the C^0 estimates for the Dirichlet problem (1.1).

Property 2.1 Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Suppose $u \in C^2(\Omega)$ is a solution of the equation (1.1) and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix $D^2 u$ with

$$\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n. \tag{2.1}$$

Then we have the following properties:

$$\eta_1 \geq \eta_2 \cdots \geq \eta_n, \tag{2.2}$$

$$\eta_1 + \eta_2 \cdots + \eta_n > 0, \tag{2.3}$$

$$|\eta_n| \leq \eta_{n-1}, \tag{2.4}$$

$$|\eta_n| \leq C_0, \tag{2.5}$$

where $C_0 = \max\left\{\tan\left\{\frac{(n-1)\pi}{2} - \min\limits_{\overline{\Omega}} \Theta(x)\right\}, \tan(\frac{\max\limits_{\overline{\Omega}} \Theta(x)}{n})\right\}$.

These properties are well-known and can be similarly found in [5, 12] and [13].

Property 2.2 Suppose $\Omega \subset \mathbb{R}^n$ is a domain and $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Let $u \in C^4(\Omega)$ be a solution of (1.1). Then for any $\xi \in \mathbb{S}^{n-1}$, we have

$$\sum_{ij=1}^{n} F^{ij} u_{ij\xi\xi} \geq \Theta_{\xi\xi} - A{\Theta_\xi}^2, \quad \text{in } \Omega, \tag{2.6}$$

where $F^{ij} = \frac{\partial \arctan \eta}{\partial u_{ij}}$ and $A = \frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)}$.

Proof of Property 2.2 For any $x \in \Omega$, we can assume $D^2 u$ is diagonal with $\lambda_i = u_{ii}$, since (2.6) is invariant under rotating the coordinates. Then we have

$$F^{ij} =: \frac{\partial \arctan \eta}{\partial u_{ij}} = \frac{\partial \arctan \eta}{\partial \lambda_p} \frac{\partial \lambda_p}{\partial u_{ij}} = \begin{cases} \frac{\partial \arctan \eta}{\partial \lambda_i} = \frac{\partial \arctan \eta}{\partial \eta_p} \frac{\partial \eta_p}{\partial \lambda_i} = \sum\limits_{p \neq i} \frac{1}{1+\eta_p^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$$
\begin{aligned}
F^{ij,kl} =: \frac{\partial^2 \arctan\eta}{\partial u_{ij}\partial u_{kl}} &= \frac{\partial^2 \arctan\eta}{\partial\lambda_p\partial\lambda_q}\frac{\partial\lambda_p}{\partial u_{ij}}\frac{\partial\lambda_q}{\partial u_{kl}} + \frac{\partial\arctan\eta}{\partial\lambda_p}\frac{\partial^2\lambda_p}{\partial u_{ij}\partial u_{kl}} \\
&= \begin{cases}
\frac{\partial^2 \arctan\eta}{\partial\lambda_i\partial\lambda_k} = \frac{\partial^2 \arctan\eta}{\partial\eta_p\partial\eta_q}\frac{\partial\eta_p}{\partial\lambda_i}\frac{\partial\eta_q}{\partial\lambda_k}, & \text{if } i=j, k=l, \\
\frac{\frac{\partial\arctan\eta}{\partial\lambda_i}-\frac{\partial\arctan\eta}{\partial\lambda_j}}{\lambda_i-\lambda_j} = -\frac{\eta_i+\eta_j}{(1+\eta_i^2)(1+\eta_j^2)}, & \text{if } i=l, j=k, i\neq j, \\
0, & \text{otherwise .}
\end{cases}
\end{aligned}
$$

From the equation (1.1), we know

$$
\Theta_\xi = \sum_{ij=1}^n F^{ij}u_{ij\xi} = \frac{\partial\arctan\eta}{\partial\lambda_p}\frac{\partial\lambda_p}{\partial u_{ij}}u_{ij\xi} = \frac{\partial\arctan\eta}{\partial\eta_p}\frac{\partial\eta_p}{\partial\lambda_i}u_{ii\xi},
$$

and

$$
\begin{aligned}
\sum_{ij=1}^n F^{ij}u_{ij\xi\xi} =& \Theta_{\xi\xi} - \sum_{ijkl=1}^n F^{ij,kl}u_{ij\xi}u_{kl\xi} = \Theta_{\xi\xi} - \sum_{i,k=1}^n F^{ii,kk}u_{ii\xi}u_{kk\xi} - \sum_{i\neq j}F^{ij,ji}u_{ij\xi}^2 \\
\geq& \Theta_{\xi\xi} - \sum_{i,k=1}^n F^{ii,kk}u_{ii\xi}u_{kk\xi}. \qquad (2.7)
\end{aligned}
$$

From the concavity lemma (Lemma 2.2 in [6]), we know

$$
\begin{aligned}
-\sum_{i,k=1}^n F^{ii,kk}u_{ii\xi}u_{kk\xi} &= -\frac{\partial^2\arctan\eta}{\partial\eta_p\partial\eta_q}\frac{\partial\eta_p}{\partial\lambda_i}\frac{\partial\eta_q}{\partial\lambda_k}\cdot u_{ii\xi}u_{kk\xi} = -\frac{\partial^2\arctan\eta}{\partial\eta_p\partial\eta_p}\cdot\frac{\partial\eta_p}{\partial\lambda_i}u_{ii\xi}\cdot\frac{\partial\eta_p}{\partial\lambda_k}u_{kk\xi} \\
&\geq -\frac{2}{\tan\left(\min\limits_{\overline{\Omega}}\Theta - \frac{(n-2)\pi}{2}\right)}\Big(\sum_{p=1}^n \frac{\partial\arctan\eta}{\partial\eta_p}\cdot\frac{\partial\eta_p}{\partial\lambda_i}u_{ii\xi}\Big)^2 \\
&= -\frac{2}{\tan\left(\min\limits_{\overline{\Omega}}\Theta - \frac{(n-2)\pi}{2}\right)}{\Theta_\xi}^2. \qquad (2.8)
\end{aligned}
$$

Hence (2.6) holds.

The C^0 estimate is easy.

Theorem 2.3 Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $\varphi\in C^0(\partial\Omega)$. Suppose $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ is the solution of (1.1) and $\Theta(x)\in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2}<\Theta(x)<\frac{n\pi}{2}$ in $\overline{\Omega}$, then we have

$$
\sup_{\overline{\Omega}}|u|\leq M_0, \qquad (2.9)
$$

where M_0 depends on n, $\mathrm{diam}(\Omega)$, $\max\limits_{\partial\Omega}|\varphi|$ and $\max\limits_{\overline{\Omega}}\Theta$.

Proof of Theorem 2.3 From (2.3), we have

$$
\Delta u = \lambda_1+\lambda_2+\cdots+\lambda_n = \frac{1}{n-1}\sum_i^n \eta_i > 0. \qquad (2.10)
$$

Then it yields from the maximum principle

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} \varphi. \tag{2.11}$$

Without loss of generality, we assume $0 \in \Omega$, and denote $B = \frac{1}{2(n-1)} \tan(\frac{\max_{\bar{\Omega}} \Theta}{n})$ and $F(D^2u) =: \arctan \eta$. Then we have

$$F(D^2u(x)) = \Theta(x) \leq \max_{\Omega} \Theta = F(D^2(B|x|^2)). \tag{2.12}$$

By the maximum principle, we can get

$$\min_{\overline{\Omega}}(u - B|x|^2) = \min_{\partial\Omega}(u - B|x|^2). \tag{2.13}$$

Hence

$$u \geq u - B|x|^2 \geq \min_{\partial\Omega}(u - B|x|^2) \geq \min_{\partial\Omega} \varphi - B\mathrm{diam}(\Omega)^2. \tag{2.14}$$

3 Global Gradient Estimate

In this section, we will prove the global gradient estimate of (1.1).

Theorem 3.1 Let $\Omega \subset \mathbb{R}^n$ be a C^2 strictly convex domain and $\varphi \in C^1(\partial\Omega)$. Suppose $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ is the solution of (1.1) and $\Theta(x) \in C^1(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$, then we have

$$\sup_{\overline{\Omega}} |Du| \leq M_1, \tag{3.1}$$

where M_1 depends on n, $\mathrm{diam}(\Omega)$, $|\varphi|_{C^1}$ and $|\Theta|_{C^1}$.

Proof of Theorem 3.1 In the following, we prove Theorem 3.1 by two steps.

Step 1 Prove $\max_{\bar{\Omega}} |Du| \leq \max_{\partial\Omega} |Du| + C$.

Consider the auxiliary function

$$P(x) = |Du(x)| + e^{u(x)}, \tag{3.2}$$

assume $P(x)$ attains its maximum at $x_0 \in \bar{\Omega}$. If $x_0 \in \partial\Omega$, then we have

$$|Du(x)| \leq \max_{\partial\Omega} |Du(x)| + e^{u(x_0)} - e^{u(x)}. \tag{3.3}$$

If $x_0 \in \Omega$, we can choose the coordinates $\{e_1, e_2, \cdots, e_n\}$ at x_0 such that

$$u_1(x_0) = |Du(x_0)| > 0, \quad \{u_{ij}(x_0)\}_{2 \leq i,j \leq n} \text{ is diagonal}. \tag{3.4}$$

Then

$$\tilde{P}(x) =: u_1(x) + e^{u(x)} \tag{3.5}$$

also attains its local maximum at x_0. Hence we have at x_0

$$0 = \tilde{P}_i = u_{1i} + e^u u_i, \tag{3.6}$$

which yields

$$u_{11} = -e^u u_1 < 0, \tag{3.7}$$

$$u_{1i} = 0, \quad i \geq 2. \tag{3.8}$$

So we know $\{D^2u(x_0)\}$ is diagonal. It follows that $F^{ij} =: \frac{\partial \arctan \eta}{\partial u_{ij}}$ is diagonal at x_0. In fact,

$$F^{ij} = \sum_p \frac{\partial \arctan \eta}{\partial \lambda_p} \frac{\partial \lambda_p}{\partial u_{ij}} = \begin{cases} \frac{\partial \arctan \eta}{\partial \lambda_i} = \sum\limits_{p \neq i} \frac{1}{1+{\eta_p}^2}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3.9}$$

Also, we have

$$0 \geq \tilde{P}_{ii} = u_{1ii} + e^u({u_i}^2 + u_{ii}), \tag{3.10}$$

and then

$$\begin{aligned} 0 \geq \sum_i F^{ii} \tilde{P}_{ii} =& \sum_i F^{ii} u_{ii1} + e^u (\sum_i F^{ii} {u_i}^2 + \sum_i F^{ii} u_{ii}) \\ =& \Theta_1 + e^u (F^{11} {u_1}^2 + \sum_p \frac{\eta_p}{1+\eta_p^2}) \\ \geq& -|D\Theta| + {u_1}^2 \frac{e^u}{1+{C_0}^2} - e^u \frac{n}{2}. \end{aligned} \tag{3.11}$$

Hence $u_1 \leq C$, and then $|Du(x)| \leq C$.

Step 2 Prove $\max\limits_{\partial\Omega} |Du| \leq C$.

From the boundary condition $u = \varphi$ on $\partial\Omega$, we know $u_\tau = \varphi_\tau$ for any tangential vector τ of $\partial\Omega$ and $|u_\tau| \leq \max |D\varphi|$.

Now we consider the normal derivative u_ν, where ν is the unit normal vector of $\partial\Omega$. Firstly, we extend φ to $\bar{\Omega}$ by

$$\begin{cases} \Delta\tilde{\varphi} = 0, & in \quad \Omega, \\ \tilde{\varphi} = \varphi, & on \quad \partial\Omega. \end{cases} \tag{3.12}$$

It is easy to know $\tilde{\varphi} \in C^\infty(\Omega) \cap C^2(\bar{\Omega})$, and

$$u \leq \tilde{\varphi}, \text{ in } \Omega; \tag{3.13}$$

$$u = \tilde{\varphi}, \text{ on } \partial\Omega. \tag{3.14}$$

Since Ω is a C^2 strictly convex domain, then there is a defining function $h \in C^2(\bar{\Omega})$ such that

$$h = 0, \text{ on } \partial\Omega; \quad h < 0 \text{ in } \Omega; \tag{3.15}$$

$$D^2 h \geq c_0 I_n > 0 \text{ in } \Omega. \tag{3.16}$$

Then we can prove

$$\underline{u}(x) =: \tilde{\varphi} + Bh(x) \tag{3.17}$$

is a subsolution for B is large enough. In fact, the eigenvalues of $D^2\underline{u}$ are $\underline{\lambda}_i \geq Bc_0 - |D^2\tilde{\varphi}|$, and $\underline{\eta}_i \geq (n-1)[Bc_0 - |D^2\tilde{\varphi}|]$. Hence for B large enough, we have

$$\underline{u} \leq u, \text{ in } \Omega; \tag{3.18}$$

$$\underline{u} = u, \text{ on } \partial\Omega. \tag{3.19}$$

which yields $|D_\nu u(x)| \leq \max\{|D\underline{u}|, |D\tilde{\varphi}|\}$ for $x \in \partial\Omega$. This completes the proof of Theorem 3.1.

4 Global Second Derivatives Estimate

We come now to the a priori estimates of global second derivatives and we obtain the following theorem.

Theorem 4.1 Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 strictly convex domain and $\varphi \in C^2(\partial\Omega)$. Let $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution of (1.1), then we have

$$\sup_{\overline{\Omega}} |D^2 u| \leq M_2, \tag{4.1}$$

where M_2 depends on n, Ω, $\min\limits_{\overline{\Omega}} \Theta$, $|u|_{C^1}$, $|\Theta|_{C^2}$ and $|\varphi|_{C^2}$.

Proof of Theorem 4.1 In the following, we prove Theorem 4.1 by two steps.

Step 1 Prove $\max\limits_{\bar{\Omega}} |D^2u| \leq C(1 + \max\limits_{\partial\Omega} |D^2u|)$.

Consider the auxiliary function

$$P(x) = \log \lambda_{\max}(D^2u) + b|x|^2, \tag{4.2}$$

where $\lambda_{\max}(D^2u(x))$ is the largest eigenvalue of $D^2u(x)$ and $b = \frac{1}{4\mathrm{diam}(\Omega)^2}$. Assume $P(x)$ attains its maximum at $x_0 \in \bar{\Omega}$. If $x_0 \in \partial\Omega$, then we have

$$\log \lambda_{\max}(D^2u(x)) \leq P(x) \leq \max_{\partial\Omega} \log \lambda_{\max}(D^2u) + b|x_0|^2, \tag{4.3}$$

hence $\max\limits_{\bar{\Omega}} |D^2u| \leq C(1 + \max\limits_{\partial\Omega} |D^2u|)$.

If $x_0 \in \Omega$, we can choose the coordinates $\{e_1, e_2, \cdots, e_n\}$ at x_0 such that

$$D^2u(x_0) \text{ is diagonal.} \tag{4.4}$$

Without loss of generality, we assume $u_{11}(x_0) \geq u_{22}(x_0) \geq \cdots \geq u_{nn}(x_0)$. Then

$$\tilde{P}(x) =: u_{11}(x) + b|x|^2 \tag{4.5}$$

also attains its local maximum at $x_0 \in \Omega$. Hence we have at x_0

$$0 = \tilde{P}_i = \frac{u_{11i}}{u_{11}} + 2bx_i, \tag{4.6}$$

hence

$$\frac{u_{11i}}{u_{11}} = -2bx_i, \quad i = 1, 2, \cdots, n. \tag{4.7}$$

Moreover, we have at x_0

$$0 \geq \tilde{P}_{ii} = \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} + 2b = \frac{u_{11ii}}{u_{11}} - 4b^2x_i^2 + 2b. \tag{4.8}$$

Hence we have at x_0

$$\begin{aligned} 0 \geq \sum_{i=1}^{n} F^{ii}\tilde{P}_{ii} =& \frac{\sum\limits_i F^{ii}u_{ii11}}{u_{11}} + 2b\sum_i F^{ii}(1-2bx_i^2) \\ \geq& \frac{1}{u_{11}}(\Theta_{11} - A\Theta_1^2) + b\sum_{i=1}^{n} F^{ii} \\ \geq& \frac{1}{u_{11}}(\Theta_{11} - A\Theta_1^2) + b\frac{1}{1+C_0^2}. \end{aligned} \tag{4.9}$$

Hence $u_{11}(x_0) \leq C$, and then $|D^2u(x_0)| \leq C$.

Step 2 Prove $\max\limits_{\partial\Omega}|D^2u| \leq C$.

For any point $x_0 \in \partial\Omega$, we assume $x_0 = 0 \in \partial\Omega$, and $\partial\Omega$ is expressed by $x_n = \rho(x')$ near $x_0 = 0$, where $x' = (x_1, \cdots, x_{n-1})$. Moreover, we can assume

$$\begin{aligned} &\rho(0) = 0, \quad D\rho(0) = 0, \\ &\rho(x') = \frac{1}{2}\sum_{i=1}^{n-1} b_i x_i^2 + O(|x'|^3). \end{aligned} \tag{4.10}$$

From the boundary condition of (1.1), we have $u(x', \rho(x)) = \varphi(x', \rho(x))$ near 0, and then for $i, j = 1, 2, \cdots, n-1$, it yields

$$\begin{aligned} &u_i + u_n\rho_i = \varphi_i + \varphi_n\rho_i, \\ &u_{ij} + u_{in}\rho_j + u_{nj}\rho_i + u_{nn}\rho_i\rho_j + u_n\rho_{ij} = \varphi_{ij} + \varphi_{in}\rho_j + \varphi_{nj}\rho_i + \varphi_{nn}\rho_i\rho_j + \varphi_n\rho_{ij}. \end{aligned} \tag{4.11}$$

Hence

$$u_{ij}(0) = \varphi_{ij}(0) + \varphi_n(0)\rho_{ij}(0) - u_n(0)\rho_{ij}(0), \tag{4.12}$$

and $|u_{ij}(0)| \leq C$ for $i, j \leq n-1$.

Now we estimate $|u_{in}(0)|$ for $i = 1, \cdots, n$. Define

$$T = \frac{\partial}{\partial x_i} + b_i(x_i\frac{\partial}{\partial x_n} - x_n\frac{\partial}{\partial x_i}),$$

then we have

$$|\sum_{i,j} F^{ij}\partial_{ij}T(u-\tilde{\varphi})| = |\sum_{i,j} F^{ij}\partial_{ij}Tu - \sum_{i,j} F^{ij}\partial_{ij}T\tilde{\varphi})| \leq |T\Theta| + C\sum_i F^{ii} \leq C, \tag{4.13}$$

in $\Omega \cap B_\varepsilon(0)$ with $\varepsilon > 0$ small, and on $\partial\Omega \cap B_\varepsilon(0)$

$$|T(u-\tilde{\varphi})| = |(\partial_i + \partial_i\rho\partial_n)(u-\tilde{\varphi}) + O(|x'|^2) - b_i x_n \partial_i(u-\tilde{\varphi})| \leq C|x'|^2. \tag{4.14}$$

Denote $w(x) = \rho(x') - x_n - a\,|x'| + x_n^2$, in $\Omega \cap B_\varepsilon(0)$, then we have

$$\sum F^{ij} w_{ij} = \sum_{i,j=1}^{n-1} F^{ij}(\rho_{ij} - a\delta_{ij}) + F^{nn} \geq \varepsilon_0 \sum_{i\leq n-1} F^{ii} + F^{nn} \geq c_0 > 0, \tag{4.15}$$

where a is a very small positive constant. Hence for K large enough, we have

$$\sum_{i,j} F^{ij}\partial_{ij}[Kw \pm T(u-\tilde{\varphi})] \geq 0, \quad \text{in} \quad \Omega \cap B\varepsilon(0). \tag{4.16}$$

Moreover, on $\partial\Omega \cap B_\varepsilon(0)$, we have $x_n = \rho(x')$, and then

$$\begin{aligned} & Kw \pm T(u-\tilde{\varphi}) \\ =& K(-a|x'|^2 + \rho(x')^2) \pm T(u-\tilde{\varphi}) \leq K(-a|x'|^2 + \rho(x')^2) + C|x'|^2 \leq 0, \end{aligned} \tag{4.17}$$

if we choose $K > 0$ large enough and $\varepsilon > 0$ small enough. On $\Omega \cap \partial B_\varepsilon(0)$, we have $\rho(x') \leq x_n$ and $x_n \geq c_0 > 0$, and then

$$\begin{aligned} & Kw \pm T(u-\tilde{\varphi}) \\ \leq& K(\rho(x') - x_n - c\rho(x') + x_n^2) \pm T(u-\tilde{\varphi}) \leq K(-cx_n + x_n^2) + C \leq 0. \end{aligned} \tag{4.18}$$

Hence we have

$$|\partial_n(T(u-\tilde{\varphi}))(0)| \leq K|\partial_n w(0)|, \tag{4.19}$$

and $|u_{in}(0)| \leq C$.

At last, we prove $|u_{nn}(0)| \leq C$. The idea is from Trudinger [11], and later used by Guan [14]. See also [6]. For any point $x \in \partial\Omega$, let $\{e_i\}_{i=1}^n$ be an orthonormal local frame defined in a neighbourhood of x such that e_n is the inner normal. For $1 \leq \alpha, \beta \leq n-1$, define $\sigma_{\alpha\beta} = \langle \nabla_{e_\alpha} e_\beta, e_n \rangle$, where ∇ denotes the covariant derivative with respect to the flat Euclidean metric. In fact, $\sigma_{\alpha\beta}$ is the second fundamental form of $\partial\Omega$. Since $u = \underline{u}$ on $\partial\Omega$ (here $\underline{u}$ is a subsolution defined in (3.17)), we have

$$u_{\alpha\beta}(x) - \underline{u}_{\alpha\beta}(x) = -(u-\underline{u})_n(x)\sigma_{\alpha\beta}(x), \quad x \in \partial\Omega, \tag{4.20}$$

where

$$u_{\alpha\beta} = \nabla_{e_\beta}(\nabla_{e_\alpha} u) - \nabla_{\nabla_{e_\beta} e_\alpha} u \tag{4.21}$$

is the Riemannian Hessian with the eigenvalues $\lambda'(u_{\alpha\beta}) = (\lambda'_1, \lambda'_2, \cdots, \lambda'_{n-1})$. As in [6], assume $g(\lambda) = -e^{-A\sum \arctan \eta_i}$ and $\psi(x) = -e^{-A\Theta(x)}$, where A is defined in Property 2.2. Then for any $x \in \partial\Omega$, we can define

$$\tilde{G}(\lambda'(u_{\alpha\beta})) = \lim_{R\to\infty} g(\lambda', R) = -e^{-A\arctan(\lambda'_1+\cdots+\lambda'_{n-1}) - A\frac{(n-1)\pi}{2}}. \tag{4.22}$$

Assume the minimum value of $\tilde{G}(\lambda'(u_{\alpha\beta}))(x) - \psi(x)$ on $\partial\Omega$ is achieved at $y_0 \in \partial\Omega$. As in [6], we can prove $|u_{nn}(y_0)| \leq C$, and then

$$\tilde{G}(\lambda'(u_{\alpha\beta}))(x) - \psi(x) \geq \tilde{G}(\lambda'(u_{\alpha\beta}))(y_0) - \psi(y_0) \geq 2c_0 > 0. \tag{4.23}$$

Hence there exists a R_0 large such that

$$g(\lambda', R_0) \geq \tilde{G}(\lambda'(u_{\alpha\beta})) - c_0 \geq \psi(x) + c_0. \tag{4.24}$$

If $|u_{nn}(0)| \geq R_{\delta_0}$, we have from Lemma 1.2 in [1]

$$\lambda_n \geq R_0, |\lambda - (\lambda', \lambda_n)| < \delta_0, \tag{4.25}$$

and then

$$(\lambda(u_{ij}))(0) \geq g(\lambda', \lambda_n) - \frac{c_0}{2} \geq g(\lambda', R_0) - \frac{c_0}{2} \geq \psi(0) + \frac{c_0}{2}, \tag{4.26}$$

which is a contradiction. Hence $|u_{nn}(0)| \leq R_{\delta_0}$.

5 Proof of Theorem 1.1

In this section, we complete the proof of the Theorem 1.1.

For the Dirichlet problem of equation (1.1), we have established the C^0, C^1 and C^2 estimates in Section 2, 3 and 4. By the global C^2 priori estimate, the equation (1.1) is uniformly elliptic in $\overline{\Omega}$. From Property 2.2, we know $-e^{-A \arctan \eta}$ is concave with respect to D^2u, where A is defined in Property 2.2. Following the discussions in the Evans-Krylov theorem [15, 16], we can get the global Hölder estimate of second derivatives,

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \tag{5.1}$$

where C and α depend on n, Ω, $\max\limits_{\overline{\Omega}} \Theta$, $\min\limits_{\overline{\Omega}} \Theta$, $|\Theta|_{C^2}$ and $|\varphi|_{C^2}$. From (5.1), one also obtains $C^{3,\alpha}(\overline{\Omega})$ estimates by differentiating the equation (1.1) and applies the Schauder theory for linear uniformly elliptic equations.

Applying the method of continuity (see [6]), the existence of the classical solution holds. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity.

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超临界相位的 Special Lagrangian 方程的狄利克雷边值问题

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摘要: 在这篇文章中, 我们介绍了一种 Special Lagrangian 方程, 并且研究此方程在超临界相位时对应的狄利克雷边值问题. 通过建立整体的 C^2 估计, 运用经典的连续性方法得到解的存在性.

关键词: Special Lagrangian 方程; 狄利克雷问题; 超临界相位

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