Vol. 42 (2022)
No. 5

数 学 杂 志
J. of Math. (PRC)

ON UQ-RINGS AND STRONG UJII-RINGS

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Abstract: In this paper, we introduce the notions of UQ-rings and strong UJII-rings which generalize the concept of UJ-rings. We provide many properties and structures of these two classes of rings by using theoretical skills in rings. The conclusions enrich the theory that is related to elements decomposition.

Keywords: UJ-ring; UQ-ring; strong UJII-ring; clean ring
2010 MR Subject Classification: 16U99; 16N20
Document code: A **Article ID:** 0255-7797(2022)05-0402-08

1 Introduction

All rings considered are associative with unity. Let R be a ring. The set of all units, the set of all idempotents and the Jacobson radical of R are denoted by $U(R)$, $idem(R)$ and $J(R)$, respectively. The symbol $M_n(R)$ stands for the $n \times n$ matrix ring over R whose identity element we write as I_n.

Rings whose elements are sums of certain special elements have been widely studied in ring theory. Recall that a ring R is called clean if every element of R is the sum of an idempotent and a unit. Clean rings were introduced by Nicholson [1] in relation to exchange rings. A ring R is called strongly clean [2] if every element of R is the sum of an idempotent and a unit that commutes. According to [3, 4], a ring R is called J-clean if for each $a \in R$, $a = e + j$ for some $e^2 = e \in R$ and $j \in J(R)$ (also called a semiboolean ring in [5]). Recently, Danchev [6] and Kosan et al. [3] called a ring R UJ if every unit of R is the sum of an idempotent and an element from $J(R)$, or equivalently, $U(R) = 1 + J(R)$. It was shown in [3] that a ring R is J-clean if and only if R is a clean UJ-ring.

Due to Harte [7], an element $a \in R$ is called quasinilpotent if $1 - ax \in U(R)$ for every $x \in \mathrm{comm}(a)$; the set of all quasinilpotents of R is denoted by R^{qnil}. Clearly, $J(R) \subseteq R^{qnil}$. Motivated by the above, we say that a unit u of a ring R is UQ if $u = 1 + q$ for some $q \in R^{qnil}$; and a ring R is UQ if every unit of R is UQ (equivalently, $U(R) = 1 + R^{qnil}$). Elementary properties of UQ-elements are studied in section 2, and some characterizations of UQ-rings are provided in section 3. In section 4, we investigate rings for which every unit

* **Received date:** 2021-04-29 **Accepted date:** 2021-10-20
Foundation item: Supported by Anhui Provincial Natural Science Foundation (2008085MA06) and the Key Project of Anhui Education Committee (gxyqZD2019009).
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is the sum of two idempotents and an element from the Jacobson radical that commute with each other.

2 On UQ-elements

Let R be a ring. We say that a unit u of R is UJ if $u = 1 + j$ for some $j \in J(R)$. Clearly, all UJ-elements are UQ. In this section, we study the properties of UQ-elements (including UJ-elements).

The following result can be obtained by a direct check.

Proposition 2.1 The product of UJ-elements is UJ.

Remark 2.2 The product of UQ-elements needs not to be UQ. For example, let $\mathbb{Z}_2$ be the ring of integers modulo 2, and let

$$u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U(M_2(\mathbb{Z}_2)).$$

Then u, v are clearly UQ. But

$$uv = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is not UQ since $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \notin U(M_2(\mathbb{Z}_2))$.

For a ring R, we denote 2×2 upper triangular matrix ring over R by $T_2(R)$.

Proposition 2.3 Let R be a ring, $u, v \in U(R)$. Then

(1) u, v are UJ if and only if $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix}$ is UJ in $T_2(R)$ for any $x \in R$.

(2) u, v are UQ if and only if, for any $x \in R$, $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix}$ is UQ in $T_2(R)$.

Proof (1) Assume that u, v are UJ. Let $u = 1 + j_1$, $v = 1 + j_2$ where $j_1, j_2 \in J(R)$. Then

$$\begin{pmatrix} u & x \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1+j_1 & x \\ 0 & 1+j_2 \end{pmatrix} = \begin{pmatrix} j_1 & x \\ 0 & j_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\begin{pmatrix} j_1 & x \\ 0 & j_2 \end{pmatrix} \in J(T_2(R))$ for any $x \in R$, we get $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix}$ is UJ in $T_2(R)$.

For the converse, since $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix}$ is UJ, we have $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u-1 & x \\ 0 & v-1 \end{pmatrix}$ where $u - 1$ and $v - 1$ are in $J(R)$. Hence u, v are UJ.

(2) Suppose that u, v are UQ. Let $u = 1 + q_1$, $v = 1 + q_2$ where $q_1, q_2 \in R^{qnil}$. Then $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q_1 & x \\ 0 & q_2 \end{pmatrix} \in T_2(R)$. Next we show that $\begin{pmatrix} q_1 & x \\ 0 & q_2 \end{pmatrix} \in (T_2(R))^{qnil}$. Let $\begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \in T_2(R)$ with $\begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \begin{pmatrix} q_1 & x \\ 0 & q_2 \end{pmatrix} = \begin{pmatrix} q_1 & x \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$. Then $aq_1 = q_1 a$, $bq_2 = q_2 b$. Since $q_1, q_2 \in R^{qnil}$, $1 + aq_1 \in U(R)$ and $1 + bq_2 \in U(R)$. So $I_2 + \begin{pmatrix} aq_1 & ax + yq_2 \\ 0 & bq_2 \end{pmatrix} = \begin{pmatrix} 1 + aq_1 & ax + yq_2 \\ 0 & 1 + bq_2 \end{pmatrix} \in U(T_2(R))$. This proves $\begin{pmatrix} u & x \\ 0 & v \end{pmatrix}$ is UQ in $T_2(R)$.

Conversely, it suffices to prove that both $u-1$ and $v-1$ are quasinilpotents in R. We may let $x=0$. Since $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is UQ in $T_2(R)$, we have $\begin{pmatrix} u-1 & 0 \\ 0 & v-1 \end{pmatrix} \in (T_2(R))^{qnil}$. By an easy computation, one gets $u-1 \in R^{qnil}$ and $v-1 \in R^{qnil}$, as required.

Proposition 2.4 Let R be a ring, and $a, b \in R$.

(1) If ab is UJ, then ba is UJ if and only if $a, b \in U(R)$.

(2) If ab is UQ, then ba is UQ if and only if $a, b \in U(R)$.

Proof The proof of (2) follows from [8, Theorem 2.11]. We give a new proof of (1).

(1) Suppose that $a, b \in U(R)$. As ab is UJ, one has $1-ab \in J(R) = \cap M_i$ where M_i is all maximal right ideal of R. If ba is not UJ, then $1-ba \notin J(R)$. So, there exists a maximal right ideal M_{i_0} such that $1-ba \notin M_{i_0}$. It follows that $R = M_{i_0} + (1-ba)R = M_{i_0} + a^{-1}a(1-ba)R = M_{i_0} + a^{-1}(1-ab)aR$. Since $1-ab \in J(R)$, $a^{-1}(1-ab)aR \subseteq J(R)$. So $M_{i_0} + J(R) = R$, which is a contradiction. Therefore, ba is UJ. The converse is trivial.

Jacobson's Lemma states that for any $a, b \in R$, $1-ab \in U(R)$ if and only if $1-ba \in U(R)$. Recall that a ring R is reversible if $ab=0$ implies $ba=0$ for any $a, b \in R$.

Proposition 2.5 Let R be a ring.

(1) For any $a, b \in R$ with $1-ab$ is UJ, then $1-ba$ is UJ if and only if $R/J(R)$ is reversible.

(2) $1-ab$ is UQ if and only if $1-ba$ is UQ.

Proof By a direct computation, we can prove (1).

For (2), we can deduce from [9, Lemma 2.1]. Here, we give a simple proof for a convenience. Note that (2) is equivalent to the comment "ab is quasinilpotent if and only if so is ba". Now we assume that $ab \in R^{qnil}$ but $ba \notin R^{qnil}$. Then there exists $y \in R$ such that $(ba)y = y(ba)$ and $1 + bay \notin U(R)$. From $bay = yba$, we obtain $ab(ay^2b) = (ay^2b)ab$. Since $ab \in R^{qnil}$, $1 - ab(ay^2b) \in U(R)$. By Jacobson's Lemma, we have $1 - babay^2 = 1 - (bay)^2 \in U(R)$, which implies $1 + bay \in U(R)$, a contradiction. So $ba \in R^{qnil}$.

3 UQ-rings

This section is devoted to the study of UQ-rings.

Proposition 3.1 Let R be a UQ-ring.

(1) For any $q_1, q_2 \in R^{qnil}, q_1 + q_2 + q_1q_2 \in R^{qnil}$.

(2) If $q_1, q_2 \in R^{qnil}$ and $q_1q_2 = q_2q_1$, then $q_1 + q_2 \in R^{qnil}$.

Proof (1) Note that $(1+q_1)(1+q_2) \in U(R)$. Since R is a UQ-ring, $1+q_1+q_2+q_1q_2 \in 1 + R^{qnil}$, and so $q_1 + q_2 + q_1q_2 \in R^{qnil}$.

(2) Let $q_1, q_2 \in R^{qnil}$ and $q_1q_2 = q_2q_1$. Then

$$1 + q_1 + q_2 = (1+q_1)(1+(1+q_1)^{-1}q_2) \in U(R) = 1 + R^{qnil}.$$

So $q_1 + q_2 \in R^{qnil}$.

Proposition 3.2 Let R be a UQ-ring. Then:

(1) $2 \in J(R)$.

(2) eRe is UQ-ring.

Proof (1) Let $-1 = 1 + x$ with $x \in R^{qnil}$. Then $x = -2$. Note that x is central. Hence, $2 \in J(R)$.

(2) Let $x \in U(eRe)$. Then there exists $y \in eRe$ such that $xy = e = yx$. So we have $[x + (1-e)][y + (1-e)] = e + 0 + 0 + 1 - e = 1 = [y + (1-e)][x + (1-e)]$, which implies $[x + (1-e)] \in U(R)$. Since R is a UQ-ring, $x + (1-e) = 1 + q$ for some $q \in R^{qnil}$. Then $x - e = q \in (R^{qnil}) \cap eRe = (eRe)^{qnil}$ by [10, Lemma 3.5(2)]. So eRe is UQ-ring.

Proposition 3.3 Let R be a UQ-ring. Then $M_n(R)$ is not UQ for any $n \geq 2$.

Proof By Proposition 3.2(2), it suffices to prove $M_2(R)$ is not UQ. Let $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $A_1, A_2 \in M_2(R)^{qnil}$. But $A_1 + A_2 + A_1A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \notin M_2(R)^{qnil}$. By Proposition 3.1(1), $M_2(R)$ is not UQ.

We next determine when a single matrix over a field is UQ. For a matrix A over a filed, we write $\mathrm{tr}(A)$ and $\det A$ for the trace and the determinate of A, respectively.

Example 3.4 Let F be a field, $U \in U(M_2(F))$. Then U is UQ if and only if $U = I_2$ or there exists an invertible matrix $P \in M_2(F)$ such that $P^{-1}UP = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

Proof By [8, Example 2.2], $(M_2(F))^{qnil}$ equals the set of all nilpotents in $M_2(F)$.

Suppose that U is UQ. By Hamilton-Cayley Theorem, we have $U^2 = \mathrm{tr}(U) \cdot U - \det U \cdot I_2$. Since U is UQ, we have $U - I_2$ is nilpotent, and so $(U - I_2)^2 = 0$. It follows that $\mathrm{tr}(U) \cdot U - \det U \cdot I_2 = 2U - I_2$, and then $(\mathrm{tr}(U) - 2)U = (\det U - 1)I_2$. Notice that $0 = \det(U - I_2) = \det U - \mathrm{tr}(U) + 1$. Thus, $\det U = \mathrm{tr}(U) - 1$. Combing this equation with $(\mathrm{tr}(U) - 2)U = (\det U - 1)I_2$, we have $(\mathrm{tr}(U) - 2)U = (\mathrm{tr}U - 2)I_2$. If $\mathrm{tr}(U) \neq 2$, then $U = I_2$ and $\det U = \mathrm{tr}(U) - 1 \neq 1$, which is a contradiction. So $\mathrm{tr}(U) = 2$, and $\det U = 1$. Assume that $U \neq I_2$. Then there exists an invertible matrix $P \in M_2(F)$ such that $P^{-1}UP = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix}$ where $a_1, a_2 \in F$. Since $\det(P^{-1}UP) = \det U$ and $\mathrm{tr}(P^{-1}UP) = \mathrm{tr}(U)$, one gets $a_1 = -1$ and $a_2 = 2$. Therefore, $P^{-1}UP = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. The converse is clear.

Proposition 3.5 Let R be a ring.

(1) R is UJ-ring if and only if $J(R) = \{x \in R : 1 + x \in U(R)\}$.

(2) R is UQ-ring if and only if $R^{qnil} = \{q \in R : 1 + q \in U(R)\}$.

Proof We only need to prove (1). The proof of (2) is similar to that of (1).

(1) Assume that R is a UJ-ring. Clearly, $J(R) \subseteq \{x \in R : 1 + x \in U(R)\}$. Let $x \in R$ be such that $1 + x \in U(R)$. Since R is UJ, $1 + x = 1 + j$ for some $j \in J(R)$. So $x = j \in J(R)$. Thus, $J(R) \supseteq \{x \in R : 1 + x \in U(R)\}$, and then $J(R) = \{x \in R : 1 + x \in U(R)\}$.

Conversely, let $u \in U(R)$. Then $u = 1 + (u - 1)$. As $J(R) = \{x : 1 + x \in U(R)\}$, we get $u - 1 \in J(R)$, which implies that u is UJ, and hence R is a UJ-ring.

Recall that a ring R is abelian if every idempotent of R is central.

Theorem 3.6 Let R be a ring. Then the following are equivalent:

(1) R is a regular UQ-ring.

(2) R is an abelian regular UQ-ring.

(3) R has the identity $x^2 = x$ (i.e., R is Boolean).

Proof (1) $\Rightarrow$ (2). Since R is regular, $J(R) = 0$ and every nonzero right ideal contains a nonzero idempotent. Let $a \in R$, if $a \neq 0, a^2 = 0$. By [11, Theorem 2.1], there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some non-trivial ring T. As R is a UQ-ring, eRe is UQ. So $M_2(T)$ is a UQ-ring, which is a contradiction. So $a = 0$, and this proves that R is reduced. As is well known, reduced rings are abelian.

(2) $\Rightarrow$ (3). Let $q \in Q(R)$. Since R is strongly regular, there exist $e^2 = e$ and $u \in U(R)$ such that $q = eu = ue$. So $1 - e = 1 - qu^{-1} \in U(R)$, and whence $e = 0$, which implies $q = 0$. Thus $Q(R) = 0$, and so $U(R) = 1 + Q(R) = 1$. Clearly, a strongly regular ring R with $U(R) = 1$ is Boolean.

(3) $\Rightarrow$ (1). Clearly, R is regular. Let $u \in U(R)$. Then $u^2 = u$, which implies that $u = 1$, and thus, R is a UQ-ring.

Corollary 3.7 Let R be a ring with $J(R) = 0$. Then the followings are equivalent:

(1) R is an exchange UQ-ring.

(2) R is a clean UQ-ring.

Proof (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2). Since R is an exchange ring with $J(R) = 0$, by Theorem 3.6, every nonzero right ideal contains a nonzero idempotent. In view of the proof (1) $\Rightarrow$ (2) of Theorem 3.6, we have R is abelian. By [1, Propsition 1.8(2)], abelian exchange rings are clean.

Theorem 3.8 Let R be a ring. Then the followings are equivalent:

(1) R is a strongly clean UQ-ring.

(2) For any $a \in R$, there exist $e^2 = e \in R$ and $q \in R^{qnil}$ such that $a = e+q$ and $ae = ea$.

Proof (2) $\Rightarrow$ (1). By (2), $a = e + q = (1 - e) + (2e - 1 + q)$ where $e^2 = e \in R$ and $q \in R^{qnil}$. Note that $2e - 1 - q = (2e - 1) - q = (2e - 1)[1 - (2e - 1)q] \in U(R)$ as $eq = qe$. Hence, R is a strongly clean ring. Further, for any $v \in U(R)$, we have $v = e + q$. Then $e = v - q = v(1 - v^{-1})q \in U(R)$, which yields $e = 1$ and $v = 1 + q$. Thus R is also a UQ-ring.

(1) $\Rightarrow$ (2). Let $a \in R$. As R is a strongly clean ring, we have $1 + a = e + u$ where $e^2 = e \in R$, $u \in U(R)$ and $ae = ea$. Since R is a UQ-ring, $u = 1 + q$ for some $q \in R^{qnil}$. Thus, $1 + a = e + u = e + 1 + q$, and so $a = e + q$ and $ae = ea$.

Recall that a ring R is Dedekind finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$.

Theorem 3.9 Every UQ-ring is Dedekind finite.

Proof We first show the following claim:

Claim If a ring R contains a set of matrix units $\{e_{ij}\}_{i,j\leq 2}$ such that $e_{ii} \neq 0$ for $i = 1, 2$, then R is not a UQ-ring.

Proof of Claim. Let $f = e_{11} + e_{22}$ and $u = e_{12} + e_{21} + e_{22}$. Then it is easy to know that f is an idempotent, $u(u - f) = (u - f)u = f$ and $fuf = u$. Hence u, $u - f \in U(fRf)$. As $(fRf)^{qnil} \cap U(fRf) = \varnothing$, we obtain $u - f \notin (fRf)^{qnil}$, that is to say fRf is not UQ-ring. By Proposition 3.2(2), R is not a UQ-ring. So the Claim follows.

Let R be a UQ-ring, and assume that R is not Dedekind finite. Let $a, b \in R$ be such that $ab = 1$ and $e = ba \neq 1$. Clearly, e is a nontrivial idempotent. In view of [12, Example 21.26], there exists a set of nonzero matrix units of the form $e_{ij} = b^i(1-e)a^j$. Then applying the above Claim, we get that R is not a UQ-ring, a contradiction.

4 Strong $UJII$-rings

We call a ring R a strong $UJII$-ring if for each $u \in U(R)$, $u = j+e+f$ where $j \in J(R)$, $e, f \in idem(R)$ and j, e, f commute with one another. UJ-rings are strong $UJII$-rings.

Proposition 4.1 Let R be a strong $UJII$-ring.

(1) $R^{qnil} = J(R)$.

(2) $J(R) = \sqrt{J(R)}$ where $\sqrt{J(R)} = \{x \in R \mid x^k \in J(R)\ for\ some\ integer\ k \geq 1\}$.

Proof (1) Let $b \in R^{qnil}$. Then $1 + b \in U(R)$. Write $1 + b = j + e + f$ where $j \in J(R)$, $e, f \in idem(R)$ and b, j, e, f all commute. So $(1 - e) - f = j - b$. Note that $(1-e)-f = [(1-e)-f]^3$. Then $(b-j)^3 = b-j$. It follows that $b-b^3 = b(1-b^2) \in J(R)$. Since $1 - b^2 \in U(R)$, we obtain $b \in J(R)$. Thus, $R^{qnil} \subseteq J(R)$, and whence $R^{qnil} = J(R)$.

(2) It suffices to show that $\sqrt{J(R)} \subseteq J(R)$. Let $x \in \sqrt{J(R)}$. Then $x^k \in J(R)$ with $k \geq 1$ and $1-x \in U(R)$. By hypothesis, $1-x = j+e+f$ for some $j \in J(R)$, $e, f \in idem(R)$, and j, e, f all commute. Clearly, $(1-e)-f = j+x$. Note that $(1-e)-f = ((1-e)-f)^{2m+1}$ for any positive integer m. Since $x^k \in J(R)$ and $jx = xj$, we obtain

$$(1-e)-f = ((1-e)-f)^{2k+1} = (j+x)^{2k+1} \in J(R) + x^{2k+1} = J(R).$$

So $1 - [(1-e)-f]^{2k} \in U(R)$. It follows from $[(1-e)-f][1-((1-e)-f)^{2k}] = 0$ that $(1-e)-f = 0$. Thus $1-e = f$, and whence $x = -j \in J(R)$. Hence $\sqrt{J(R)} = J(R)$.

Corollary 4.2 Let R be a ring. Then $M_n(R)$ is not strong $UJII$ for any $n \geq 2$.

Proof Note that any matrix unit E_{ij} is quasinilpotent when $i \neq j$. However, $E_{ij} \notin J(M_n(R))$. By Proposition 4.1(1), $M_n(R)$ is not strong $UJII$.

Proposition 4.3 If R is a strong $UJII$-ring, then $6 \in J(R)$.

Proof Write $-1 = j+e+f$, where $j \in J(R)$, $e, f \in \mathrm{idem}(R)$ and j, e, f all commute. Then

$$1+2j+j^2 = (-1-j)^2 = (e+f)^2 = e+f+2ef = (-1-j)+2e(-1-j-e) = -1-j-4e-2ej.$$

We get $2+4e = -3j-j^2-2ej$, and then $6e = (2+4e)e = (-3j-j^2-2ej)e = -(5e+je)j \in J(R)$. Similarly, $6f \in J(R)$. Then $-6 = 6j + 6e + 6f \in J(R)$, so $6 \in J(R)$.

Clearly, a direct product of rings is strong $UJII$ if and only if each ring is strong $UJII$.

Corollary 4.4 Let R be a ring with $J(R) = 0$. Then R is a strong $UJII$-ring if and only if $R = A \oplus B$ where A, B are strong $UJII$-rings, and $2 = 0$ in A and $3 = 0$ in B.

Proof Suppose that R is a strong $UJII$-ring. Since $J(R) = 0$, by Proposition 4.3, $6 = 0$. By Chinese Remainder Theorem, $R \cong R/2R \oplus R/3R$. Let $A = R/2R$, $B = R/3R$. So A, B are strong $UJII$-rings, and $2 = 0$ in A, $3 = 0$ in B. The other direction is obvious.

A ring R is 2-UJ [13] if for any $u \in U(R)$, $u^2 = 1 + j$ for some $j \in J(R)$. Recall that a ring R is called reduced if R contains no nonzero nilpotent elements (equivalently, $a^2 = 0$ implies $a = 0$ for any $a \in R$).

Proposition 4.5 If R is a strong $UJII$-ring, then R is 2-UJ and $R/J(R)$ is reduced.

Proof Let $u \in U(R)$. Write $u = j + e + f$ where $j \in J(R)$, $e, f \in idem(R)$ and j, e, f all commute. Then $u - j = e + f \in U(R)$. Set $u^{'} = u - j = e - f + 2f$. Then $u^{'} - 2f = e - f$, and we get $(u^{'} - 2f)^3 = u^{'3} - 6u^{'2}f + 12u^{'}f - 8f = u^{'3} - 2f + 6(u^{'2}f + 2u^{'}f) = u^{'} - 2f$. By Proposition 4.3, $u^{'3} - u^{'} = 6(u^{'2}f + 2u^{'}f) \in J(R)$, and thus $u^{'2} - 1 \in J(R)$. It follows that $u^2 - 1 = (u^{'2} - 1) + 2u^{'}j + j^2 \in J(R)$. Hence R is a 2-UJ ring. Next we show that $R/J(R)$ is reduced. Let $x \in R$ with $x^2 \in J(R)$. It is easy to see that $x \in R^{qnil}$. In view of Proposition 4.1(2), we get $x \in J(R)$, which implies that $R/J(R)$ is reduced.

Theorem 4.6 Let R be a strong $UJII$-ring. Then the followings are equivalent:

(1) For any $u \in U(R)$, there exists a unique $j \in J(R)$ such that $u = e + f + j$ where $e, f \in idem(R)$.

(2) R is abelian.

Proof (1) $\Rightarrow$ (2). For any $e^2 = e \in R$ and $r \in R$, we have $er(1 - e) \in N(R)$. By Proposition 4.1(1), $er(1 - e) \in J(R)$. So $1 + er(1 - e) \in U(R)$. Write $t = 1 + er(1 - e)$. Then $t = 1 + 0 + er(1 - e) = [e + er(1 - e)] + (1 - e) + 0$ are two $UJII$ expressions of t. By assumption, $er(1 - e) = 0$, and thus, $er = ere$. Similarly, we can deduce that $re = ere$, and so $er = re$. Therefore, R is abelian.

(2) $\Rightarrow$ (1). Let $u \in U(R)$. We may assume that

$$u = e_1 + f_1 + j_1 \tag{1}$$

and

$$u = e_2 + f_2 + j_2, \tag{2}$$

where $e_i, f_i \in \mathrm{idem}(R)$, $j_i \in J(R)$ and $i = 1, 2$. Multiplying the above two equations by $1 - e_1$, we get $u(1 - e_1) = f_1(1 - e_1) + j_1(1 - e_1)$ and $u(1 - e_1) = e_2(1 - e_1) + f_2(1 - e_1) + j_2(1 - e_1)$. Note that $f_1(1 - e_1) \in U((1 - e_1)R(1 - e_1)) \cap \mathrm{idem}((1 - e_1)R(1 - e_1))$. Thus, $f_1(1 - e_1) = 1 - e_1$. It follows that

$$\begin{aligned} 0 &= [f_1(1 - e_1) + j_1(1 - e_1)] - [e_2(1 - e_1) + f_2(1 - e_1) + j_2(1 - e_1)] \\ &= [(1 - e_1) + j_1(1 - e_1)] - [e_2(1 - e_1) + f_2(1 - e_1) + j_2(1 - e_1)] \\ &= (1 - e_1)(1 - e_2 - f_2) + j_1(1 - e_1) - j_2(1 - e_1). \end{aligned}$$

So one has $(1 - e_1)(1 - e_2 - f_2) \in J(R)$. Since $1 - e_1 \in idem(R)$, $(1 - e_2 - f_2)^3 = 1 - e_2 - f_2$ and $(1 - e_1)(1 - e_2 - f_2)$ are a tripotent. A direct check will reveal that $(1 - e_1)(1 - e_2 - f_2) = 0$. Similarly, multiplying equations (1) and (2) by $1 - e_2$, we obtain $(1 - e_2)(1 - e_1 - f_1) = 0$. Thus, $(1 - e_1)f_2 = (1 - e_2)f_1$, and whence $f_1 - f_2 = e_2 f_1 - e_1 f_2$. One may note that e_1, e_2 and f_1, f_2 are parallel. So we can also get $e_1 - e_2 = f_2 e_1 - f_1 e_2$. Therefore, $e_1 + f_1 = e_2 + f_2$. This shows that $j_1 = j_2$, and we are done.

Theorem 4.7 Let R be a ring. Then the followings are equivalent:

(1) R is a strong $UJII$-ring and $2 \in J(R)$.

(2) R is a UJ ring.

Proof (1) $\Rightarrow$ (2). Let $u \in U(R)$. Then $u = j + e + f$ where $j \in J(R)$, $e, f \in idem(R)$ and j, e, f all commute. Write $g := e + f - 2ef$ and $c := j + 2ef$. Then $g^2 = g$ and $c \in J(R)$ as $2 \in J(R)$. So $u = g + c$ and $ug = gu$. It follows that $g = u - c \in U(R)$. Thus $g = 1$, and therefore, $u = 1 + c$, which implies that R is a UJ ring.

(2) $\Rightarrow$ (1). By hypothesis, $-1 = 1 + j$. So $2 \in J(R)$. It is clear that R is a strong $UJII$-ring.

We finish this short paper with following problems:

Problem (1) If R is a strong $UJII$-ring, is eRe strong $UJII$ for any $e^2 = e \in R$?
(2) Characterize when a group ring is strong $UJII$.

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UQ-环和强UJII-环

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摘要: 本文引入了UQ-环和UJII-环的概念, 推广了UJ-环. 利用环论中元素的技巧, 研究了UQ-环和UJII-环的性质和结构, 相关结果丰富了环中关于元素分解的理论.

关键词: UJ-环; UQ-环; 强UJII-环; clean环

MR(2010)主题分类号: 16U99; 16N20 **中图分类号:** O153.3