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RESEARCH ANNOUNCEMENTS ON "ON TIDAL ENERGY IN NEWTONIAN TWO-BODY MOTION WITH INFINITE INITIAL SEPARATION"

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1 Introduction and Main Results

In [7], we have studied the evolution of tidal energy for the motion of two gravitating incompressible fluid balls with free boundaries obeying the Euler-Poisson equations

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\mathbf{p} - \nabla(\psi_1+\psi_2), & \text{in } \mathcal{B}_j(t),\\ \nabla\cdot\mathbf{v}=0, \quad \nabla\times\mathbf{v}=\mathbf{0}, & \text{in } \mathcal{B}_j(t),\\ \mathbf{p}=\mathbf{0}, & \text{on } \partial\mathcal{B}_j(t),\\ (1,\mathbf{v})\in T(t,\partial\mathcal{B}_j(t)), & \text{on } \partial\mathcal{B}_j(t). \end{cases} \tag{1.1}$$

Here, $\mathbf{v}(t,\mathbf{x})$ and $\mathbf{p}(t,\mathbf{x})$ denote the fluid velocity and fluid pressure with $t\in\mathbb{R}$ and $\mathbf{x}\in\mathcal{B}_j(t)(j=1,2)$, respectively. $\psi_j(j=1,2)$ are the gravitational potentials

$$\psi_j(t,\mathbf{x}) := -G\rho\int_{\mathcal{B}_j(t)}\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|},$$

so that

$$\Delta\psi_j(t,\mathbf{x}) = 4\pi G\rho\chi_{\mathcal{B}_j(t)}.$$

In addition, $\mathcal{B}_1$ and $\mathcal{B}_2$ are assumed to be two fluid bodies of equal mass with constant density ρ and volume $\frac{4\pi}{3}R^3$, which are initially round spheres of radius R and approach each other from a very far distance with initial velocities $-v_0$ and v_0, respectively.

Such a problem stems from the observation of Kepler and the analysis of Newton, who discovered that the orbit of the point masses must be a conic curve. In particular, if the initial distance of the two point masses is very large, then the mechanical energy of the point masses is positive and conserved, and the orbit of the two point masses is a hyperbola (see

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[7] for more details). That is to say, the two bodies get closer until they reach their minimum distance and then start to move apart. If the point masses are replaced by the fluid bodies obeying (1.1), it is shown in [7] that for a class of initial configuration, the orbital energy which describes center-of-mass motion of the bodies, becomes from positive to negative as the bodies approach their first minimum distance. This reveals the possibility that the center of mass orbit, which is unbounded initially, may become bounded during the evolution.

This result in [7] is based on a quantitative relation between the tidal energy and the distance of two bodies. However, this relation only holds when the two-body distance is within multiples of the first closest approach, due to the fact that initially the tidal energy vanishes but the two-body distance is finite. In [8], based on the a priori estimates established in [7], we construct a solution to the same two-body problem as in [7] but with infinite initial separation. Therefore the above mentioned quantitative relation holds during the entire evolution up to the first closest approach.

To describe the above results more precisely, we recall some basic setup in [7]. Assume that at time T_0, the center of masses of the two bodies $\mathcal{B}_1(T_0)$ and $\mathcal{B}_2(T_0)$ which are balls of radius R are denoted by $\mathbf{x}_1=(x_1,y_1,z_1)=(-b,R_0,0)$ and $\mathbf{x}_2=(x_2,y_2,z_2)=(b,-R_0,0)$, respectively, with $R_0\gg b\gg R$. We let $\widetilde{R}_0:=\sqrt{b^2+R_0^2}$ be the initial distance of the center of mass of the entire system to the initial position of the center of mass of each body and suppose that the initial velocities of $\mathbf{x}_1$ and $\mathbf{x}_2$ are

$$\mathbf{x}_1'(T_0)=\mathbf{v}_0:=(0,-v_0,0),\quad \mathbf{x}_2'(T_0)=-\mathbf{v}_0:=(0,v_0,0),\quad v_0>0.$$

According to the symmetry, the center of mass of the entire system is always at the origin. Note that, the center of mass of each body is given by

$$\mathbf{x}_j=\frac{\rho}{M}\int_{\mathcal{B}_j}\mathbf{x}d\mathbf{x},$$

which yields

$$\mathbf{x}_j''=\frac{\rho}{M}\int_{\mathcal{B}_j}(\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v})d\mathbf{x}=-\frac{\rho}{M}\int_{\mathcal{B}_j}(\nabla\psi_1+\nabla\psi_2)d\mathbf{x}.$$

By symmetry, we only look at the body $\mathcal{B}_1$. The conserved total energy is

$$\mathscr{E}(t):=\frac{1}{2}\int_{\mathcal{B}_1}|\mathbf{v}(t,\mathbf{x})|^2d\mathbf{x}+\frac{1}{2}\int_{\mathcal{B}_1}\psi_1(t,\mathbf{x})d\mathbf{x}+\frac{1}{2}\int_{\mathcal{B}_1}\psi_2(t,\mathbf{x})d\mathbf{x},$$

which we modify and decompose it as

$$\widetilde{\mathscr{E}}:=\frac{1}{|\mathcal{B}_1|}\mathscr{E}+\frac{3GM}{5R}=\widetilde{\mathscr{E}_{\mathrm{orbital}}}+\widetilde{\mathscr{E}_{\mathrm{tidal}}},$$

where

$$\widetilde{\mathscr{E}_{\mathrm{tidal}}}=\frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}|\mathbf{v}(t,\mathbf{x})|^2d\mathbf{x}-\frac{1}{2}|\mathbf{x}_1'|^2+\frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\psi_1(t,\mathbf{x})d\mathbf{x}+\frac{3GM}{5R}$$

and

$$\widetilde{\mathscr{E}_{\mathrm{orbital}}}=\frac{1}{2}|\mathbf{x}_1'|^2+\frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\psi_2(t,\mathbf{x})d\mathbf{x}.$$

By introducing $\xi : \mathbb{R} \times S_R \to \mathcal{B}_1$ being the Lagrangian parametrization of $\mathcal{B}_1$ such that $\xi(T_0, \iota) = \iota$ for all ι in S_R and defining

$$h(t,\iota) = |\xi(t,\iota) - \mathbf{x}_1(t)| - R, \quad p := \frac{GM}{bv_0^2}, \quad r_+ := \frac{2b}{p},$$

and

$$\eta := \frac{R}{|\mathbf{x}_1(t)|}, \quad \eta_+ := \frac{R}{r_+}, \quad \beta := \frac{b}{R},$$

the main result in [7] can be stated as below.

Theorem 1.1 Suppose $r_+ \geq CR$ where $C > 0$ is sufficiently large. Then $|\mathbf{x}_1(t)|$ is decreasing on any time interval $I_T := [T_0, T)$ such that $|\mathbf{x}_1(t)| \geq \frac{3}{2} r_+$ for all $t \in I_T$, and a classical solution to (1.1) exists on the longest time interval on which $|\mathbf{x}_1|$ is decreasing. Moreover, there exist universal constants c_1 and c_2 such that, with r_0 denoting the first local minimum of $|\mathbf{x}_1|$, $c_1 \frac{GM}{R} \eta^6 \leq \widetilde{\mathcal{E}_{\mathrm{tidal}}} \leq c_2 \frac{GM}{R} \eta^6$ if $|\mathbf{x}_1| \in (r_0, 2r_0)$, and $\widetilde{\mathcal{E}_{\mathrm{tidal}}}$ is related to the height function h as

$$\widetilde{\mathcal{E}_{\mathrm{tidal}}} \approx \frac{GM}{R^5} \|h\|^2_{H^1(S_R)} + \frac{1}{R^2} \|\partial_t h\|^2_{L^2(S_R)}, \tag{1.2}$$

where the constants c_1 and c_2 as well as the implicit constants in (1.2) are independent of the initial time T_0 and the initial separation $\widetilde{R}_0$. In particular, if $\eta_+^5 p^2 \gtrsim 1$, then for some $m > 2$

$$\widetilde{\mathcal{E}_{\mathrm{tidal}}} \geq m \widetilde{\mathscr{E}}$$

when $|\mathbf{x}_1(t)| \in (r_0, 2r_0)$.

Although the proof of Theorem 1.1 is based on the fixed initial time T_0 and initial separation $\widetilde{R}_0$, the constants c_1 and c_2 as well as the implicit constant in (1.2) do not depend on them. This fact should allow us to take the limit $|T_0|, R_0 \to +\infty$ which means there exists a unique classical solution to (1.1) for any $t \in (-\infty, T)$ for some T. In [8], we give a rigorous justification for this fact. Our main result is:

Theorem 1.2 Suppose $r_+ \geq CR$ where $C > 0$ is sufficiently large. Then $|\mathbf{x}_1(t)|$ is decreasing on any time interval $I_T := [-\infty, T)$ such that $|\mathbf{x}_1(t)| \geq \frac{3}{2} r_+$ for all $t \in I_T$, and a classical solution to (1.1) exists on the longest time interval on which $|\mathbf{x}_1|$ is decreasing. Moreover, there exist universal constants c_1' and c_2' such that, with r_0 denoting the first local minimum of $|\mathbf{x}_1|$, $c_1' \frac{GM}{R} \eta^6 \leq \widetilde{\mathcal{E}_{\mathrm{tidal}}} \leq c_2' \frac{GM}{R} \eta^6$ for any $|\mathbf{x}_1| \in (r_0, +\infty]$. In particular, as in Theorem 1.1, the tidal energy can be made arbitrarily large relative to the conserved total energy for $|\mathbf{x}_1(t)| \in (r_0, 2r_0)$ if $\eta_+^5 p^2 \gtrsim 1$.

Remark 1.3 The novelty of Theorem 1.2 is not only we can prove that (1.1) admits a classical solution in the time interval $I_T = [-\infty, T)$ but also we give a precise description of the evolution of the tidal energy $\widetilde{\mathcal{E}_{\mathrm{tidal}}}$ during this time. Compared with Theorem 1.1, the relation $c_1' \frac{GM}{R} \eta^6 \leq \widetilde{\mathcal{E}_{\mathrm{tidal}}} \leq c_2' \frac{GM}{R} \eta^6$ holds for all $|\mathbf{x}_1| \in (r_0, +\infty]$. This fact is invalid for the setup in [7], where $\widetilde{\mathcal{E}_{\mathrm{tidal}}} = 0$ and $\eta > 0$ initially when the bodies are perfect balls.

Now we outline the main steps of the proof for Theorem 1.2. Consider a decreasing sequence $T_m < 0$ and an increasing sequence $R_m > 0$ such that $\lim_{m\to\infty} T_m = -\infty$ and

$\lim_{m\to\infty} R_m = \infty$. For each such T_m and R_m, by the results in [7], there exists a classical solution $(\mathbf{v}^m, \psi_j^m, \mathbf{p}^m)$ to (1.1) for $t \in [T_m, T^*)$ and some $T^* < 0$ with the center of masses of the two bodies at $(-b, R_m, 0)$ and $(b, -R_m, 0)$ (Figure 1). In particular, there exists a solution $(u^m, \mathbf{x}_1^m(t))$ to the derived quasi-linear system for all $t \in [T_m, T^*)$ which was firstly introduced in [10, 11, 12] for the water wave equations. We then extend u^m to the time interval $t \in (-\infty, T^*)$ which we denote by

$$\widetilde{u^m}(t,p) := \begin{cases} 0, & -\infty < t \le T_m, \\ u^m(t,p), & T_m < t \le T^*, \end{cases} \tag{1.3}$$

for some fixed time $T^* < 0$ and using the uniform estimates derived in [7] to prove the convergence of $\widetilde{u^m}$. Finally, we prove that the limit $\widetilde{u^\infty}(t,p)$ which is defined for $t \in (-\infty, T^*)$ satisfies the derived quasi-linear equations. The analysis tools contain Clifford analysis (c.f. [4]), the theory of layered potentials (c.f. [5, 6, 9]) as well as singular integral estimates (c.f. [1, 2, 3, 7]).

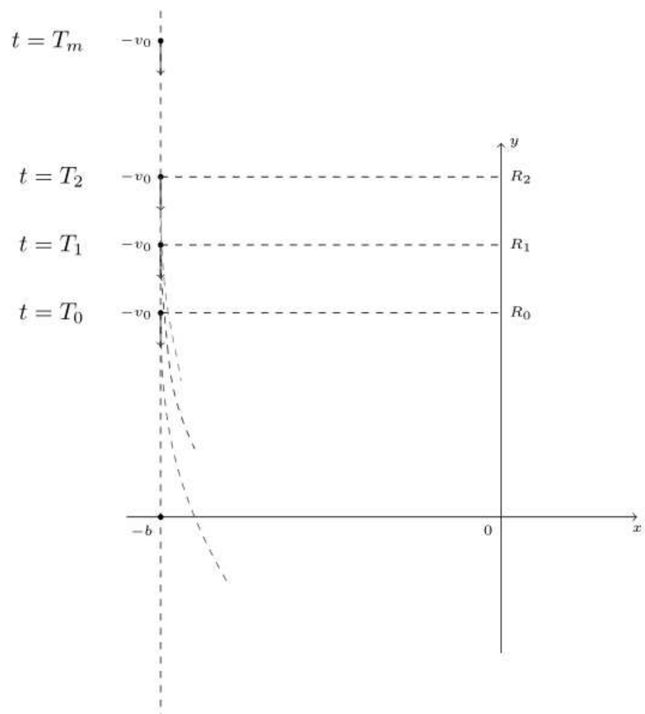

Figure 1

The detailed proof for Theorem 1.2 is given in [8].

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