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SYMPLECTIC CRITICAL SURFACES WITH CIRCULAR ELLIPSE OF CURVATURE IN TWO-DIMENSIONAL COMPLEX SPACE FORMS

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Abstract: We study the symplectic critical surfaces in two-dimensional complex space forms with the property that the ellipse of curvature is always a circle. By using the method of moving frame, we prove that such surfaces are minimal. The results enrich the contents of symplectic critical surface.

Keywords: symplectic critical surfaces; ellipse of curvature; minimal surfaces; complex space forms

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1 Introduction

An interesting notion that comes up in the study of surfaces in higher codimension is that of the ellipse of curvature. This is the image in the normal space of the unit circle in the tangent plane under the second fundamental form.

Using this concept, Guadalupe-Rodriguez(cf.[1]) obtained some inequalities relating the area of compact surfaces in (real) space forms and the integral of the square of the norm of the mean curvature vector with topological invariants. When the ellipse of curvature is a circle, restrictions on the Gaussian and normal curvatures gave them some rigidity results.

Castro(cf.[2]) classified the Lagrangian orientable surfaces in complex space forms with the property that the ellipse of curvature is always a circle. As a consequence, they obtained new characterizations of the Clifford torus in the complex projective plane and of the Whitney spheres in the complex projective, complex Euclidean and complex hyperbolic planes.

Other works which use the ellipse of curvature as a tool in the study of surfaces in (real) space forms can be found in these articles.(cf.[3–5])

In this article we attach the circular ellipse of curvature condition to symplectic surfaces in two-dimensional complex space forms.

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Let M be a complex two-dimensional Kähler manifold with Kähler form ω. Let Σ be a Riemann surface and we consider an isometric immersion $f : \Sigma \to M$ from Σ into M. Chern-Wolfson(cf.[6]) defined the Kähler angle θ of Σ in M by

$$f^*\omega = \cos\theta d\mu_\Sigma,$$

where $d\mu_\Sigma$ is the area element of Σ in the induced metric. It is said that Σ is a holomorphic curve if $\cos\theta = 1$, Σ is a Lagrangian surface if $\cos\theta = 0$ and Σ is a symplectic surface if $\cos\theta > 0$.

Let $\mathbf{H}$ be the mean curvature vector field of f, which is defined by

$$\mathbf{H} = \sum_{\alpha,i} h^\alpha_{ii} e_\alpha, \tag{1.1}$$

where h^α_{ij}'s are the components of the second fundamental form of f, and e_i and e_α are adapted frames along f.

A symplectic minimal surface is a critical point of the area of surfaces, which is symplectic. Han-Li(cf.[7]) considered generally the critical point of the functional

$$L = \int_\Sigma \frac{1}{\cos\theta} d\mu_\Sigma, \tag{1.2}$$

in the class of symplectic surfaces. The Euler-Lagrange equation of this functional is

$$\cos^3\theta \mathbf{H} = (J(J\nabla\cos\theta)^\top)^\perp, \tag{1.3}$$

where $()^\top$ and $()^\perp$ mean tangential components and normal components of $()$ respectively. Such a surface is called a symplectic critical surface.

Many interesting results about symplectic critical surfaces have been obtained by Han et al.(cf.[7–11]). In this paper we will focus on the explicit characterization of symplectic critical surfaces from the viewpoint of differential geometry. It follows from Eq.(1.3) that a minimal surface with constant Kähler angle that values in $[0, \frac{\pi}{2})$ is a symplectic critical surface. There are few examples of symplectic critical surfaces that is non-minimal. Han-Li-Sun(cf.[11]) gave a two-parameters family of symplectic critical surfaces in two-dimensional complex plane $\mathbb{C}^2$, which is rotationally symmetric. Later, He-Li(cf.[12]) showed the symplectic critical surfaces with parallel normalized mean curvature vector in $\mathbb{C}^2$ must be the above examples, and there does not exist any symplectic critical surface with parallel normalized mean curvature vector in two-dimensional complex space forms of non-zero constant holomorphic sectional curvature. So, it is natural to considering what kind of symplectic critical surface must be minimal.

In Sec.2, we introduce the concept of ellipse of curvature. In Sec.3, we study the fundamental equations of symplectic critical surfaces with circular ellipse of curvature in two-dimensional complex space forms by using the method of moving frame(cf.[6]). In Sec.4, we study the equations under the condition $\rho = 0$ and get all solutions of the equations explicitly in this case. In Sec.5, we give a geometric result. Concretely, we prove that the symplectic critical surfaces with circular ellipse of curvature in two-dimensional complex space forms are minimal(cf. Theorem 5.1).

2 The Ellipse of Curvature

Suppose that M is a 4-dimensional Riemannian manifold. Let Σ be a Riemann surface and $f : \Sigma \to M$ be an isometric immersion. Let $\mathbf{H}$ be the mean curvature vector field of f. We denote the metric of M as well as the induced metric in Σ by $<,>$. If $A : T\Sigma \times T\Sigma \to T^{\perp}\Sigma$ is the second fundamental form of f, the ellipse of curvature is the subset of the normal plane defined as $\{A(v,v) \in T_p^{\perp}\Sigma :< v,v >= 1, v \in T_p\Sigma, p \in \Sigma\}$. To see that it is an ellipse, we consider an arbitary orthogonal tangent frame $\{v_1, v_2\}$, denote $h_{ij} = A(v_i, v_j), i,j = 1,2$, and look at the following formula for $v = \cos\tau v_1 + \sin\tau v_2$:

$$A(v,v) = \frac{\mathbf{H}}{2} + \cos 2\tau \frac{h_{11} - h_{22}}{2} + \sin 2\tau h_{12}, \tag{2.1}$$

where $\mathbf{H} = traceA$ is the mean curvature vector. As v goes once around the unit tangent circle, $A(v,v)$ goes twice around the ellipse.

From Eq.(2.1), it is not difficult to deduce that the ellipse of curvature is a circle if and only if

$$\frac{|h_{11} - h_{22}|^2}{4} = |h_{12}|^2, < h_{11} - h_{22}, h_{12} >= 0. \tag{2.2}$$

The property that the ellipse is a circle is a conformal invariant. Of course this ellipse could degenerate into a line segment or a point. And the following properties are equivalent at a point of the immersed surface: (i) the ellipse degenerates into a line segment or a point, (ii) $(h_{11} - h_{22})/2$ and h_{12} are linearly dependent, (iii) the normal bundle is flat, and (iv) if v_α (α=3,4) is an orthonormal normal frame, the second fundamental forms A_{v_α} are simultaneously diagonalizable. Here, A_{v_α} is the symmetric endomorphism of $T\Sigma$ defined by $< A(X,Y), v_\alpha >=< A_{v_\alpha}X, Y >$, where $X, Y \in T\Sigma$.

3 The Fundamental Equations of the Surfaces

Suppose that M is a complex two-dimensional Kähler manifold of constant holomorphic sectional curvature 4ρ. Let $\{\omega_i\}$ be a local field of unitary coframes on M, so that the Kähler metric is represented by $\sum \omega_i \overline{\omega}_i$. Here and in what follows, we will agree on the following range of indices: $1 \le i,j,k \le 2$. We denote by ω_{ij} the unitary connection forms with respect to $\{\omega_i\}$. So we have

$$\begin{aligned} d\omega_i &= \sum \omega_{ij} \wedge \omega_j,\ \omega_{ij} + \overline{\omega}_{ji} = 0, \\ d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \\ \Omega_{ij} &= -\rho\left(\omega_i \wedge \overline{\omega}_j + \delta_{ij} \sum \omega_k \wedge \overline{\omega}_k\right). \end{aligned} \tag{3.1}$$

Let Σ be a Riemann surface and $f : \Sigma \to M$ be an isometric immersion. Let $\mathbf{H}$ be the mean curvature vector field of f. We assume $\mathbf{H}$ has no zeros on Σ. We can construct a unique system of global orthonormal vector fields $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ along Σ such that $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to Σ by the following: First we set the normal vector field $\tilde{e}_3$ of $T^\perp\Sigma$ arbitrarily, then the normal vector field $\tilde{e}_4$ of $T^\perp\Sigma$ is uniquely determined by choosing it to be compatible with the fixed orientations of Σ and M. The system of vectors $\{\tilde{e}_3, \tilde{e}_4, \mathbf{J}\tilde{e}_3, \mathbf{J}\tilde{e}_4\}$ is linearly independent, because f is neither holomorphic nor anti-holomorphic. Here the angle of $\mathbf{J}\tilde{e}_4$ and $\tilde{e}_3$ is equal to the Kähler angle θ which is defined in Sec.1. In fact, set

$$\tilde{e}_1 = -\frac{\mathbf{J}\tilde{e}_4 - \langle \mathbf{J}\tilde{e}_4, \tilde{e}_3\rangle \tilde{e}_3}{\|\mathbf{J}\tilde{e}_4 - \langle \mathbf{J}\tilde{e}_4, \tilde{e}_3\rangle \tilde{e}_3\|},\ \tilde{e}_2 = \frac{\mathbf{J}\tilde{e}_3 - \langle \mathbf{J}\tilde{e}_3, \tilde{e}_4\rangle \tilde{e}_4}{\|\mathbf{J}\tilde{e}_3 - \langle \mathbf{J}\tilde{e}_3, \tilde{e}_4\rangle \tilde{e}_4\|}.$$

Then $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to Σ. A straightforward calculation shows

$$\langle \mathbf{J}\tilde{e}_4, \tilde{e}_3\rangle = \langle \mathbf{J}\tilde{e}_1, \tilde{e}_2\rangle = \cos\theta.$$

It is easy to see that $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an adapted frame on Σ in M, that is, $\tilde{e}_1$ and $\tilde{e}_2$ are sections on $T\Sigma$ and $\tilde{e}_3$ and $\tilde{e}_4$ are sections on $T^\perp\Sigma$. The complex structure $\mathbf{J}$ is represented under the frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ as follows:

$$\begin{aligned}
\mathbf{J}\tilde{e}_1 &= \cos\theta \cdot \tilde{e}_2 + \sin\theta \cdot \tilde{e}_4,\\
\mathbf{J}\tilde{e}_2 &= -\cos\theta \cdot \tilde{e}_1 - \sin\theta \cdot \tilde{e}_3,\\
\mathbf{J}\tilde{e}_3 &= \sin\theta \cdot \tilde{e}_2 - \cos\theta \cdot \tilde{e}_4,\\
\mathbf{J}\tilde{e}_4 &= -\sin\theta \cdot \tilde{e}_1 + \cos\theta \cdot \tilde{e}_3.
\end{aligned}$$

Moreover, we define vector fields e_1 and e_3 as follows:

$$\begin{aligned}
e_1 &= \frac{\tilde{e}_1 - \mathbf{J}\tilde{e}_2}{\|\tilde{e}_1 - \mathbf{J}\tilde{e}_2\|} = \cos\frac{\theta}{2} \cdot \tilde{e}_1 + \sin\frac{\theta}{2} \cdot \tilde{e}_3,\\
e_3 &= \frac{\tilde{e}_1 + \mathbf{J}\tilde{e}_2}{\|\tilde{e}_1 + \mathbf{J}\tilde{e}_2\|} = \sin\frac{\theta}{2} \cdot \tilde{e}_1 - \cos\frac{\theta}{2} \cdot \tilde{e}_3.
\end{aligned}$$

and put

$$\begin{aligned}
e_2 &= \mathbf{J}e_1 = \cos\frac{\theta}{2} \cdot \tilde{e}_2 + \sin\frac{\theta}{2} \cdot \tilde{e}_4,\\
e_4 &= \mathbf{J}e_3 = -\sin\frac{\theta}{2} \cdot \tilde{e}_2 + \cos\frac{\theta}{2} \cdot \tilde{e}_4.
\end{aligned}$$

Then $\{e_1, e_2, e_3, e_4\}$ is a $\mathbf{J}$-canonical frame along f. We extend $\{\tilde{e}_A\}$ and $\{e_A\}$ to a neighbourhood of Σ in M, where A, B and C run from 1 to 4.

Let $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$ be the dual coframes of $\{\tilde{e}_A\}$ and $\{e_A\}$ respectively. Let $\tilde{\theta}_{AB}$ and θ_{AB} be the Riemannian connection forms with respect to the canonical 1-forms $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$ respectively and put

$$\omega_j = \theta_{2j-1} + \mathtt{i}\theta_{2j},$$

$$\omega_{jk} = \theta_{2j-1,2k-1} + \mathtt{i}\theta_{2j,2k-1},\ \text{where } \mathtt{i} = \sqrt{-1}.$$

Then we have the following relations

$$\begin{aligned} \tilde{\theta}_1 + \mathtt{i}\tilde{\theta}_2 &= \cos\frac{\theta}{2}\omega_1 + \sin\frac{\theta}{2}\overline{\omega}_2, \\ \tilde{\theta}_3 + \mathtt{i}\tilde{\theta}_4 &= \sin\frac{\theta}{2}\omega_1 - \cos\frac{\theta}{2}\overline{\omega}_2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \tilde{\theta}_{12} &= \mathtt{i}\left(\cos^2\frac{\theta}{2}\omega_{11} - \sin^2\frac{\theta}{2}\omega_{22}\right), \\ \tilde{\theta}_{34} &= \mathtt{i}\left(\sin^2\frac{\theta}{2}\omega_{11} - \cos^2\frac{\theta}{2}\omega_{22}\right), \\ \tilde{\theta}_{13} + \mathtt{i}\tilde{\theta}_{23} &= -\left\{\omega_{12} + \frac{1}{2}\left[d\theta - \sin\theta(\omega_{11} + \omega_{22})\right]\right\}, \\ \tilde{\theta}_{14} + \mathtt{i}\tilde{\theta}_{24} &= \mathtt{i}\left\{\omega_{12} - \frac{1}{2}\left[d\theta - \sin\theta(\omega_{11} + \omega_{22})\right]\right\}. \end{aligned} \tag{3.3}$$

We denote the restriction of $\{\tilde{\theta}_A\}$ to Σ by the same letters. Then we have $\tilde{\theta}_3 = 0 = \tilde{\theta}_4$ on Σ. Putting $\phi = \tilde{\theta}_1 + \mathtt{i}\tilde{\theta}_2$, the induced metric of Σ is written as $ds^2 = \phi\overline{\phi}$. By taking the exterior derivative of Eq.(3.2) restricted to Σ, we get

$$\begin{aligned} &\frac{1}{2}\left[d\theta + \sin\theta(\omega_{11} + \omega_{22})\right] = a\phi + b\overline{\phi}, \\ &\omega_{12} = b\phi + c\overline{\phi}, \end{aligned} \tag{3.4}$$

where a, b and c are complex-valued smooth functions defined locally on Σ. Let $\{h^\alpha_{ij}\}$ be the components of the second fundamental form so that $\tilde{\theta}_{i\alpha} = \sum_j h^\alpha_{ij}\tilde{\theta}_j$. By using Eqs.(3.3) and (3.4) , all h^α_{ij}'s can be expressed in terms of a, b and c. Indeed, we have

$$\begin{aligned} h^3_{11} &= -\frac{1}{2}\left[a + \bar{a} + 2(b + \bar{b}) + c + \bar{c}\right], \\ h^3_{12} &= \frac{\mathtt{i}}{2}\left(-a + \bar{a} + c - \bar{c}\right), \\ h^3_{22} &= \frac{1}{2}\left[a + \bar{a} - 2(b + \bar{b}) + c + \bar{c}\right], \\ h^4_{11} &= \frac{\mathtt{i}}{2}\left[a - \bar{a} + 2(b - \bar{b}) + c - \bar{c}\right], \\ h^4_{12} &= \frac{1}{2}\left(-a - \bar{a} + c + \bar{c}\right), \\ h^4_{22} &= \frac{\mathtt{i}}{2}\left[-a + \bar{a} + 2(b - \bar{b}) - c + \bar{c}\right]. \end{aligned} \tag{3.5}$$

Put $\tilde{e}_3 = -\frac{\mathbf{H}}{\|\mathbf{H}\|}$, then $\mathbf{H} = -\|\mathbf{H}\|\tilde{e}_3 = (h^3_{11} + h^3_{22})\tilde{e}_3 + (h^4_{11} + h^4_{22})\tilde{e}_4$, it follows from Eq.(3.5) that $b = \bar{b}$, and $\|\mathbf{H}\| = 4b$.

Let K be the Gauss curvature of Σ, then

$$d\tilde{\theta}_{12} = -K\tilde{\theta}_1 \wedge \tilde{\theta}_2 = -\frac{\mathtt{i}}{2}K\phi \wedge \overline{\phi}.$$

By taking the exterior derivative of the first formula of Eq.(3.3), using Eq.(3.1) and Eq.(3.4), we have

$$K = (1 + 3\cos^2\theta)\rho - 2(|a|^2 - 2b^2 + |c|^2). \tag{3.6}$$

Let K_N be the normal curvature of f defined by $d\tilde{\theta}_{34} = -K_N\tilde{\theta}_1 \wedge \tilde{\theta}_2 = -\frac{\mathtt{i}}{2}K_N\phi \wedge \overline{\phi}$. By taking the exterior derivative of the second formula of Eq.(3.3), using Eq.(3.1) and Eq.(3.4), we have

$$K_N = 2(|a|^2 - |c|^2) - (3\cos^2\theta - 1)\rho. \tag{3.7}$$

Since

$$\begin{aligned}(\mathbf{J}\nabla\cos\theta)^\top &= (\nabla_{\tilde{e}_1}\cos\theta\cdot\mathbf{J}\tilde{e}_1 + \nabla_{\tilde{e}_2}\cos\theta\cdot\mathbf{J}\tilde{e}_2)^\top\\ &= \nabla_{\tilde{e}_1}\cos\theta\cdot\cos\theta\cdot\tilde{e}_2 - \nabla_{\tilde{e}_2}\cos\theta\cdot\cos\theta\cdot\tilde{e}_1,\end{aligned}$$

then

$$\begin{aligned}(\mathbf{J}(\mathbf{J}\nabla\cos\theta)^\top)^\perp &= \nabla_{\tilde{e}_1}\cos\theta\cdot\cos\theta\cdot(\mathbf{J}\tilde{e}_2)^\perp - \nabla_{\tilde{e}_2}\cos\theta\cdot\cos\theta\cdot(\mathbf{J}\tilde{e}_1)^\perp\\ &= -\sin\theta\cos\theta\nabla_{\tilde{e}_1}\cos\theta\cdot\tilde{e}_3 - \sin\theta\cos\theta\nabla_{\tilde{e}_2}\cos\theta\cdot\tilde{e}_4.\end{aligned}$$

Hence, in particular, $\sin\theta \neq 0$. From the symplectic critical surface equation Eq.(1.3), we get

$$4b\cos^3\theta = \sin\theta\cos\theta\nabla_{\tilde{e}_1}\cos\theta, \tag{3.8}$$

and

$$\nabla_{\tilde{e}_2}\cos\theta = 0. \tag{3.9}$$

It follows from the first formula of Eq.(3.4) that

$$d\theta = (a+b)\phi + (\overline{a}+b)\overline{\phi} = (a+\overline{a}+2b)\tilde{\theta}_1 + \mathtt{i}(a-\overline{a})\tilde{\theta}_2. \tag{3.10}$$

Combining Eqs.(3.9) and (3.10), we have

$$a = \overline{a}, \tag{3.11}$$

which implies

$$d\theta = 2(a+b)\tilde{\theta}_1. \tag{3.12}$$

Substituting Eqs.(3.12) into (3.8), we obtain

$$a = -(1 + 2\cot^2\theta)b. \tag{3.13}$$

Next, we study the fundamental equations of symplectic critical surfaces with circular ellipse of curvature.

Using Eq.(2.2), we can obtain

$$\begin{aligned}&h^3_{12}(h^3_{11} - h^3_{22}) + h^4_{12}(h^4_{11} - h^4_{22}) = 0,\\ &(h^3_{11} - h^3_{22})^2 + (h^4_{11} - h^4_{22})^2 = 4((h^3_{12})^2 + (h^4_{12})^2).\end{aligned}$$

From Eq.(3.5), using the above two equations, we can get $a=0$ or $c=0$ at any point $p\in\Sigma$. When $a=0$, using Eq.(3.13), we can know that $b=0$, so $\mathbf{H}=0$ at p, i.e. p is a minimal point. When $c=0$, we obtain the following proposition:

Proposition 3.1 If Σ is a symplectic critical surface with circular ellipse of curvature in M, let $U=\{p\in\Sigma|\theta(p)\neq 0\}$, then we have, on U,

$$\begin{aligned}
&\tilde{\theta}_{12}=\mathtt{i}(2b\cot^3\theta-b\cot\theta+\frac{3\rho}{8b}\sin^3\theta\cos\theta)(\phi-\overline{\phi}),\\
&d\theta=-2b\cot^2\theta(\phi+\overline{\phi}),\\
&db=-(4b^2\cot^3\theta+b^2\cot\theta+\frac{3\rho}{8}\sin^3\theta\cos\theta)(\phi+\overline{\phi}),\\
&\mathbf{H}=-4b\tilde{e}_3.
\end{aligned}\tag{3.14}$$

Proof From the first formula of Eq.(3.4), we have

$$a\phi+b\overline{\phi}=\frac{1}{2}[d\theta+\sin\theta(\omega_{11}+\omega_{22})],\tag{3.15}$$

where a,b,θ are all real. Taking the exterior derivative of Eq.(3.15), we get

$$\begin{aligned}
&da\wedge\phi+db\wedge\overline{\phi}\\
=&\mathtt{i}a\tilde{\theta}_{12}\wedge\phi-\mathtt{i}b\tilde{\theta}_{12}\wedge\overline{\phi}+\frac{1}{2}[\cos\theta d\theta\wedge(\omega_{11}+\omega_{22})+\sin\theta(d\omega_{11}+d\omega_{22})].
\end{aligned}\tag{3.16}$$

From Eq.(3.15), using

$$a=-(1+2\cot^2\theta)b,\tag{3.17}$$

we get

$$\begin{aligned}
&d\theta=-2\cot^2\theta\, b(\phi+\overline{\phi}),\\
&\sin\theta(\omega_{11}+\omega_{22})=(a-b)(\phi-\overline{\phi})=\frac{-2b}{\sin^2\theta}(\phi-\overline{\phi}),
\end{aligned}$$

which implies

$$\omega_{11}+\omega_{22}=-\frac{2b}{\sin^3\theta}(\phi-\overline{\phi}).\tag{3.18}$$

From Eq.(3.2), using $\tilde{\theta}_3=0=\tilde{\theta}_4$, we have

$$\cos\frac{\theta}{2}\omega_1+\sin\frac{\theta}{2}\overline{\omega}_2=\phi,\quad \sin\frac{\theta}{2}\omega_1-\cos\frac{\theta}{2}\overline{\omega}_2=0,$$

which implies

$$\omega_1=\cos\frac{\theta}{2}\phi,\quad \overline{\omega}_2=\sin\frac{\theta}{2}\phi.\tag{3.19}$$

Then using Eq.(3.1), we get

$$d\omega_{11}+d\omega_{22}=-3\rho(\omega_1\wedge\overline{\omega}_1+\omega_2\wedge\overline{\omega}_2)=-3\rho\cos\theta\phi\wedge\overline{\phi}.$$

Hence,

$$
\begin{aligned}
\frac{1}{2}\cos\theta d\theta\wedge(\omega_{11}+\omega_{22}) &= -4b^2\frac{\cot^3\theta}{\sin^2\theta}\phi\wedge\overline{\phi},\\
\frac{1}{2}\sin\theta(d\omega_{11}+d\omega_{22}) &= -\frac{3\rho}{2}\sin\theta\cos\theta\phi\wedge\overline{\phi}.
\end{aligned} \tag{3.20}
$$

Using Eqs.(3.16), (3.17) and (3.20), we have

$$
\begin{aligned}
&-(1+2\cot^2\theta)db\wedge\phi+db\wedge\overline{\phi}+\mathtt{i}(1+2\cot^2\theta)b\tilde{\theta}_{12}\wedge\phi+\mathtt{i}b\tilde{\theta}_{12}\wedge\overline{\phi}\\
=&(-12b^2\frac{\cot^3\theta}{\sin^2\theta}-\frac{3\rho}{2}\sin\theta\cos\theta)\phi\wedge\overline{\phi}.
\end{aligned} \tag{3.21}
$$

From the second formula of Eq.(3.4), we have

$$\omega_{12}=b\phi+c\overline{\phi}. \tag{3.22}$$

Taking the exterior derivative of Eq.(3.22), we get

$$db\wedge\phi+dc\wedge\overline{\phi}=\mathtt{i}b\tilde{\theta}_{12}\wedge\phi-\mathtt{i}c\tilde{\theta}_{12}\wedge\overline{\phi}+d\omega_{12}. \tag{3.23}$$

Since $c=0$, then we have

$$\omega_{12}=b\phi,\quad db\wedge\phi=\mathtt{i}b\tilde{\theta}_{12}\wedge\phi+d\omega_{12}. \tag{3.24}$$

From the second formula of Eq.(3.1), we have

$$d\omega_{12}=(\omega_{11}-\omega_{22})\wedge\omega_{12}-\rho\omega_1\wedge\overline{\omega}_2.$$

Using Eqs.(3.19) and (3.24), we get

$$db\wedge\phi=\mathtt{i}b\tilde{\theta}_{12}\wedge\phi+b(\omega_{11}-\omega_{22})\wedge\phi. \tag{3.25}$$

By the conjugate of the above equation, we have

$$db\wedge\overline{\phi}=-\mathtt{i}b\tilde{\theta}_{12}\wedge\overline{\phi}+b(\overline{\omega}_{11}-\overline{\omega}_{22})\wedge\overline{\phi}. \tag{3.26}$$

Combining Eqs.(3.21), (3.25) and (3.26), we have

$$-(1+2\cot^2\theta)b(\omega_{11}-\omega_{22})\wedge\phi+b(\overline{\omega}_{11}-\overline{\omega}_{22})\wedge\overline{\phi}=(-12b^2\frac{\cot^3\theta}{\sin^2\theta}-\frac{3\rho}{2}\sin\theta\cos\theta)\phi\wedge\overline{\phi}.$$

Taking the conjugate of the above equation, we have

$$-(1+2\cot^2\theta)b(\overline{\omega}_{11}-\overline{\omega}_{22})\wedge\overline{\phi}+b(\omega_{11}-\omega_{22})\wedge\phi=(12b^2\frac{\cot^3\theta}{\sin^2\theta}+\frac{3\rho}{2}\sin\theta\cos\theta)\phi\wedge\overline{\phi}.$$

Using the above two equations, we get

$$\omega_{11}-\omega_{22}=(6b\cot^3\theta+\frac{3\rho}{4b}\sin^3\theta\cos\theta)(\phi-\overline{\phi}). \tag{3.27}$$

Since $\omega_{11}+\omega_{22}=-\frac{2b}{\sin^3\theta}(\phi-\overline{\phi})$, hence we have

$$\begin{aligned}
\omega_{11}&=(3b\cot^3\theta+\frac{3\rho}{8b}\sin^3\theta\cos\theta-\frac{b}{\sin^3\theta})(\phi-\overline{\phi}),\\
\omega_{22}&=(-\frac{b}{\sin^3\theta}-3b\cot^3\theta-\frac{3\rho}{8b}\sin^3\theta\cos\theta)(\phi-\overline{\phi}),\\
\omega_{11}-\omega_{22}&=(6b\cot^3\theta+\frac{3\rho}{4b}\sin^3\theta\cos\theta)(\phi-\overline{\phi}).
\end{aligned}\tag{3.28}$$

Using Eq.(3.3), we get the first formula of Eq.(3.14). Then using Eqs.(3.25) and (3.28), we have

$$\begin{aligned}
db\wedge\phi&=\mathtt{i}b\tilde{\theta}_{12}\wedge\phi+b(\omega_{11}-\omega_{22})\wedge\phi\\
&=(4b^2\cot^3\theta+b^2\cot\theta+\frac{3\rho}{8}\sin^3\theta\cos\theta)\phi\wedge\overline{\phi},
\end{aligned}$$

then we get the third formula of Eq.(3.14).

Thus, we finish our proofs.

Remark 3.2 Next, we discuss the case of $U=\emptyset$. In fact, if $U=\emptyset$, then $\theta\equiv 0$ on Σ, which implies Σ is a holomorphic curve in M. Of course it is a minimal surface.

Set $\phi=\lambda dz$, where λ is a non-zero complex-valued function on a simply connected domain $U_1\subset U$ with complex coordinate z. Then the set of the first three formulas of Eq.(3.14) is rewritten as the following system of differential equations:

$$\begin{aligned}
\frac{\partial\lambda}{\partial\bar{z}}&=-|\lambda|^2(2b\cot^3\theta-b\cot\theta+\frac{3\rho}{8b}\sin^3\theta\cos\theta),\\
\frac{\partial\theta}{\partial\bar{z}}&=-2\overline{\lambda}b\cot^2\theta,\\
\frac{\partial b}{\partial\bar{z}}&=-\overline{\lambda}(4b^2\cot^3\theta+b^2\cot\theta+\frac{3\rho}{8}\sin^3\theta\cos\theta).
\end{aligned}\tag{3.29}$$

In the following we give a lemma about the existence of isothermal coordinate.

Lemma 3.3 Suppose Σ is a symplectic critical surface with circular ellipse of curvature in M. Then there exists a complex coordinate w on a neighborhood of a point of $U\subset\Sigma$ such that $\phi=\mu dw$, where μ is real-valued.

Proof Since θ is not constant, we claim that b is a function of θ. In fact, canceling out $(\phi+\overline{\phi})$ in the second and third formula of Eq.(3.14), we get a differential equation in b for θ. Using the claim, we write $b=b(\theta)$, and define a real-valued function

$$F(\theta)=-2\tan\theta+\cot\theta+\frac{3\rho}{8b^2}\tan\theta\sin^4\theta.$$

Taking the partial derivative of the second formula of Eq.(3.29) with respect to z and using Eq.(3.29), we have a second-order partial differential equation $\frac{\partial^2\theta}{\partial z\partial\bar{z}}-F(\theta)\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial\bar{z}}=0$. It follows that $\frac{\partial(\theta_z\exp(-\int F(\theta)d\theta))}{\partial\bar{z}}=0$. Hence, there exists a holomorphic function $G(z)$ on U such that $\frac{\partial\theta}{\partial z}=G(z)\exp\left(\int F(\theta)d\theta\right)$. Setting

$$w=\int G(z)dz,\ \mu=-\frac{\exp\left(\int F(\theta)d\theta\right)}{2b\cot^2\theta},$$

the lemma is proved by the conjugate of the second formula of Eq.(3.29).

Hence, for a neighbourhood U of a point of Σ, there exists an isothermal coordinate $z = u + iv$ such that

$$ds^2 = \lambda^2 dz d\bar{z},$$

where λ is a positive function defined on U, and we have

$$\phi = \lambda dz.$$

This implies that λ, θ and b are functions of single variable, and Eq.(3.29) is seen to be a system of ordinary differential equations. Consequently, if Σ is a symplectic critical surface with circular ellipse of curvature in M, then there exist real-valued smooth functions of single variable λ, θ and b which are defined locally on Σ and satisfy the system of ordinary differential equations (cf.Eq.(3.30)).

Theorem 3.4 Let M be a two-dimensional complex space form of constant holomorphic sectional curvature 4ρ. If Σ is a symplectic critical surface with circular ellipse of curvature in M, then there exist a system of local coordinates (u, v) on Σ and real-valued smooth functions $\lambda(u), \theta(u)$ and $b(u)$ of single variable u which are defined on an interval I of u, such that they satisfy a system of ordinary differential equations

$$\begin{aligned} \frac{d\lambda}{du} &= -2\lambda^2(2b\cot^3\theta - b\cot\theta + \frac{3\rho}{8b}\sin^3\theta\cos\theta),\ \lambda(u) > 0, \\ \frac{d\theta}{du} &= -4\lambda\cot^2\theta\, b, \\ \frac{db}{du} &= -2\lambda\left\{(\cot\theta + 4\cot^3\theta)b^2 + \frac{3}{8}\rho\sin^3\theta\cos\theta\right\}. \end{aligned} \tag{3.30}$$

4 Analysis of the Overdetermined System: $\rho = 0$ Case

When $\rho = 0$, we get all solutions of the system Eq.(3.30) as follows.

Lemma 4.1 Assume that $\rho = 0$. Then all solutions of the system Eq.(3.30) are given by

$$\lambda(\theta) = c_1 \sin\theta\sqrt{\cos\theta}, \quad b(\theta) = c_2\frac{\sin^2\theta}{\sqrt{\cos\theta}}, \tag{4.1}$$

for any positive constants c_1 and c_2.

Proof Since both $\theta(u)$ and $b(u)$ are not constants, regarding θ as variable, we get from Eq.(3.30) that

$$\begin{aligned} \frac{d\lambda}{d\theta} &= \lambda(\theta)(\cot\theta - \frac{1}{2\cot\theta} + \frac{3\rho\sin^5\theta}{16b^2(\theta)\cos\theta}), \\ \frac{db}{d\theta} &= (\frac{\tan\theta}{2} + 2\cot\theta)b(\theta) + \frac{3\rho\tan\theta\sin^4\theta}{16b(\theta)}. \end{aligned} \tag{4.2}$$

Since $\rho = 0$, the equations above reduce to

$$\frac{d\lambda}{d\theta} = \lambda(\theta)(\cot\theta - \frac{1}{2\cot\theta}), \quad \frac{db}{d\theta} = (\frac{\tan\theta}{2} + 2\cot\theta)b(\theta). \tag{4.3}$$

The integration of the above equations give us the solution of $\lambda(\theta)$ and $b(\theta)$ as follows:

$$\lambda(\theta) = c_1 \sin\theta\sqrt{\cos\theta}, \quad b(\theta) = c_2 \frac{\sin^2\theta}{\sqrt{\cos\theta}}, \tag{4.4}$$

for any positive constants c_1 and c_2. Hence we finish our proof.

5 The Geometric Result

In this section, we show a geometric result.

Theorem 5.1 Let M be a two-dimensional complex space form of constant holomorphic sectional curvature 4ρ. If Σ is a symplectic critical surface with circular ellipse of curvature in M, then Σ is a minimal surface in M.

Proof First, we prove our result in the case of $\rho \neq 0$:

We already know that

$$K = -\frac{\Delta \log\lambda}{\lambda^2} = -\frac{\frac{d^2\log\lambda}{du^2}}{\lambda^2}. \tag{5.1}$$

and that

$$K = (1 + 3\cos^2\theta)\rho - 2(|a|^2 - 2b^2 + |c|^2). \tag{5.2}$$

Using the first equation of Eq.(3.30) and Eq.(5.1), we can get

$$K = (-48\cot^6\theta + 24\cot^4\theta + \frac{48\cos^4\theta}{\sin^6\theta} - \frac{8\cos^2\theta}{\sin^4\theta})b^2 + (9\sin^2\theta\cos^2\theta - 12\cos^4\theta)\rho. \tag{5.3}$$

Using Eqs.(3.13) and (5.2), since $c = 0$, we have

$$K = \rho + 3\rho\cos^2\theta - 8b^2\cot^4\theta - 8b^2\cot^2\theta + 2b^2. \tag{5.4}$$

Combining Eqs.(5.3) and (5.4), we get

$$\begin{aligned}&(-48\cot^6\theta + 32\cot^4\theta + \frac{48\cos^4\theta}{\sin^6\theta} - \frac{8\cos^2\theta}{\sin^4\theta} + 8\cot^2\theta - 2)b^2\\ &+ (9\sin^2\theta\cos^2\theta - 12\cos^4\theta - 3\cos^2\theta - 1)\rho = 0.\end{aligned} \tag{5.5}$$

Regarding θ as variable, taking the derivative of Eq.(5.5) and using the second equation of Eq.(4.2), we have

$$\begin{aligned}&(-192\cot^7\theta + 80\cot^5\theta + 64\cot^3\theta - \frac{272\cos^3\theta}{\sin^5\theta} - \frac{8\cos\theta}{\sin^3\theta} + \frac{192\cos^5\theta}{\sin^7\theta} - 2\tan\theta)b^2 +\\ &(-\frac{18\cos^5\theta}{\sin\theta} + \frac{18\cos^3\theta}{\sin\theta} - \frac{3\sin^5\theta}{4\cos\theta} + 78\sin\theta\cos^3\theta - 15\sin^3\theta\cos\theta + 3\sin\theta\cos\theta)\rho = 0.\end{aligned} \tag{5.6}$$

Set $x = \sin\theta$. Using Eq.(5.5), we have

$$b^2 = \frac{\rho x^4(21x^4 - 36x^2 + 16)}{2(35x^4 - 72x^2 + 36)}. \tag{5.7}$$

Taking Eq.(5.7) into Eq.(5.6), we get

$$\frac{\rho x(6615x^8-28224x^6+43896x^4-29632x^2+7344)}{4\sqrt{1-x^2}(35x^4-72x^2+36)}=0. \tag{5.8}$$

Hence, x is constant, then θ is constant. So $b=0$ by the second formula of Eq.(3.30), i.e. $\mathbf{H}=0$. Thus, we finish the first part of our proofs.

Now, we prove our result in the case of $\rho=0$:

When $\rho=0$, using the second formula of Eq.(4.1) and Eq.(5.3), we have

$$K=-48c_2^2\frac{\cos^5\theta}{\sin^2\theta}+48c_2^2\frac{\cos^3\theta}{\sin^2\theta}+24c_2^2\cos^3\theta-8c_2^2\cos\theta. \tag{5.9}$$

Using the second formula of Eq.(4.1) and Eq.(5.4), we get

$$K=-8c_2^2\cos^3\theta-8c_2^2\sin^2\theta\cos\theta+2c_2^2\frac{\sin^4\theta}{\cos\theta}. \tag{5.10}$$

Combining Eq.(5.9) and Eq.(5.10), we can have

$$-\frac{48\cos^5\theta}{\sin^2\theta}+\frac{48\cos^3\theta}{\sin^2\theta}-\frac{2\sin^4\theta}{\cos\theta}+8\sin^2\theta\cos\theta+32\cos^3\theta-8\cos\theta=0.$$

Set $x=\sin\theta$, then we get

$$\frac{2(35x^4-72x^2+36)}{\sqrt{1-x^2}}=0,$$

hence x is constant, then θ is constant. So $b=0$ by the second formula of Eq.(3.30), i.e. $\mathbf{H}=0$. We finish our proofs.

Remark 5.2 The coordinate of b^2 in Eq.(5.5) doesn't equal to 0. Setting $x=\sin\theta$, from the calculation by Mathematica, we can know that

$$-48\cot^6\theta+32\cot^4\theta+\frac{48\cos^4\theta}{\sin^6\theta}-\frac{8\cos^2\theta}{\sin^4\theta}+8\cot^2\theta-2=\frac{2(35x^4-72x^2+36)}{x^4}=0,$$

hence, $x=\sqrt{\frac{6}{7}}$.

If $x=\sqrt{\frac{6}{7}}$, the above equation equals to zero, then from Eq.(5.5), we have

$$9\sin^2\theta\cos^2\theta-12\cos^4\theta-3\cos^2\theta-1=-21x^4+36x^2-16=0,$$

and solve the equation by Mathematica, but we can't have the solution in $(0,1)$. It's a contradiction. So the coordinate of b^2 in Eq.(5.5) doesn't equal to 0.

Remark 5.3 From the discussion in Remark 5.2, we can know that the denominator in Eq.(5.8) doesn't equal to 0.

As we already know that any closed symplectic minimal surface in a Kähler-Einstein surface with non-negative scalar curvature is holomorphic, we have the following Liouville theorem:

Corollary 5.4 Any closed symplectic critical surfaces with circular ellipse of curvature in two-dimensional complex space forms with non-negative holomorphic sectional curvature must be holomorphic.

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复二维空间形式中曲率椭圆是圆的辛临界曲面

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摘要: 本文研究了复二维空间形式中曲率椭圆是圆的辛临界曲面. 利用活动标架法, 获得了这类曲面是极小曲面的结果, 丰富了辛临界曲面的内容.

关键词: 辛临界曲面; 曲率椭圆; 极小曲面; 复空间形式

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