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DIFFERENTIAL MIXED EQUILIBRIUM PROBLEMS IN BANACH SPACE

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Abstract: In this paper, we investigate a new class of differential mixed equilibrium problems ((DME), for short) in Banach space. By using Fan-KKM theorem and Ky Fan's minmax inequality, we respectively prove the existence of solutions for mixed equilibrium problems under some suitable conditions. Moreover, we prove the superpositional measurability and upper semicontinuity for a class of set-valued mappings. Finally, by using the theory of semigroups and Filippov implicit function lemma, we obtain the existence theorem concerned with the mild solutions for (DME) and discuss the compactness of the solution set. The results enrich and extend the theory of equilibrium.

Keywords: differential mixed equilibrium problems; Banach space; Fan-KKM theorem; Ky Fan's minmax inequality; Filippov implicit function lemma

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1 Introduction and Preliminaries

It was well known that Pang and Stewart introduced and studied differential variational inequality in a finite-dimensional Euclidean space (see [1]). Recently, the existence of solutions for different types of differential variational inequalities problems (see [2-7]) is considered by many authors.

In this paper, we introduce a class of differential mixed equilibrium problem. Under various conditions, we obtain the existence theorem concerned with the mild solutions for this class of problems.

Now we introduce some preliminaries which will be used in the paper. For any nonempty set Y, $P(Y)$ denotes the family of all nonempty subsets of Y. We denote

$$K(Y) := \{D \in P(Y) | D \text{ is compact}\},$$

$$K_v(Y) := \{D \in P(Y) | D \text{ is compact and convex}\}.$$

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Lemma 1.1 [8] (Fan-KKM) Let K be a nonempty subset of a Hausdorff topological vector space E_1 and let $G : K \to P(E_1)$ be a set-valued mapping with the properties:

(i) G is a KKM mapping;

(ii) $G(v)$ is closed in E_1 for every $v \in K$;

(iii) $G(v_0)$ is compact in E_1 for some $v_0 \in K$.

Then one has $\bigcap\limits_{v \in K} G(v) \neq \varnothing$.

Lemma 1.2 [9] Let K be a nonempty compact and convex subset of a Banach space X, let $\varphi : K \times K \to \mathbb{R}$ be a mapping. Suppose the following conditions hold:

(i) $x \mapsto \varphi(x, y)$ is lower semicontinuous for every $y \in K$;

(ii) $y \mapsto -\varphi(x, y)$ is convex for every $x \in K$;

(iii) $\varphi(y, y) \leq 0$ for every $y \in K$.

Then there exists $\overline{x} \in K$ such that $\varphi(\overline{x}, y) \leq 0$ for every $y \in K$.

Definition 1.1 Let E_1 be a topological vector space and let P be a pointed convex cone in E_1. $\preceq$ is a partial order relation on E_1: $x \preceq y$ if and only if $y - x \in P$. A mapping $Q : E_1 \to \mathbb{R}$ is said to be order weak monotone increasing if for each $x_1, x_2 \in E_1$ satisfying $x_1 \preceq x_2$, it holds $Q(x_2 - x_1) > 0$.

Lemma 1.3 [10] Let $F : X \to P(Y)$ be a set-valued mapping, with X and Y be topological spaces. The statements below are equivalent:

(i) F is upper semicontinuous;

(ii) for every closed set $C \subset Y$, the set $F^{-1}(C)$ is closed in X, where

$$F^{-1}(C) := \{x \in X : F(x) \cap C \neq \emptyset\};$$

(iii) for every open set $O \subset Y$, the set $F^{+1}(O)$ is open in X, where

$$F^{+1}(O) := \{x \in X : F(x) \subset O\}.$$

(i') F is lower semicontinuous;

(ii') for every closed set $C \subset Y$, the set $F^{+1}(C)$ is closed in X, where

$$F^{+1}(C) := \{x \in X : F(x) \subset C\};$$

(iii') for every open set $O \subset Y$, the set $F^{-1}(O)$ is open in X, where

$$F^{-1}(O) := \{x \in X : F(x) \cap O \neq \emptyset\}.$$

Definition 1.2 [11] Let X be a Banach space. A set-valued mapping $F : [0, T] \to P(X)$ is said to be measurable if for every closed set $C \subset X$, the set $F^{-1}(C) := \{x \in [0, T] : F(x) \cap C \neq \emptyset\}$ on $\mathbb{R}$ is measurable.

Definition 1.3 [12] Let E and E_1 be Banach spaces, and let an interval $I \subset \mathbb{R}$. We say that a mapping $F : I \times E \to P(E_1)$ is super volitionally measurable if, for every measurable set-valued mapping $Q : I \to K(E)$, the composition $\phi : I \to P(E_1)$ given by $\phi(t) = F(t, Q(t))$ is measurable.

Lemma 1.4 [11] Let E and E_1 be Banach spaces, and let an interval $I \subset \mathbb{R}$. Assume E is separable. If the mapping $F : I \times E \to K(E_1)$ satisfies the Carathéodory condition or is upper or lower semicontinuous, then F is superpositionally measurable.

Lemma 1.5 [11] Let E and E_1 be Banach spaces. Suppose that the set-valued mapping $G : [0,T] \times E \to K(E_1)$ satisfies:

(i) for every $x \in E$, $G(\cdot, x) : [0,T] \to K(E_1)$ has a strongly measurable selection;

(ii) for a.e. $t \in [0,T]$, $G(t, \cdot) : E \to K(E_1)$ is upper semicontinuous.

Then for every strongly measurable function $q : [0,T] \to E$, there exists a strongly measurable selection $g : [0,T] \to E_1$ of the composition $M : [0,T] \to K(E_1)$ given by $M(t) = G(t, q(t))$ for a.e. $t \in [0,T]$.

Now we recall the classical definition of measure of noncompactness.

Definition 1.4 [11] Let X be a Banach space and $(\mathfrak{A}, \preceq)$ be a partially ordered set. A mapping $\beta : P(X) \to \mathfrak{A}$ is called a measure of noncompactness(MNC, for short) in X if $\beta(\overline{co}\Omega) = \beta(\Omega)$ for every $\Omega \in P(X)$. A measure of noncompactness β is called:

(i) monotone if $\Omega_0, \Omega_1 \in P(X), \Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$;

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X, \Omega \in P(X)$.

An example is the Hausdorff MNC$_\chi$, which is defined by

$$\chi(\Omega) := \inf\{\varepsilon > 0 : \exists \Omega_i \in P(E), \mathrm{diam}(\Omega_i) \le \varepsilon, i = 1, \cdot, \cdot, n \text{ s.t. } \Omega \subseteq \bigcup_{i=1}^{n} \Omega_i\},$$

Another example we mentioned here is the monotone nonsingular MNC in the space $C([0,T]; E)$. Namely, for every nonempty bounded set $\Omega \subset C([0,T]; E)$, it is equal to

$$v(\Omega) := \max_{\omega \in \Delta(\Omega)} (\gamma(\omega), \mathrm{mod}_C(\omega)), \tag{1.1}$$

where $\Delta(\Omega)$ denotes the collection of all countable subsets of Ω, and

$$\gamma(\omega) := \sup_{t \in [0,T]} e^{-Lt} \chi(\omega(t)),$$

$$\mathrm{mod}_C(\omega) := \lim_{\delta \to 0} \sup_{x \in \omega} \max_{|t_1 - t_2| \le \delta} \|x(t_1) - x(t_2)\|_E.$$

Definition 1.5 [11] Let X be a closed subset of a Banach space E and let β be a MNC in E. A set-valued mapping $F : X \to K(E)$ is said to be β-condensing if there exists some $0 \le k \le 1$ such that $\beta(F(\Omega)) \le k\beta(\Omega)$ for every $\Omega \in P(X)$.

Lemma 1.6 [11] Let E be a Banach space and $M \subset E$ be a nonempty closed and convex subset. If $F : M \to K_v(M)$ is a closed β-condensing set-valued mapping with β be a nonsingular MNC β in E, then the set Fix$F$ of fixed points of F is nonempty.

Lemma 1.7 [11] Let E be a Banach space and let CCE be a nonempty closed subset. $F : C \to K(E)$ is a closed set-valued mapping, which is β-condensing on every bounded subset of E with a monotone MNC β in E. If Fix$F$ is bounded, then it is compact.

2 The Introduction of Some New Problems

Let E and E_1 be real Banach spaces and let K be a nonempty compact and convex subset of E_1. Let $A : D(A) \subset E \to E$ be the infinitesimal generator of a C_0-semigroup e^{At} in E and let $\phi : E_1 \to \mathbb{R}$ be a convex, lower semicontinuous functional. Let $f : [0,T] \times E \times E_1 \to E$ and $g : [0,T] \times E \times E_1 \to \mathbb{R}$ be two fixed mappings with some constant $T > 0$. In this paper, we investigate a new class of differential mixed equilibrium problems((DME), for short):

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), & t \in [0,T] \\ u(t) \in \mathrm{SOL}(K, g(t, x(t), \cdot), \phi), & t \in [0,T] \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $\mathrm{SOL}(K, g(t, x(t), \cdot), \phi)$ stands for the solution set of the mixed equilibrium problem ((MEP), for short): find $u : [0,T] \to K$ such that

$$g(t, x(t), v - u(t)) + \phi(v) - \phi(u(t)) \geq 0, \forall v \in K. \tag{2.2}$$

Definition 2.1 A pair of functions (x, u), with $x \in C([0,T]; E)$ and $u : [0,T] \to K$ measurable, is said to be a mild solution of problem (DME) if

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} f(s, x(s), u(s))\, ds, \quad t \in [0,T], \tag{2.3}$$

and $u(s) \in \mathrm{SOL}(K, g(s, x(s), \cdot), \phi), s \in [0,T]$. If (x, u) is a mild solution of problem (DME), then x is called the mild trajectory and u is called the variational control trajectory.

3 Existence and Properties of Solution Sets for Mixed Equilibrium Problems

Let E_1 be a real Banach space and let K be a nonempty subset of E_1. Let $Q : E_1 \to \mathbb{R}$ and $\phi : E_1 \to \mathbb{R}$ be two fixed mappings. Now we consider the following mixed equilibrium problem: find $u \in K$ such that

$$Q(v - u) + \phi(v) - \phi(u) \geq 0, \forall v \in K. \tag{3.1}$$

In this section, we will use Fan-KKM theorem and Ky Fan's minmax inequality separately to prove the existence and properties of solutions for (3.1).

Theorem 3.1 Let $(E_1, \preceq)$ be a totally ordered real Banach space and let K be a nonempty compact and convex subset of E_1. $Q : E_1 \to \mathbb{R}$ is an order weak monotone increasing mapping with $Q(0) = 1$. Assume that:

(i) Q is concave, continuous and $Q(x) - Q(-x) = 0$ on E_1;

(ii) $\phi : E_1 \to \mathbb{R}$ is convex and lower semicontinuous on E_1.

Then the solution set of (3.1) is nonempty, convex and closed in K.

Proof We consider the set-valued mapping $G : K \to P(K)$ defined by

$$G(v) := \{u \in K : Q(v - u) + \phi(v) - \phi(u) \geq 0\}, \forall v \in K.$$

For each $v \in K, G(v)$ is nonempty since $v \in G(v)$.

First, we claim that $G(v)$ is closed in K for all $v \in K$. Indeed, we take a sequence $\{u_n\} \subset G(v)$ such that $u_n \to u_0$ as $n \to \infty$. We get

$$Q(v - u_n) + \phi(v) - \phi(u_n) \geq 0, \forall n \in \mathbb{N}.$$

Applying the continuity of Q and the lower semicontinuity of ϕ, let $n \to \infty$, then we have

$$Q(v - u_0) + \phi(v) - \phi(u_0) \geq 0,$$

which means $u_0 \in G(v)$, so $G(v)$ is closed in K.

Second, we claim that G is a KKM mapping. Arguing by contradiction, we assume that there exist a finite subset $\{v_1, v_2, \cdot, \cdot, \cdot, v_n\} \subset K$ and $u_0 = \sum\limits_{i=0}^{n} \lambda_i v_i (\sum\limits_{i=1}^{n} \lambda_i = 1,\ 0 \leq \lambda_i \leq 1)$ such that $u_0 \notin \bigcup\limits_{i=1}^{n} G(v_i)$, then we have

$$Q(v_i - u_0) + \phi(v_i) - \phi(u_0) < 0, \forall i \in \{1, 2, \cdots, n\},$$

which implies

$$\sum_{i=1}^{n} \lambda_i Q(v_i - u_0) + \sum_{i=1}^{n} \lambda_i \phi(v_i) - \phi(u_0) < 0.$$

Since ϕ is convex, we further obtain $\sum\limits_{i=1}^{n} \lambda_i Q(v_i - u_0) < 0$, which means $\exists j \in \{1, 2, \cdots, n\}$ such that $Q(v_j - u_0) < 0$. Since $Q(x) - Q(-x) = 0$ on E_1, we have

$$Q(u_0 - v_j) = Q(v_j - u_0) < 0. \tag{3.2}$$

Since E_1 is totally ordered, we have either $v_j \preceq u_0$ or $u_0 \preceq v_j$ holds. Since Q is order weak monotone increasing, we get

$$Q(u_0 - v_j) > 0 \text{ or } Q(v_j - u_0) > 0,$$

which contradicts (3.2). Therefore G is a KKM mapping.

Third, for any $v_0 \in K$, since $G(v_0)$ is a closed subset of the compact set K, we know $G(v_0)$ is a compact set.

Using Lemma 1.1, we derive

$$\bigcap_{v \in K} G(v) \neq \emptyset,$$

which ensures that the solution set of (3.1) is nonempty.

Finally, we verify that the solution set of problem (3.1) is closed and convex.

Let $\{u_n\}$ be a sequence in the solution set satisfying $u_n \to u_0$ as $n \to \infty$, then we have

$$Q(v - u_n) + \phi(v) - \phi(u_n) \geq 0, \forall v \in K,$$

which yields in the limit

$$Q(v-u_0)+\phi(v)-\phi(u_0)\geq 0,\forall v\in K.$$

Therefore u_0 solves problem (3.1), thus the solution set of (3.1) is closed.

Let u_1 and u_2 be arbitrary points in the solution set of (3.1), $\lambda\in[0,1]$. Since Q is concave and ϕ is convex, for any $v\in K$, we have

$$\begin{aligned}
&Q(v-(\lambda u_1+(1-\lambda)u_2))+\phi(v)-\phi(\lambda u_1+(1-\lambda)u_2)\\
=&Q(\lambda(v-u_1)+(1-\lambda)(v-u_2))+\lambda\phi(v)+(1-\lambda)\phi(v)-\phi(\lambda u_1+(1-\lambda)u_2)\\
\geq&\lambda Q(v-u_1)+(1-\lambda)Q(v-u_0)+\lambda(\phi(v)-\phi(u_1))+(1-\lambda)(\phi(v)-\phi(u_2))\\
=&\lambda(Q(v-u_1)+\phi(v)-\phi(u_1))+(1-\lambda)(Q(v-u_2)+\phi(v)-\phi(u_2))\\
\geq&0.
\end{aligned}$$

This implies that $\lambda u_1+(1-\lambda)u_2$ solves problem (3.1), thus the solution set of (3.1) is convex. The proof is complete.

Example 3.1 Let $E_1=\mathbb{R}$, $K=[0,1]$ and $P=[0,+\infty)$ be a cone, we define the mappings $Q:E_1\to\mathbb{R}$ and $\phi:E_1\to\mathbb{R}$ by

$$Q(x)=\sqrt{|x|}+1\ \ and\ \ \phi(x)=x.$$

We can check that the solution set of (3.1) is $[0,1]$.

Theorem 3.2 Let E_1 be a real Banach space and let K be a nonempty compact and convex subset of E_1. Let $Q:E_1\to\mathbb{R}$ be a mapping satisfies $Q(0)=0$. Assume that:

(i) Q is concave, continuous on E_1;

(ii) $\phi:E_1\to\mathbb{R}$ is convex and lower semicontinuous on E_1.

Then the solution set of (3.1) is nonempty, convex and closed in K.

Proof We consider the mapping $\varphi:K\times K\to R$ defined by

$$\varphi(u,v):=-Q(v-u)+\phi(u)-\phi(v).$$

One can check that $\varphi(\cdot,\cdot)$ satisfies conditions (i),(ii) and (iii) in Lemma 1.2.

Thus, by Lemma 1.2, we obtain that there exists $\overline{u}\in K$, such that

$$\varphi(\overline{u},v)=-Q(v-\overline{u})+\phi(\overline{u})-\phi(v)\leq 0,\forall v\in K,$$

or equivalently,

$$Q(v-\overline{u})+\phi(v)-\phi(\overline{u})\geq 0,\forall v\in K.$$

Hence, the solution set of (3.1) is nonempty.

In fact, the solution set of (3.1) is convex and closed. The proof can be done by arguing in the same way as Theorem 3.1, so we omit it.

Example 3.2 Let $E_1 = \mathbb{R}$, $K = [0,1]$, we define the mappings $Q : E_1 \to \mathbb{R}$ and $\phi : E_1 \to \mathbb{R}$ by

$$Q(x) = x \ \ and \ \ \phi(x) = x(x-2).$$

We can check that the solution set of (3.1) is $\{0\}$.

Remark 3.1 The hypotheses of Theorem 3.1 and Theorem 3.2 are different, while we obtain the same conclusion. The reason is that the method we use is different, we use Fan-KKM theorem to prove Theorem 3.1, while using Ky Fan's minmax inequality in the proof of Theorem 3.2.

4 Continuity and Superpositional Measurability for a Class of Set-Valued Mappings

Theorem 4.1 Let $(E_1, \preceq)$ be a totally ordered real Banach space and let K be a nonempty compact and convex subset of E_1. E is a real separable Banach space. The mappings $Q(\cdot) := g(t,x,\cdot)$ and $\phi : E_1 \to \mathbb{R}$ satisfy hypotheses in Theorem 3.1. Assume that:

(i) $(t,x) \mapsto g(t,x,u)$ is continuous on $[0,T] \times E$;

(ii) $u \mapsto g(t,x,u)$ is upper semicontinuous on K.

Then the set-valued mapping $U : [0,T] \times E \to K_v(K)$ defined by

$$U(t,x) := \{u \in K : g(t,x,v-u) + \phi(v) - \phi(u) \geq 0, \forall v \in K\} \tag{4.1}$$

is upper semicontinuous and superpositionally measurable.

Proof Theorem 3.1 guarantees that for every $(t,x) \in [0,T] \times E$, the set $U(t,x)$ is nonempty, convex and compact in K. Thus the mapping $U(\cdot,\cdot)$ is well defined.

Now we claim that $U(\cdot,\cdot)$ is upper semicontinuous. By Lemma 1.3, it's sufficient to prove that $U^{-1}(C) := \{(t,x) \in [0,T] \times E : U(t,x) \cap C \neq \emptyset\}$ is closed for each closed subset C of K. In fact, if $(t_n, x_n) \in U^{-1}(C)$ and $(t_n, x_n) \to (t_0, x_0)$. Since $(t_n, x_n) \in U^{-1}(C)$, there exists $u_n \in U(t_n, x_n) \cap C, \forall n \in \mathbb{N}$. Therefore,

$$g(t_n, x_n, v - u_n) + \phi(v) - \phi(u_n) \geq 0, \forall v \in K. \tag{4.2}$$

Since C is a closed subset of the compact set K, we obtain C is also a compact set. Hence there is a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u_0 \in K$. Let $k \to \infty$ in (4.2), by the conditions (i) and (ii), we have

$$g(t_0, x_0, v - u_0) + \phi(v) - \phi(u_0) \geq 0 \ , \forall v \in K.$$

In other words, we proved $u_0 \in U(t_0, x_0) \cap C$, so $(t_0, x_0) \in U^{-1}(C)$. Therefore, $U(\cdot,\cdot)$ is upper semicontinuous.

Finally, we conclude that $U(t,x)$ is superpositionally measurable by applying Lemma 1.4. This completes the proof.

5 Existence of Mild Solutions for a Class of Differential Mixed Equilibrium Problems

In this section, we will show the existence of solutions for (DME). First, we assume the following hypotheses on the mapping $f:[0,T]\times E\times E_1\to E$ in (2.1):

$(f1)$ for every $(t,x)\in[0,T]\times E$, the set $f(t,x,D)$ is convex in E for every convex set $D\subset K$;

$(f2)$ there exists $\psi\in L^1([0,T])$ such that

$$\|f(t,x,u)\|_E\le\psi(t)(1+\|x\|_E),\forall(t,x,u)\in[0,T]\times E\times K;$$

$(f3)$ $f(\cdot,x,u):[0,T]\to E$ is measurable for every $(x,u)\in E\times E_1$;

$(f4)$ $f(t,\cdot,\cdot):E\times E_1\to E$ is continuous for a.e. $t\in[0,T]$;

$(f5)$ there exists $k\in L^1([0,T])$ such that

$$\|f(t,x_0,u)-f(t,x_1,u)\|_E\le k(t)\|x_0-x_1\|_E$$

for a.e. $t\in[0,T],\forall x_0,x_1\in E,\forall u\in K$.

We next study the properties of the set-valued mapping $F:[0,T]\times E\to P(E)$ given by

$$F(t,x)=f(t,x,U(t,x)),$$

with U introduced in (4.1).

Lemma 5.1 [2] Let E and E_1 be real Banach space, with E separable, and let $K\subseteq E_1$ be a nonempty compact and convex subset. We assume that the hypotheses of Theorem 4.1 and conditions $(f1)-(f5)$ are fulfilled. Then we have:

(i) $F(t,x)\in K_v(E)$ for all $(t,x)\in[0,T]\times E$;

(ii) $F(\cdot,x)$ has a strongly measurable selection for every $x\in E$;

(iii) $F(t,\cdot)$ is upper semicontinuous for a.e. $t\in[0,T]$;

(iv) for every bounded subset $D\subset E$, there exists $l\in L^1([0,T])$ such that

$$\chi(F(t,D))\le l(t)\chi(D),\ \text{for a.e. } t\in[0,T],$$

where χ is the Hausdorff measure of noncompactness in E.

By Lemma 5.1 we can define the set-valued mapping $P_F:C([0,T];E)\to P(L^1([0,T];E))$ by

$$P_F(q):=\{g|g \text{ is strongly measurable and } g(t)\in F(t,q(t)) \text{ for a.e. } t\in[0,T]\}.$$

Furthermore, we can introduce $\Gamma:C([0,T];E)\to K_v(C([0,T];E))$ by

$$\Gamma x:=\{y\in C([0,T];E):y(t)=e^{At}x_0+\int_0^t e^{A(t-s)}h(s)\,ds,h\in P_F(x)\},\tag{5.1}$$

where $A:D(A)\subset E\to E$ is the infinitesimal generator of a C_0-semigroup e^{At} in E as given in (2.1).

Lemma 5.2 [13] Under the hypotheses of Lemma 5.1, the set-valued mapping Γ in (5.1) is upper semicontinuous and v-condensing in the sense of Definition 1.5 on every closed bounded subset of $C([0,T];E)$, with v constructed in (1.1).

Theorem 5.1 Under the hypotheses of Lemma 5.2, the solution set of problem (DME) in the sense of Definition 2.1 is nonempty and the set of all mild trajectories x of (DME) is compact in $C([0,T];E)$.

Proof We introduce the following evolutionary differential inclusion((EDI), for short):

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t,x(t)), \quad t \in [0,T] \\ x(0) = x_0, \end{cases} \tag{5.2}$$

Here $F(t,x) = f(t,x,U(t,x))$ with $U(t,x)$ is defined in (4.1). The proof of the theorem is divided into three parts.

Step 1 We note that the solution set of (EDI) is nonempty if and only if the set of fixed points $\mathrm{Fix}\Gamma$ of Γ is nonempty. By Lemma 5.2, the set-valued mapping $\Gamma : C([0,T];E) \to K_v(C[0,T];E)$ in (5.1) is upper semicontinuous and v-condensing on every bounded subset of $C([0,T];E)$.

Let L be a positive constant such that

$$\int_0^t e^{-L(t-s)}\psi(s)\,ds < \frac{1}{M}\ , \ \forall t \in [0,T], \tag{5.3}$$

where $\psi \in L^1([0,T])$ and $M := \max\limits_{t\in[0,T]} \|e^{At}\|$.

By (5.3), there exists $r > 0$ such that

$$M\|x_0\|_E + Mr\int_0^t e^{-L(t-s)}\psi(s)\,ds \le r, \ \forall t \in [0,T]. \tag{5.4}$$

Next, we introduce the equivalent norm on the space $C([0,T];E)$ by

$$\|x\|_* := \max_{t\in[0,T]} e^{-Lt}\|x(t)\|_E,$$

we denote the closed ball centered at 0 with radius r in $C([0,T];E)$ by

$$\overline{B_r}(0) := \{x \in C([0,T];E) : \| x \|_* \le r\}.$$

Now we claim that $\Gamma(\overline{B_r}(0)) \subset \overline{B_r}(0)$. Let $x \in \overline{B_r}(0)$ and $y \in \Gamma x$. From (5.1), there exists $h \in P_F(x)$ such that

$$y(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}\psi(s)h(s)\,ds, \ \forall t \in [0,T].$$

Using $(f2)$, we obtain

$$\begin{aligned} e^{-Lt}\|y(t)\|_E =& e^{-Lt}\|e^{At} + \int_0^t e^{A(t-s)}h(s)\,ds\|_E \\ \le& e^{-Lt}\|e^{At}x_0\| + e^{-Lt}\int_0^t \|e^{A(t-s)}\|\|h(s)\|_E\,ds \\ \le& M(\|x_0\|_E + \|\psi\|_{L^1([0,T])}) + M\|x\|_*\int_0^t e^{-L(t-s)}\psi(s)\,ds. \end{aligned}$$

Since $x \in \overline{B_r}(0)$ and using (5.4), it follows that for every $t \in [0,T]$,

$$e^{-Lt}\|y(t)\|_E \le M(\|x_0\|_E + \|\psi\|_{L^1([0,T])}) + Mr\int_0^t e^{-L(t-s)}\psi(s)\,ds \le r,$$

which implies $\|y\|_* \le r$. Therefore, $\Gamma(\overline{B_r}(0)) \subset \overline{B_r}(0)$.

Applying Lemma 5.2 and Lemma 1.6 with $M = \overline{B_r}(0)$ and $F = \Gamma$, it follows that $\mathrm{Fix}\Gamma \neq \emptyset$. Hence the solution set of (EDI) is nonempty.

Step 2 We claim that the solution set of (EDI) is compact in $C([0,T];E)$. Let $x \in C([0,T];E)$ be a solution of (EDI), then we have

$$\|x(t)\|_E \le \|e^{At}\|\|x_0\|_E + \int_0^t \|e^{A(t-s)}\|\|h(s)\|_E\,ds,\ \forall t \in [0,T],$$

where $h \in P_F(x)$. From the condition $(f2)$ we obtain

$$\begin{aligned}\|x(t)\|_E &\le M\|x_0\|_E + M\int_0^t \psi(s)(1+\|x(s)\|_E)\,ds\\ &\le M(\|x_0\|_E + \|\psi\|_{L^1([0,T])} + \int_0^t \psi(s)\|x(s)\|_E\,ds).\end{aligned}$$

Using Gronwall inequality, we have the following estimate

$$\|x(t)\|_E \le M(\|x_0\|_E + \|\psi\|_{L^1([0,T])})e^{M\|\psi\|_{L^1([0,T])}}.$$

Therefore, $\mathrm{Fix}\Gamma$ is bounded in $C([0,T];E)$.

Applying Lemma 5.2 and Lemma 1.7 with $F = \Gamma$, we know the solution set of problem (EDI), which equals to $\mathrm{Fix}\Gamma$, is compact in $C([0,T];E)$.

Step 3 Note that the set-valued mapping U is superpositionally measurable from Theorem 4.1. Therefore, by Filippov implicit function lemma (see [11]), we deduce that for every solution x of (EDI), there exists a measurable selection $u(t) \in U(t,x(t))$ such that $\dot{x}(t) = Ax(t) + f(t,x(t),u(t)), t \in [0,T]$. Hence, (x,u) is a mild solution of problem (DME) in the sense of Definition 2.1, which implies the set of all mild trajectories of problem (DME) is consistent with the solution set of problem (EDI). This completes the proof.

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Banach空间上的微分混合均衡问题

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摘要: 本文研究了Banach空间中一类新的微分混合均衡问题(简记为(DME)). 利用Fan-KKM 定理和Ky Fan 极大极小不等式, 分别证明了在某些合适条件下混合均衡问题解的存在性. 此外, 证明了一类集值映射的叠加可测性和上半连续性. 最后, 利用半群理论和Filippov 隐函数引理, 获得了关于(DME) 问题温和解的存在性定理并讨论了解集合的紧性. 所得结果丰富并扩展了均衡理论.

关键词: 微分混合均衡问题; Banach空间; Fan-KKM 定理; Ky Fan 极大极小不等式; Filippov隐函数引理

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