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THE REPRESENTATION CATEGORIES OF DIAGONAL CROSSED PRODUCTS OF INFINITE-DIMENSIONAL COFROBENIUS HOPF ALGEBRAS

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Abstract: The categorical interpretations on representations of diagonal crossed products of infinite-dimensional coFrobenius Hopf algebras are studied in this paper. By the tools of multiplier Hopf algebra and homological algebra theories, we get that the unital representation category of a diagonal crossed product of an infinite-dimensional coFrobenius Hopf algebra is isomorphic to its generalized Yetter-Drinfeld category, which generalizes the results of Panaite et al. in finite-dimensional case.

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1 Introduction

A Yetter-Drinfel'd module over a Hopf algebra, firstly introduced by Yetter (crossed bimodule in [1]), is a module and a comodule satisfying a certain compatibility condition. The main feature is that Yetter-Drinfel'd modules form a pre-braided monoidal category. Under favourable conditions (e.g. if the antipode of the Hopf algebra is bijective), the category is even braided (or quasisymmetric). Via a (pre-) braiding structure, the notion of Yetter-Drinfel'd module plays a part in the relations between quantum groups and knot theory.

When a Hopf algebra is finite-dimensional, the generalized (anti) Yetter-Drinfel'd module category was studied in [2]. The authors showed that ${}_H\mathcal{YD}^H(\alpha,\beta) \cong {}_{H^*\bowtie H(\alpha,\beta)}\mathcal{M}$, where $H^* \bowtie H(\alpha,\beta)$ is the diagonal crossed product algebra. Then one main question naturally arises: Does this isomorphism still hold for some infinite-dimensional Hopf algebra?

For this question, we first recall from our paper [3] the diagonal crossed product of an infinite-dimensional coFrobenius Hopf algebra, then we consider the representation category

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of the diagonal crossed product, and show that for a coFrobenius Hopf algebra H with its dual multiplier Hopf algebra $\widehat{H}$, the unital $\widehat{H} \bowtie H(\alpha,\beta)$-module category is isomorphic to (α,β)-Yetter-Drinfeld module category introduced in [2, 4], i.e., ${}_H\mathcal{YD}^H(\alpha,\beta) \cong {}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$. Moreover, as braided T-categories the representation category $Rep(\bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha,\beta))$ is isomorphic to $\mathcal{YD}(H)$ introduced in [2].

The paper is organized in the following way. In section 2, we recall some notions which will be used in the following, such as multiplier Hopf algebras and (α,β)-quantum double of an infinite dimensional coFrobenius Hopf algebra.

In section 3, we show that for a coFrobenius Hopf algebra H, the unital $\widehat{H} \bowtie H(\alpha,\beta)$-module category ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$ is isomorphic to ${}_H\mathcal{YD}^H(\alpha,\beta)$. And as braided T-categories the representation theory $Rep(\mathcal{A})$ is isomorphic to $\mathcal{YD}(H)$ introduced in [2], generalizing the classical result in [2, 5].

2 Preliminaries

We begin this section with a short introduction to multiplier Hopf algebras.

Throughout this paper, all spaces we considered are over a fixed field K (such as the field $\mathbb{C}$ of complex numbers). Algebras may or may not have units, but always should be non-degenerate, i.e., the multiplication maps (viewed as bilinear forms) are non-degenerate. Recalling from the appendix in [6], the multiplier algebra $M(A)$ of an algebra A is defined as the largest algebra with unit in which A is a dense ideal.

2.1 Multiplier Hopf Algebras

Now, we recall the definition of a multiplier Hopf algebra (see [6] for details). A comultiplication on an algebra A is a homomorphism $\Delta : A \longrightarrow M(A\otimes A)$ such that $\Delta(a)(1\otimes b)$ and $(a\otimes 1)\Delta(b)$ belong to $A\otimes A$ for all $a, b \in A$. We require Δ to be coassociative in the sense that

$$(a\otimes 1\otimes 1)(\Delta\otimes\iota)(\Delta(b)(1\otimes c)) = (\iota\otimes\Delta)((a\otimes 1)\Delta(b))(1\otimes 1\otimes c)$$

for all $a, b, c \in A$ (where ι denotes the identity map).

A pair (A,Δ) of an algebra A with non-degenerate product and a comultiplication Δ on A is called a multiplier Hopf algebra, if the maps $T_1, T_2 : A\otimes A \longrightarrow M(A\otimes A)$ defined by

$$T_1(a\otimes b) = \Delta(a)(1\otimes b), \qquad T_2(a\otimes b) = (a\otimes 1)\Delta(b) \tag{2.1}$$

have range in $A\otimes A$ and are bijective.

A multiplier Hopf algebra (A,Δ) is called regular if (A,Δ^{cop}) is also a multiplier Hopf algebra, where Δ^{cop} denotes the co-opposite comultiplication defined as $\Delta^{cop} = \tau\circ\Delta$ with τ the usual flip map from $A\otimes A$ to itself (and extended to $M(A\otimes A)$). In this case, $\Delta(a)(b\otimes 1)$ and $(1\otimes a)\Delta(b) \in A\otimes A$ for all $a, b \in A$.

Multiplier Hopf algebra (A, Δ) is regular if and only if the antipode S is bijective from A to A (see [7], Proposition 2.9). In this situation, the comultiplication is also determined by the bijective maps $T_3, T_4 : A \otimes A \longrightarrow A \otimes A$ defined as follows

$$T_3(a \otimes b) = \Delta(a)(b \otimes 1), \qquad T_4(a \otimes b) = (1 \otimes a)\Delta(b) \tag{2.2}$$

for all $a, b \in A$.

In this paper, all the multiplier hopf algebras we considered are regular. We will use the adapted Sweedler notation for regular multiplier Hopf algebras (see [8]). We will e.g., write $\sum a_{(1)} \otimes a_{(2)}b$ for $\Delta(a)(1 \otimes b)$ and $\sum ab_{(1)} \otimes b_{(2)}$ for $(a \otimes 1)\Delta(b)$, sometimes we omit the $\sum$.

Define two linear operators $\mathcal{T}$ and $\mathcal{T}'$ acting on $A \otimes A$ introduced by the formulae

$$\mathcal{T}(a \otimes b) = b_{(2)} \otimes aS(b_{(1)})b_{(3)},$$
$$\mathcal{T}'(a \otimes b) = b_{(1)} \otimes S(b_{(2)})ab_{(3)}$$

for any $a, b \in A$.

These two operators above are well-defined, since

$$\mathcal{T}(a \otimes b) = T_4(S \otimes \iota)T_3(\iota \otimes S^{-1})\tau(a \otimes b),$$
$$\mathcal{T}'(a \otimes b) = (\iota \otimes S)T_4\tau(\iota \otimes S^{-1})T_4(a \otimes b).$$

They are obviously invertible, and the inverses can be written as follows

$$\mathcal{T}^{-1}(a \otimes b) = bS^{-1}(a_{(3)})a_{(1)} \otimes a_{(2)},$$
$${\mathcal{T}'}^{-1}(a \otimes b) = a_{(3)}bS^{-1}(a_{(2)}) \otimes a_{(1)}.$$

For any $a, b \in A$, $\mathcal{T} \circ T_2 = T_4$. If A is commutative, then $\mathcal{T}' = \tau$, and if A is cocommutative, then $\mathcal{T} = \tau$.

Proposition 2.1 Operators $\mathcal{T}$ and $\mathcal{T}'$ satisfy the braided equation

$$(\mathcal{T} \otimes \iota)(\iota \otimes \mathcal{T})(\mathcal{T} \otimes \iota) = (\iota \otimes \mathcal{T})(\mathcal{T} \otimes \iota)(\iota \otimes \mathcal{T}),$$
$$(\mathcal{T}' \otimes \iota)(\iota \otimes \mathcal{T}')(\mathcal{T}' \otimes \iota) = (\iota \otimes \mathcal{T}')(\mathcal{T}' \otimes \iota)(\iota \otimes \mathcal{T}').$$

2.2 Diagonal Crossed Product of an Infinite Dimensional coFrobenius Hopf Algebra

Let H be a Hopf algebra, and $\alpha, \beta \in Aut_{Hopf}(H)$. Denote $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$, a group with multiplication $(\alpha, \beta) * (\gamma, \delta) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$. The unit is (ι, ι) and $(\alpha, \beta)^{-1} = (\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1})$.

Let H be a coFrobenius Hopf algebra with its dual multiplier Hopf algebra $\widehat{H}$. Then $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in G} \mathcal{A}_{(\alpha,\beta)} = \bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$ is a G-cograded multiplier Hopf algebra with the following strucrures:

- For any $(\alpha,\beta)\in G$, $\mathcal{A}_{(\alpha,\beta)}$ has the multiplication given by

$$(p\bowtie h)(q\bowtie l)=p\big(\alpha(h_{(1)})\blacktriangleright q\blacktriangleleft S^{-1}\beta(h_{(3)})\big)\bowtie h_{(2)}l$$

for $p,q\in\widehat{H}$ and $h,l\in H$.

- The comultiplication on $\mathcal{A}$ is given by:

$$\Delta_{(\alpha,\beta),(\gamma,\delta)}:\mathcal{A}_{(\alpha,\beta)*(\gamma,\delta)}\longrightarrow M(\mathcal{A}_{(\alpha,\beta)}\otimes\mathcal{A}_{(\gamma,\delta)}),$$
$$\Delta_{(\alpha,\beta),(\gamma,\delta)}(p\bowtie h)=\Delta^{cop}(p)(\gamma\otimes\gamma^{-1}\beta\gamma)\Delta(h).$$

- The counit $\varepsilon_{\mathcal{A}}$ on $\mathcal{A}_{(\iota,\iota)}=D(H)$ is the counit on the Drinfel'd double of H.

- For any $(\alpha,\beta)\in G$, the antipode is given by

$$S:\mathcal{A}_{(\alpha,\beta)}\longrightarrow\mathcal{A}_{(\alpha,\beta)^{-1}},$$
$$S_{(\alpha,\beta)}(p\bowtie h)=T(\alpha\beta S(h)\otimes S^{-1}(p))\text{ in }\mathcal{A}_{(\alpha,\beta)^{-1}}=\mathcal{A}_{(\alpha^{-1},\alpha\beta^{-1}\alpha^{-1})}.$$

- A crossing action $\xi:G\longrightarrow Aut(\mathcal{A})$ is given by

$$\xi_{(\alpha,\beta)}^{(\gamma,\delta)}:\mathcal{A}_{(\gamma,\delta)}\longrightarrow\mathcal{A}_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}}=\mathcal{A}_{(\alpha\gamma\alpha^{-1},\alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})},$$
$$\xi_{(\alpha,\beta)}^{(\gamma,\delta)}(p\bowtie h)=p\circ\beta\alpha^{-1}\bowtie\alpha\gamma^{-1}\beta^{-1}\gamma(h).$$

Let H be a coFrobenius Hopf algebra with a left integral φ, and $t\in A$ is a cointegral in A such that $\varphi(t)=1$. Recalling from [5] there is an element

$$u\otimes v=:\sum t(\cdot\varphi_{(2)})\otimes S^{-1}(\varphi_{(1)})\in M(H\otimes\widehat{H}).$$

Following from Lemma 9 in [5] we have:

- For any $p\in\widehat{H}$ and $h\in H$,

$$v\langle p,u\rangle=p,\qquad u\langle v,h\rangle=h.\tag{2.3}$$

- Let $u\otimes v=u'\otimes v'$, then

$$(\Delta\otimes\iota)(u\otimes v)=u\otimes u'\otimes vv',\qquad(\iota\otimes\Delta)(u\otimes v)=uu'\otimes v\otimes v'.\tag{2.4}$$

And from [3] $\mathcal{A}=\bigoplus_{(\alpha,\beta)\in G}\widehat{H}\bowtie H(\alpha,\beta)$ is a quasitriangular G-cograded multiplier Hopf algebra with a generalized R-matrix given by

$$R=\sum_{(\alpha,\beta),(\gamma,\delta)\in G}R_{(\alpha,\beta),(\gamma,\delta)}=\sum_{(\alpha,\beta),(\gamma,\delta)\in G}\varepsilon\bowtie\beta^{-1}(u)\otimes v\bowtie 1.$$

3 Representation Category of the Diagonal Crossed Product

Recalling from [2], a (α,β)-Yetter-Drinfel'd module over H is a vector space M, such that M is a left H-module (with notation $h\otimes m\mapsto h\cdot m$) and a right H-comodule (with notation $M\to M\otimes H, m\mapsto m_{(0)}\otimes m_{(1)}$) with the following compatibility condition:

$$(h\cdot m)_{(0)}\otimes(h\cdot m)_{(1)}=h_{(2)}\cdot m_{(0)}\otimes\beta(h_{(3)})m_{(1)}\alpha S^{-1}(h_{(1)}). \tag{3.1}$$

We denote by ${}_H\mathcal{YD}^H(\alpha,\beta)$ the category of (α,β)-Yetter-Drinfel'd modules, morphism being the H-linear H-colinear maps. If H is "finite-dimensional", then

$${}_H\mathcal{YD}^H(\alpha,\beta)\cong{}_{H^*\bowtie H(\alpha,\beta)}\mathcal{M},$$

where $H^*\bowtie H(\alpha,\beta)$ is the diagonal crossed product.

One main question naturally arises: Does this isomorphism also hold for some "infinite-dimensional" Hopf algebras? In the following, we will give a positive answer to an infinite-dimensional coFrobenius Hopf algebra case.

Recalling from Lemma 11 in [5], If M is a left unital $\widehat{H}$-module, then

$$\begin{aligned}\rho:\quad & M\longrightarrow M\otimes H,\\ & m\mapsto\sum S^{-1}(\varphi_{(1)})\cdot m\otimes t(\cdot\varphi_{(2)})=v\cdot m\otimes u\end{aligned}$$

gives the H-comodule structure on M. Following this lemma, we get the following proposition.

Proposition 3.1 If $M\in{}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$, then $M\in{}_H\mathcal{YD}^H(\alpha,\beta)$ with structures

$$\begin{aligned}&h\cdot m=(\varepsilon\bowtie h)\cdot m,\\ &m\mapsto m_{(0)}\otimes m_{(1)}=(S^{-1}(\varphi_{(1)})\bowtie 1)\cdot m\otimes t(\cdot\varphi_{(2)}).\end{aligned}$$

Proof Here we treat $\widehat{H}$ and H as subalgebras of $\widehat{H}\bowtie H(\alpha,\beta)$ in the usual way, then it is easy to get M is an H-module and H-comodule.

To show $M\in{}_H\mathcal{YD}^H(\alpha,\beta)$, i.e., $\rho(h\cdot m)=h_{(2)}\cdot m_{(0)}\otimes\beta(h_{(3)})m_{(1)}\alpha S^{-1}(h_{(1)})$, it is enough to verify that

$$(S^{-1}(\varphi_{(1)})\bowtie h)\otimes t(\cdot\varphi_{(2)})=(\varepsilon\bowtie h_{(2)})(S^{-1}(\varphi_{(1)})\bowtie 1)\otimes\beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)}).$$

Viewing $\widehat{H}\bowtie H(\alpha,\beta)\otimes H$ as a subspace of $Hom(\widehat{H},(\widehat{H}\bowtie H)(\alpha,\beta))$ in a natural way, we only need to check that

$$\begin{aligned}p\bowtie h &\overset{(2.3)}{=} (S^{-1}(\varphi_{(1)})\bowtie h)t(p\varphi_{(2)})\\ &= (\varepsilon\bowtie h_{(2)})(S^{-1}(\varphi_{(1)})\bowtie 1)\langle p,\beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)})\rangle\end{aligned}$$

holds for any $p\in\widehat{H}$. Indeed, for any $p'\in\widehat{H}$,

$$\begin{aligned}&(p'\bowtie h_{(2)})(S^{-1}(\varphi_{(1)})\bowtie 1)\langle p,\beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)})\rangle\\ =\;&(p'\bowtie h_{(2)})(S^{-1}(\varphi_{(1)})\bowtie 1)\langle p,\beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)})\rangle\end{aligned}$$

$$
\begin{aligned}
&= (p'\bowtie h_{(2)})(p_{(2)}\bowtie 1)\langle p_{(1)},\beta(h_{(3)})\rangle\langle p_{(3)},\alpha S^{-1}(h_{(1)})\rangle\\
&= \langle p_{(1)},\beta(h_{(5)})\rangle\langle p_{(3)},\alpha S^{-1}(h_{(1)})\rangle\langle p_{(2)},S^{-1}\beta(h_{(4)})\rangle\langle p_{(4)},\alpha(h_{(1)})\rangle(p'p_{(3)}\bowtie h_{(3)})\\
&= p'p\bowtie h.
\end{aligned}
$$

This completes the proof.

Proposition 3.2 If $M\in {}_H\mathcal{YD}^H(\alpha,\beta)$, then $M\in {}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$ with the structure

$$(p\bowtie h)\cdot m = p((h\cdot m)_{(1)})(h\cdot m)_{(0)}.$$

Proof It is straightforward to check that $(p\bowtie h)\cdot\big((q\bowtie l)\cdot m\big)=\big((p\bowtie h)(q\bowtie l)\big)\cdot m$. In fact,

$$
\begin{aligned}
&(p\bowtie h)\cdot\big((q\bowtie l)\cdot m\big)\\
=\ &(p\bowtie h)\cdot\big(q((l\cdot m)_{(1)})(q\cdot l)_{(0)}\big)\\
=\ &(p\bowtie h)\cdot\big(q(\beta(l_{(3)})m_{(1)}\alpha S^{-1}(l_{(1)}))l_{(2)}\cdot m_{(0)}\big)\\
=\ &q\big(\beta(l_{(3)})m_{(1)}\alpha S^{-1}(l_{(1)})\big)p\big((hl_{(2)}\cdot m_{(0)})_{(1)}\big)(hl_{(2)}\cdot m_{(0)})_{(0)}\\
=\ &q\big(\beta(l_{(5)})m_{(2)}\alpha S^{-1}(l_{(1)})\big)p\big(\beta(h_{(3)}l_{(4)})m_{(1)}\alpha S^{-1}(h_{(1)}l_{(2)})\big)(h_{(2)}l_{(3)})\cdot m_{(0)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big((p\bowtie h)(q\bowtie l)\big)\cdot m\\
=\ &\big(\langle q_{(1)},S^{-1}\beta(h_{(3)})\rangle\langle q_{(3)},\alpha(h_{(1)})\rangle(pq_{(2)}\bowtie h_{(2)}l)\big)\cdot m\\
=\ &\langle q_{(1)},S^{-1}\beta(h_{(3)})\rangle\langle q_{(3)},\alpha(h_{(1)})\rangle(pq_{(2)})((h_{(2)}l\cdot m)_{(1)})\otimes(h_{(2)}l\cdot m)_{(0)}\\
=\ &\langle q_{(1)},S^{-1}\beta(h_{(5)})\rangle\langle q_{(3)},\alpha(h_{(1)})\rangle(pq_{(2)})\big(\beta(h_{(4)}l_{(3)})m_{(1)}\alpha S^{-1}(h_{(2)}l_{(1)})\big)(h_{(3)}l_{(2)})\cdot m_{(0)}\\
=\ &\underline{\langle q_{(1)},S^{-1}\beta(h_{(7)})\rangle\langle q_{(3)},\alpha(h_{(1)})\rangle}\langle p,\beta(h_{(5)}l_{(4)})m_{(1)}\alpha S^{-1}(h_{(3)}l_{(2)})\rangle\\
&\underline{\langle q_{(2)},\beta(h_{(6)}l_{(5)})m_{(2)}\alpha S^{-1}(h_{(2)}l_{(1)})\rangle}(h_{(4)}l_{(3)})\cdot m_{(0)}\\
=\ &\langle q,\beta(l_{(5)})m_{(2)}\alpha S^{-1}(l_{(1)})\rangle\langle p,\beta(h_{(3)}l_{(4)})m_{(1)}\alpha S^{-1}(h_{(1)}l_{(2)})\rangle(h_{(2)}l_{(3)})\cdot m_{(0)}.
\end{aligned}
$$

Next, we get the main result of this section, generalizing the conclusion in [2] and giving an answer to the question introduced in Section 1.

Theorem 3.3 For a coFrobenius Hopf algebra H,

$$ {}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}\cong {}_H\mathcal{YD}^H(\alpha,\beta). \tag{3.2}$$

Proof The correspondence easily follows from Proposition 3.1 and 3.2. Let $f:M\to N$ be a morphism in ${}_H\mathcal{YD}^H(\alpha,\beta)$, i.e., f is a module and comodule map. Then in ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$,

$$
\begin{aligned}
(f\otimes\iota)\rho(m) &= f\big((S^{-1}(\varphi_{(1)})\bowtie 1)\cdot m\big)\otimes t(\cdot\varphi_{(2)})\\
\rho(f(m)) &= (S^{-1}(\varphi_{(1)})\bowtie 1)\cdot f(m)\otimes t(\cdot\varphi_{(2)}).
\end{aligned}
$$

$(f\otimes\iota)\rho(m)=\rho(f(m))$ implies f is a $\widehat{H}\bowtie 1$-module map, and so a ${}_H\mathcal{YD}^H(\alpha,\beta)$-module map. We define a functor $F_{(\alpha,\beta)}: {}_H\mathcal{YD}^H(\alpha,\beta)\longrightarrow {}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$ as follows,

$$F_{(\alpha,\beta)}(M)=M,\qquad \text{and}\qquad F_{(\alpha,\beta)}(f)=f.$$

Conversely, if $f : M \to N$ be a morphism in ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$, then

$$\begin{aligned}(f\otimes\iota)\rho(m) &= f\big((S^{-1}(\varphi_{(1)})\bowtie 1)\cdot m\big)\otimes t(\cdot\varphi_{(2)})\\ &= (S^{-1}(\varphi_{(1)})\bowtie 1)\cdot f(m)\otimes t(\cdot\varphi_{(2)})\\ &= \rho(f(m)).\end{aligned}$$

This shows that f is a H-comodule map. Then we similarly define a functor $G_{(\alpha,\beta)}$: ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M} \longrightarrow {}_H\mathcal{YD}^H(\alpha,\beta)$ by $G_{(\alpha,\beta)}(M) = M, \text{and} G_{(\alpha,\beta)}(f) = f$.

From above, F and G preserve the morphisms from each other. Also $F_{(\alpha,\beta)}G_{(\alpha,\beta)} = 1_{(\alpha,\beta)}$ and $G_{(\alpha,\beta)}F_{(\alpha,\beta)} = 1_{(\alpha,\beta)}$. We have established the equivalence between ${}_H\mathcal{YD}^H(\alpha,\beta)$ and ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$.

Corollary 3.4 Let H be a coFrobenius Hopf algebra and $\alpha,\beta,\gamma\in Aut(H)$, then

$${}_{\widehat{H}\bowtie H(\alpha\beta,\gamma\beta)}\mathcal{M} \cong {}_{\widehat{H}\bowtie H(\alpha,\gamma)}\mathcal{M}.$$

Proof It follows straightforwardly from the fact ${}_H\mathcal{YD}^H(\alpha\beta,\gamma\beta) \cong {}_H\mathcal{YD}^H(\alpha,\gamma)$.

Example 3.5 (1) When $\alpha=\beta=\iota$, then $\widehat{H}\bowtie H(\iota,\iota) = D(H)$ the quantum double of a coFrobenius Hopf algebra. Then we have the following result, which is the main result in [5], i.e., for a coFrobenius Hopf algebra H,

$${}_H\mathcal{YD}^H \cong {}_{\widehat{H}\bowtie H}\mathcal{M}.$$

(2) When $\alpha = S^2$ and $\beta=\iota$, ${}_H\mathcal{YD}^H(S^2,\iota)$ is exactly the category of anti-Yetter-Drinfeld modules defined in [9]. Then we have

$${}_H\mathcal{YD}^H(S^2,\iota) \cong {}_{\widehat{H}\bowtie H(S^2,\iota)}\mathcal{M}.$$

Let $\mathcal{YD}(H)$ be the disjoint union of ${}_H\mathcal{YD}^H(\alpha,\beta)$ for every $(\alpha,\beta)\in G$. Then following Section 3 in [2] or [4, 10] (H is a special multiplier Hopf algebra), we have that ${}_H\mathcal{YD}^H$ is a braided T-category with the structures as follows

- Tensor product: if $V\in {}_H\mathcal{YD}^H(\alpha,\beta)$ and $W\in {}_H\mathcal{YD}^H(\gamma,\delta)$ with $\alpha,\beta,\gamma,\delta\in Aut(H)$, then $V\otimes W\in {}_H\mathcal{YD}^H(\alpha\gamma,\delta\gamma^{-1}\beta\gamma)$, with the structures as follows:

$$h\cdot(v\otimes w) = \gamma(h_{(1)})\cdot v\otimes\gamma^{-1}\beta\gamma(h_{(2)})\cdot w,$$
$$v\otimes w\mapsto (v\otimes w)_{(0)}\otimes(v\otimes w)_{(1)} = (v_{(0)}\otimes w_{(0)})\otimes w_{(1)}v_{(1)}$$

  for all $v\in V, w\in W$.

- Crossed functor: Let $W\in {}_H\mathcal{YD}^H(\gamma,\delta)$, we define $\xi_{(\alpha,\beta)}(W) = {}^{(\alpha,\beta)}W = W$ as vector space, with structures: for all $a,a'\in A$ and $w\in W$

$$a\rightharpoonup w = \gamma^{-1}\beta\gamma\alpha^{-1}(a)\cdot w,$$
$$w\mapsto w_{<0>}\otimes w_{<1>} = w_{(0)}\otimes\alpha\beta^{-1}(w_{(1)}).$$

  Then ${}^{(\alpha,\beta)}W \in {}_H\mathcal{YD}^H((\alpha,\beta)\#(\gamma,\delta)\#(\alpha,\beta)^{-1}) = {}_H\mathcal{YD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})$. The functor $\xi_{(\alpha,\beta)}$ acts as identity on morphisms.

- Braiding: If $V \in {}_H\mathcal{YD}^H(\alpha,\beta)$, and $W \in {}_H\mathcal{YD}^H(\gamma,\delta)$. Taking ${}^VW = {}^{(\alpha,\beta)}W$, we define a map $C_{V,W} : V \otimes W \longrightarrow {}^VW \otimes V$ by

$$C_{(\alpha,\beta),(\gamma,\delta)}(v \otimes w) = w_{(0)} \otimes \beta^{-1}(w_{(1)}) \cdot v$$

for all $v \in V$ and $w \in W$.

Following from Theorem 3.3, we obtain the following result, generalizing Theorem 3.10 in [2].

Theorem 3.6 For a coFrobenius Hopf algebra H and its G-cograded multiplier Hopf algebra $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha,\beta)$, $Rep(\mathcal{A})$ and $\mathcal{YD}(H)$ are isomorphic as braided T-categories over G.

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无限维余Frobenius Hopf代数对角交叉积的表示范畴

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摘要: 本文研究了无限维余Frobenius Hopf代数对角交叉积表示范畴刻画的问题. 利用乘子Hopf代数以及同调代数理论中的方法, 获得了无限维余Frobenius Hopf代数对角交叉积的表示范畴与广义Yetter-Drinfeld范畴同构的结果, 推广了Panaite等人在有限维Hopf代数中的结果.

关键词: 余Frobenius Hopf代数; 对角交叉积; Yetter-Drinfel'd模

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