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STRONGLY QUASI-NIL CLEAN 2×2 MATRICES OVER COMMUTATIVE LOCAL RINGS

HU Die, YIN Xiao-bin, SHA Ling-yu

(*Department of Mathematics, Anhui Normal University, Wuhu 241002, China*)

Abstract: In this paper, we determine when a 2×2 matrix over a commutative local ring is strongly quasi-nil clean. We provide an effective answer to this question by using the characteristic equation of the matrix. This conclusion enrichs and generalizes the theory of strongly quasi-nil clean rings.

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1 Introduction

Throughout this paper, rings are associative with identity. Let R be a ring. The set of all units and the Jacobson radical of R are denoted by $U(R)$ and $J(R)$, respectively. The symbol $M_n(R)$ stands for the $n \times n$ matrix ring over R whose identity element is written as I_n. The commutant and double commutant of $a \in R$ are denoted respectively by $\mathrm{comm}(a) = \{x \in R : ax = xa\}$ and $\mathrm{comm}^2(a) = \{x \in R : yx = xy \text{ for any } y \in \mathrm{comm}(a)\}$.

Rings in which every element is the sum of certain special elements were frequently studied in ring theory. A ring R is called strongly clean if all of its elements are a sum of an idempotent and a unit which commutes (see [1]). Following Diesl [2], a ring R is strongly nil clean if for every element a of R, there exists $e^2 = e \in R$ and a nilpotent $b \in R$ such that $a = e + b$ and $e \in \mathrm{comm}(a)$. Chen et al. [3] introduced the concept of a perfect J-clean ring. A ring R is perfectly J-clean if for each $a \in R$, there exists $e^2 = e \in \mathrm{comm}^2(a)$ such that $a - e \in J(R)$. Recall that $a \in R$ is quasi-nilpotent if $1 - ax \in U(R)$ for any $x \in \mathrm{comm}(a)$, and the set of all quasi-nilpotent elements of R is denoted by R^{qnil}. It is clear that all nilpotent elements and $J(R)$ are contained in R^{qnil}. As a generalization of strongly nil clean rings and perfectly J-clean rings, Cui and Yin [4] introduced the notion of strongly quasi-nil clean rings. A ring R is strongly quasi-nil clean if for every $a \in R$ there exists $e^2 = e \in \mathrm{comm}^2(a)$ such that $a - e \in R^{\mathrm{qnil}}$, it was shown that $M_n(R)$ was not strongly quasi-nil clean for any ring R and any $n \geq 2$.

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Biography: Hu Die (1994 –), female, born at Fuyang Anhui, postgraduate, major in ring theory.

Corresponding author: Yin Xiaobin

In this paper, we determine when a single matrix $A \in M_2(R)$ is strongly quasi-nil clean where R is a commutative local ring. It is shown that over a commutative local ring R, a matrix $A \in M_2(R)$ is strongly quasi-nil clean if and only if either $A \in (M_2(R))^{\rm qnil}$ or $I_2 - A \in (M_2(R))^{\rm qnil}$ or the equation $x^2 - ({\rm tr}A)x + {\rm det}A = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

2 Main Results

We begin with some primary facts, which will be used freely.

Lemma 2.1 [4, Proposition2.1] Let R be a ring and $u \in U(R)$. Then

(1) $a \in R^{\rm qnil}$ is strongly quasi-nil clean.

(2) $a \in R$ is strongly quasi-nil clean if and only if so is $u^{-1}au$.

(3) $a \in U(R)$ is strongly quasi-nil clean if and only if $1 - a \in R^{\rm qnil}$.

According to [5], an element $a \in R$ is called quasipolar if there exists $p^2 = p \in R$ such that $p \in {\rm comm}^2(a)$, $a + p \in U(R)$, and $ap \in R^{\rm qnil}$. Any idempotent p satisfying the above conditions is called a spectral idempotent of a, which is uniquely determined by the element of a, and is denoted by a^π. By [4, Proposition2.3], every strongly quasi-nil clean element is quasipolar. In addition, if $a = e + q$ is a strongly quasi-nil expression of $a \in R$, then a is quasipolar with $1 - e$ the spectral idempotent.

Corollary 2.2 Let R be a ring and $A \in M_2(R)$. Then A is quasi-nilpotent if and only if A is quasipolar and $A^\pi = I_2$.

Let R be a commutative ring. For a square matrix $A \in M_2(R)$, the notation ${\rm det}A$ and ${\rm tr}A$ denoted the determinant and the trace of A, respectively.

Lemma 2.3 [6, Lemma4.1] Let R be a commutative local ring. The following statements are equivalent for $A \in M_2(R)$.

(1) $A \in (M_2(R))^{\rm qnil}$.

(2) ${\rm det}A \in J(R)$ and ${\rm tr}A \in J(R)$.

(3) $A^2 \in J(M_2(R))$.

Lemma 2.4 [7, Lemma4.4] Let R be a commutative local ring and $E \in M_2(R)$. Then $E^2 = E$ if and only if E is one of the cases of the following statements:

(1) $E = 0$ or $E = I_2$.

(2) $E = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ with $bc = a - a^2$.

It is easy to see that A is a unit if and only if ${\rm det}A \in U(R)$, A is quasinilpotent if and only if ${\rm det}A \in J(R)$ and ${\rm tr}A \in J(R)$. Naturally we may consider the case of ${\rm det}A \in J(R)$ and ${\rm tr}A \in U(R)$.

Proposition 2.5 Let R be a commutative local ring and $A \in M_2(R)$ satisfies that $A \notin (M_2(R))^{\rm qnil}$ and ${\rm det}A \in J(R)$. The following statements are equivalent.

(1) A is strongly quasi-nil clean.

(2) A is similar to a matrix $\begin{pmatrix} j & 0 \\ 0 & u \end{pmatrix}$ with $j \in J(R), u \in 1 + J(R)$.

(3) The characteristic equation of A, $x^2 - {\rm tr}Ax + {\rm det}A = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Proof (1)$\Rightarrow$(2) Assume that A is strongly quasi-nil clean. It is clear that $\mathrm{tr}A \in U(R)$. By Lemma 2.4, we may let $E \doteq I_2 - A^{\pi} = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right) \in M_2(R)$, where $bc = a - a^2$. As R is local, either $a \in U(R)$ or $1 - a \in U(R)$. Let $U_1 = \left(\begin{smallmatrix} b & a \\ -a & c \end{smallmatrix}\right)$ and $U_2 = \left(\begin{smallmatrix} a-1 & b \\ c & 1-a \end{smallmatrix}\right)$. Since $\mathrm{det}U_1 = a, \mathrm{det}U_2 = a - 1$, we have at least one of U_1 and U_2 which is invertible. We may as well suppose that U_1 is invertible. We write $B = (b_{ij}) = U_1^{-1}AU_1$. Then B is strongly quasi-nil clean with $E^{'} \doteq I_2 - B^{\pi} = U_1^{-1}EU_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. As $E^{'} \in \mathrm{comm}(B)$, we have $b_{12} = b_{21} = 0$, which implies that B is diagonal. Since $\mathrm{det}B = \mathrm{det}A \in J(R)$ and $\mathrm{tr}B = \mathrm{tr}U_1^{-1}AU_1 = \mathrm{tr}A \in U(R)$, so $b_{11} \in J(R), b_{22} \in U(R)$ or $b_{11} \in U(R), b_{22} \in J(R)$. We may as well suppose that it is the first case. Let $b_{11} = j \in J(R)$ and $b_{22} = u \in U(R)$. From Lemma 2.4, we have $E^{'} = \left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right)$, where $yz = x - x^2$. Then $BE^{'} = E^{'}B$ implies that $y = z = 0$, so $x = x^2$ is an idempotent. Since R is local, it follows that $x = 0$ or $x = 1$. If $x = 1$, we have $B - E^{'} = \left(\begin{smallmatrix} j-1 & 0 \\ 0 & u \end{smallmatrix}\right) \in U(M_2(R)) \bigcap (M_2(R))^{\mathrm{qnil}} = \emptyset$. Hence $x = 0$ and $B - E^{'} = \left(\begin{smallmatrix} j & 0 \\ 0 & u-1 \end{smallmatrix}\right) \in ((M_2(R)))^{\mathrm{qnil}}$. By Lemma 2.3, $(u-1)^2 \in J(R)$, thus $u - 1 \in J(R)$, and $B = \left(\begin{smallmatrix} j & 0 \\ 0 & u \end{smallmatrix}\right)$ where $j \in J(R), u \in 1 + J(R)$.

(2)$\Rightarrow$ (3) is clear.

(3)$\Rightarrow$ (1) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$ with $\mathrm{tr}A \in U(R)$ and $\mathrm{det}A \in J(R)$. Assume that λ_1 and λ_2 are two roots of the equation $x^2 - (\mathrm{tr}A)x + \mathrm{det}A = 0$, where $\lambda_1 \in J(R), \lambda_2 \in 1 + J(R)$. We note that $\mathrm{tr}A \in U(R)$. So we may let $a \in U(R)$. We write $V = \left(\begin{smallmatrix} b & a-\lambda_1 \\ \lambda_1 - a & c \end{smallmatrix}\right)$. Then $V \in U(M_2(R))$ because of $\mathrm{det}V = bc - (a - \lambda_1)(\lambda_1 - a) = a\mathrm{tr}A + (\lambda_1^2 - 2a\lambda_1 - \mathrm{det}A) \in U(R)$. Since $\lambda_1^2 - (\mathrm{tr}A)\lambda_1 + \mathrm{det}A = 0$, we have $V^{-1}AV = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right) \doteq B$. We assume that $\lambda_2 = 1 + j$ for some $j \in J(R)$. Writing $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $E^2 = E$ and $B = E + \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & j \end{smallmatrix}\right) \in ((M_2(R)))^{\mathrm{qnil}}$. For any $C = (c_{ij}) \in M_2(R)$, if $CB = BC$, then $c_{12} = c_{21} = 0$, so $EC = CE$. Thus $E \in \mathrm{comm}^2(B)$. Therefore B is strongly quasi-nil clean. By Lemma 2.1, A is strongly quasi-nil clean.

As a consequence of the above results, we obtain the following conclusion about 2×2 matrixes over commutative local rings to be strongly quasi-nil clean.

Theorem 2.6 Let R be a commutative local ring and let $A \in M_2(R)$. Then A is strongly quasi-nil clean if and only if one of the following statements holds:

(1) $A \in (M_2(R))^{\mathrm{qnil}}$ or $I_2 - A \in (M_2(R))^{\mathrm{qnil}}$.

(2) The equation $x^2 - (\mathrm{tr}A)x + \mathrm{det}A = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Combining Theorem 2.6 with Lemma 2.3, we have the following corollary:

Corollary 2.7 Let R be a commutative local ring. The following are equivalent for $A \in M_2(R)$:

(1) $A \in M_2(R)$ is strongly quasi-nil clean.

(2) One of the following statements holds:

(i) $A^2 \in J(M_2(R))$ or $(I_2 - A)^2 \in J(M_2(R))$.

(ii) The equation $x^2 - (\mathrm{tr}A)x + \mathrm{det}A = 0$ has a root in $J(R)$ and a root in $1 + J(R)$

Corollary 2.8 Let R be a commutative local ring and $A \in M_2(R)$. If A is strongly quasi-nil clean, then one of the following statements holds:

(1) A is quasipolar and $A^{\pi} = 0$ or $A^{\pi} = I_2$.

(2) A is quasipolar and A^π is similar to a matrix $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

By Theorem 2.6, we can figure out all strongly quasi-nil clean elements of a ring $M_2(\mathbb{Z}_2)$.

Example 2.9 Let $R = M_2(\mathbb{Z}_2)$. Then all elements of R are strongly quasi-nil except for $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$.

For a ring R, we use $T_n(R)$ to denote the $n \times n$ upper triangular matrix ring over R.

Corollary 2.10 Let R be a commutative local domain with $J(R) = 2R$ and $R/J(R) \cong \mathbb{Z}_2$. The following are equivalent for $A \in M_2(R)$:

(1) A is strongly quasi-nil clean.

(2) $A \in (M_2(R))^{\text{qnil}}$ or $I_2 - A \in (M_2(R))^{\text{qnil}}$ or there exists some $u \in U(R)$ such that $(\text{tr}A)^2 - 4\text{det}A = u^2$.

Proof If A and $I_2 - A$ are all not in $(M_2(R))^{\text{qnil}}$, then $\text{tr}A \in U(R)$ and $\text{det}A \in J(R)$. Since $R/J(R) \cong \mathbb{Z}_2$, $U(R) = -1 + J(R)$. By [8, Theorem 2.13], $x^2 - (\text{tr}A)x + \text{det}A = 0$ is solvable in R if and only if $(\text{tr}A)^2 - 4\text{det}A = u^2$ for some $u \in U(R)$. By Theorem 2.6, we complete the proof.

It follows from Corollary 2.7.

Corollary 2.11 Let R be a commutative local ring and $A = \left(\begin{smallmatrix}a_{11}&a_{12}\\0&a_{22}\end{smallmatrix}\right) \in M_2(R)$. The following statements are equivalent:

(1) A is strongly quasi-nil clean.

(2) One of the following holds:

(i) $a_{11}, a_{22} \in 1 + J(R)$ or $a_{11}, a_{22} \in J(R)$.

(ii) One of a_{11} and a_{22} is in $J(R)$, and the other is in $1 + J(R)$.

Let R be a commutative local ring. By Corollary 2.11, the matrix $A = \left(\begin{smallmatrix}j_1&x\\0&j_2\end{smallmatrix}\right) \in M_2(R)$ where $j_1, j_2 \in J(R)$ and $x \in R$ is strongly quasi-nil clean.

For $a \in R$, we recall that an element $a \in R$ is said to be perfectly J-clean [3] if there exists $e^2 = e \in \text{comm}^2(a)$ such that $a - e \in J(R)$. A ring R is called perfectly J-clean if every element in R is perfectly J-clean. We use l_a and r_a to denote the abelian group endomorphisms of R given by left and right multiplication by a, respectively. Following [9], a local ring R is called bleached if $l_u - r_j$ and $l_j - r_u$ are both surjective for any $j \in J(R)$ and any $u \in U(R)$. Commutative local rings are well-known examples of bleached local rings. In [10, Example 4.4], the author demonstrated that over a bleached local ring R, $(T_2(R))^{\text{qnil}} = J(T_2(R))$.

Corollary 2.12 If R is a bleached local ring. Then $A \in T_2(R)$ is strongly quasi-nil clean if and only if A is perfectly J-clean.

proposition 2.13 Let R be a ring. The following statements are equivalent:

(1) R is local and strongly quasi-nil clean.

(2) R is perfectly J-clean with the only idempotents 0 and 1.

Proof (1) $\Rightarrow$ (2) Clearly, 0 and 1 are the only idempotents of R. Since R is local, R is quasipolar and $R^{\text{qnil}} = J(R)$. Whence R is perfectly J-clean.

(2) $\Rightarrow$ (1) Since R is perfectly J-clean, $R/J(R)$ is boolean by [3, Theorem 4.1]. It follows that $R/J(R) \cong \mathbb{Z}_2$ from the only trivial idempotent 0 and 1 of R. Thus R is local.

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交换局部环上的强拟诣零clean 2×2 矩阵

胡 蝶, 殷晓斌, 沙玲玉

(安徽师范大学数学与统计学院, 安徽 芜湖 241002)

摘 要: 本文研究了交换局部环上的2阶矩阵的强拟诣零clean性, 利用矩阵的特征方程的方法, 给出了交换局部环上的2阶矩阵是强拟诣零clean的具体判别方法, 所得结果丰富并推广了强拟诣零clean的研究.

关键词: 强拟诣零clean元; 矩阵; 局部环

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