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GRADIENT ESTIMATE FOR POSITIVE SOLUTIONS OF THE PME UNDER GEOMETRIC FLOW

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Abstract: In this paper, we derive a local gradient estimate of the Aronson-Bénilan type with Laplace operator and drifting Laplace operator for positive solutions of porous medium equations posed on Riemainnian manifolds with bounded symmetric tensor by using Li-Yau method. These results extend Zhu Xiao-bao's and Deng Yi-hua's results.

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1 Introduction

The porous medium equation(PME for short)

$$\partial_t u = \Delta u^m, \quad m > 1 \tag{1.1}$$

appears in the description of different natural phenomena, and its theory and properties depart strongly from the heat equation, $u_t = \Delta u$, it's most famous relative. There are kinds of physical applications where we can use this model, mainly to characterize process involving fluid flow, heat transfer or diffusion. For more knowledge, we recommend the book [1] to the reader.

Among typical nonlinear problems, the mathematical theory of PME is also based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form [2]

$$\sum_i \frac{\partial}{\partial x^i}(mu^{m-2}\frac{\partial u}{\partial x^i}) \geq \frac{\kappa}{t}, \quad \kappa := \frac{n}{n(m-1)+2} \tag{1.2}$$

which applies to all positive smooth solutions of (1.1) defined on the whole Euclidean space on the condition that $m > m_c := 1 - 2/n$. The theory of PME on manifolds is rare. In 2008, Lu, Ni, Vázquez and Villani studied the PME on manifolds [3]. They got the following local Aronson-Bénilan estimate.

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Biography: Zhao Helei (1994–), male, born at Wenzhou, Zhejiang, postgraduate, major in geometric flow.

Theorem 1 (see [3]) Let u be a positive solution to PME (1.1), $m > 1$ on the cylinder $Q := B(\mathcal{O}, R) \times [0, T]$. Let $v := mu^{m-1}/(m-1)$ be the pressure and $v_{\max}^{R,T} := \max_{B(\mathcal{O},R)\times[0,T]} v$.

Assume that Ric $\geq -(n-1)K^2$ on $B(\mathcal{O}, R)$ for some $K \geq 0$. Then, for any α>1, we have that on $Q' := B(\mathcal{O}, R/2) \times [0, T]$

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{t} \leq a\alpha^2 (\frac{1}{t} + \widetilde{C_3(\alpha)} K^2 v_{\max}^{R,T}) + a\alpha^2 \frac{v_{\max}^{R,T}}{R^2} (\widetilde{C_2(\alpha)} + \widetilde{C_1'(KR)}). \tag{1.3}$$

Here, $a = \frac{n(m-1)}{n(m-1)+2}$, $\widetilde{C_2(\alpha)}$, $\widetilde{C_3(\alpha)}$ and $\widetilde{C_1'(KR)}$ depend on m and n.

Next, we will discuss PME with drifting Laplacian operator. Smooth metric measure spaces carry a similar operator to the Laplace-Beltrami operator Δ, the f-Laplacian, which is also called drifting Laplacian or Witten-Laplacian, defined for a function u by $\Delta_f u = \Delta u - g(\nabla f, \nabla u) = \Delta u - \langle \nabla f, \nabla v \rangle$. The N-Barkry-Emery Ricci tensor is defined by $Ric_f^N = Ric + Hess f - \frac{1}{N} df \otimes df$. A natural question about smooth metric measure space is which of the results about the Ricci tensor and the Laplace-Beltrami operator can be extended to the N-Bakry-Emery Ricci tensor and f-Laplacian. For example, In [3], Lu et al. derived some gradient estimates for the PME equation on Riemannian manifolds with Ricci curvature bounded from below:

$$u_t = \Delta u^m, \tag{1.4}$$

where m>1. In [4], Huang and Li got a better result in [3]. In [5], Huang and Li researched the following porous medium type equation,

$$u_t = \Delta_f u^m, \tag{1.5}$$

on smooth metric measure space. Under the assumption that the N-dimensional Bakery-Emery Ricci curvature is bounded from below, Huang and Li obtained some gradient estimates that generalized the results in [3] and [5] .

In this paper, we will follow closely [3] and derive local gradient estimates for positive bounded solutions of PME on Riemannian manifolds under general geometric flow. The general geometric flow equation where h_{ij} is a second-order symmetric tensor is as follows:

$$\frac{\partial g_{ij}}{\partial t} = 2h_{ij}. \tag{1.6}$$

The idea is from the Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$, which was introduced by Hamilton [6] in 1982.

Then, we will get a similar result with drifting Laplacian operator on PME. Also, our idea comes from Huang and Ma in [7], who considered gradient estimate for the following parabolic equation

$$\frac{\partial u}{\partial t} = \Delta_f u + aulogu + bu,$$

on smooth metric measure spaces. Inspired by the research of harmonic function and positive solution to linear heat flow on Riemannian manifolds, this paper extends corresponding gradient estimate from a fixed Riemannian metric to the case that the metric evolves by a general geometric flow.

Our first result states the gradient estimate of the pressure function v.

Main Theorem 1 Let $g(t)$ be a solution to the general geometric flow on a Riemannian manifold $M^n (n \geq 2)$ for t in some time interval $[0,T]$. Let M be complete under the initial metric $g(0)$. Let u be a positive smooth solution to PME (1.1), $m > 1$ on the cylinder $Q := B(\mathcal{O}, R) \times [0,T]$. Note $v := \frac{m}{m-1} u^{m-1}$ is the pressure, and write $v_{\max}^{R,T} := \max\limits_{B(\mathcal{O},R)\times[0,T]} v$ and $v_{\min}^{R,T} := \min\limits_{B(\mathcal{O},R)\times[0,T]} v$.

Assume that $-(n-1)K_0^2 \leq Ric$, $-(n-1)K_1^2 \leq h \leq (n-1)K_2^2$, $|\nabla h| \leq K_3$ on $B(\mathcal{O}, R)$ for some $K_0, K_1, K_2, K_3 \geq 0$. Then, for any $\alpha > 1$, we have that on $Q' := B(\mathcal{O}, \frac{R}{2}) \times [0,T]$,

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq a\alpha^2 \Big(C_4 \frac{v_{\max}^{R,T}}{R^2} + C_2 + \frac{1}{t} \Big) + \alpha \sqrt{a(C_5 + C_6)} v_{\max}^{R,T} + \alpha \sqrt{a(C_7 + C_8)}. \tag{1.7}$$

C_i is a constant depending on $m,\ n,\ a,\ \alpha,\ K_0,\ K_1,\ K_2,\ K_3,\ K_4,\ R$.

Remark When $h = -2Ric$, (1.6) is the Ricci flow equation. In this case our results reduce to [8]. Note that for Ricci flow the assumption $|\nabla Ric| \leq K_3$ is not needed because of the contracted second Bianchi identity [[9], section 4].

As an application, we get the following result.

Corollary 1.1 v is the pressure, then for any $x_1,\ x_2 \in B(\mathcal{O}, R/6)$ and any $\alpha > 1$,

$$\begin{aligned} \frac{v(x_2,t_2)}{v(x_1,t_1)} \geq & (\frac{t_1}{t_2})^{a\alpha} \exp(-\frac{\alpha \int_{t_1}^{t_2} |\gamma'(s)|_s^2 ds}{4 v_{\min}^{\frac{R}{2},T}}) \times \exp\Big[(-a\alpha)(t_2 - t_1) \Big(C_2 + C_4 \frac{v_{\max}^{R,T}}{R^2} \Big) \\ & + \alpha \sqrt{a(C_5 + C_6)} v_{\max}^{R,T} + \alpha \sqrt{a(C_7 + C_8)} \Big], \end{aligned}$$

where $\gamma(s)$ is a smooth curve connected x_1 and x_2 with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$, $|\gamma'(s)|_s$ is the length of the vector $\gamma'(s)$ at time s.

We extend the Laplacian operator to the drifting Laplacian operator, and we can get similar gradient estimate.

Main Theorem 2 Let (M^n, g, dv) be a smooth metric measure space. Suppose that u is a positive solution to (1.5). If $|\nabla f| \leq c_0$, $|\nabla^2 v| \leq c_1$, $Ric_f^N(B_p(2R)) \geq -k_0$, $-k_1 \leq h \leq k_2$, $|\nabla h| \leq k_3$. Here, $k_0, k_1, k_2, k_3 \geq 0$, then on the ball $B_p(R)$ we have

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{\alpha c_4}{c_3} + \alpha \sqrt{\frac{c_2}{c_3}},$$

where c_2, c_3, c_4 are constants depending on $a,\ c_0,\ c_1,\ \alpha,\ m,\ n,\ H,\ R,\ T$, and

$$H = \sup_{B_p(2R)\times[0,T]} (m-1)v, \quad \alpha(t) = e^{2Hk_0 t}.$$

Corollary 2.1 Let (M^n, g, dv) be a smooth metric measure space with $|\nabla f| \leq c_0$, $|\nabla^2 v| \leq c_1$, $Ric_f^N \geq -k_0$, $-k_1 \leq h \leq k_2$, $|\nabla h| \leq k_3$. Here, $k_0, k_1, k_2, k_3 \geq 0$ suppose that (M^n, g) is a complete non-compact Riemannian manifold and u is a positive solution to (1.5), then

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{\alpha}{c_3 T} + \alpha \sqrt{\frac{c_2}{c_3}},$$

where c_i is a constant depending on $a,\ \alpha,\ m,\ n,\ S,\ T$, and $S = \sup\limits_{M^n\times[0,T]}(m-1)v, \alpha(t) = e^{2Sk_0t}$.

Remark When g is independent of t, our results reduce to [10].

2 Proof of Main Theorems

Note that the pressure $v := \frac{m}{(m-1)}u^{m-1}$ satisfies

$$\partial_t v = (m-1)v\Delta v + |\nabla v|^2. \tag{2.1}$$

Assuming that $u > 0$, we introduce the quantities $y = \frac{|\nabla v|^2}{v}$, $z = \frac{v_t}{v}$ and the differential operator: $L := \partial_t - (m-1)v\Delta$. Let $F_\alpha := \alpha z - y$. From equation (2.1) we know

$$F_\alpha = (m-1)\Delta v + (\alpha-1)\frac{v_t}{v} = \alpha(m-1)\Delta v + (\alpha-1)\frac{|\nabla v|^2}{v}. \tag{2.2}$$

First, let us calculate a formula for $L(F_\alpha)$.

Lemma 2.1

$$\frac{\partial}{\partial t}\Delta v = \Delta\frac{\partial}{\partial t}v - 2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle.$$

Proof Recall that $\frac{\partial}{\partial t}\Gamma_{ij}^k = g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})$. Thus,

$$\begin{aligned}
\frac{\partial}{\partial t}\Delta v &= \frac{\partial}{\partial t}\{g^{ij}(\frac{\partial^2 v}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial v}{\partial x_k})\}\\
&= -2h^{ij}(\frac{\partial^2 v}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial v}{\partial x_k}) + \Delta\frac{\partial}{\partial t}v - g^{ij}(\frac{\partial}{\partial t}\Gamma_{ij}^k)\frac{\partial v}{\partial x_k}\\
&= -2\langle h, \nabla^2 v\rangle + \Delta\frac{\partial v}{\partial t} - g^{ij}g^{kl}\{\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\}\nabla_k v\\
&= \Delta\frac{\partial v}{\partial t} - 2\langle h, \nabla^2 v\rangle - 2g^{kl}\{g^{ij}\nabla_i h_{jl} - \frac{1}{2}\nabla_l(tr_g h)\}\nabla_k v\\
&= \Delta\frac{\partial v}{\partial t} - 2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle.
\end{aligned}$$

Lemma 2.2 Let $g(t)$ be a solution to the geometric flow on a Riemannian manifold $M^n(n \geq 2)$ for t in some time interval $[0, T]$. Let M be complete under the initial metric $g(0)$. Let u be a positive smooth solution to (1.1) on manifold $(M^n, g(t))$ for some $m > 0$, and let $v := \frac{m}{m-1}u^{m-1}$ be the pressure. Then we have

$$L(v_t) = 2\langle\nabla v, \nabla v_t\rangle + F_1 v_t - 2h^{ij}v_iv_j + (m-1)v(-2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle), \tag{2.3}$$

$$L(|\nabla v|^2) = 2|\nabla v|^2 F_1 + 2\langle\nabla(|\nabla v|^2), \nabla v\rangle - 2(m-1)vv_{ij}^2 - 2(m-1)vR_{ij}v_iv_j - 2h^{ij}v_iv_j. \tag{2.4}$$

Proof Calculate directly by using the Lemma 2.1.

Proposition 2.1 Let u and v be as in Lemma 2.2. Then

$$\begin{aligned}
L(F_\alpha) =& 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j + 2m\langle\nabla F_\alpha, \nabla v\rangle + (\alpha-1)(\frac{v_t}{v})^2\\
&+ F_1^2 - 2\frac{\alpha-1}{v}h^{ij}v_iv_j + \alpha(m-1)(-2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle).
\end{aligned} \tag{2.5}$$

Proof The following formula is helpful in the calculation:

$$L(\frac{f}{g}) = \frac{1}{g}L(f) - \frac{f}{g^2}L(g) + 2(m-1)v\langle\nabla(\frac{f}{g}), \nabla \log g\rangle. \tag{2.6}$$

Using (2.6) and Lemma 2.2 we have

$$\begin{aligned} L(\frac{|\nabla v|^2}{v}) =& \frac{1}{v}(2|\nabla v|^2 F_1 + 2\langle\nabla(|\nabla v|^2), \nabla v\rangle) - 2(m-1)v_{ij}^2 - 2(m-1)R_{ij}v_iv_j \\ & - \frac{2}{v}h^{ij}v_iv_j - \frac{|\nabla v|^4}{v^2} + 2(m-1)v\langle\nabla(\frac{|\nabla v|^2}{v}), \nabla \log v\rangle), \end{aligned} \tag{2.7}$$

$$\begin{aligned} L(\frac{v_t}{v}) =& \frac{1}{v}\Big(2\langle\nabla v, \nabla v_t\rangle + F_1 v_t - 2h^{ij}v_iv_j\Big) - 2(m-1)\langle h, \nabla^2 v\rangle \\ & - 2(m-1)\langle \mathrm{div} h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle - \frac{v_t}{v}\frac{|\nabla v|^2}{v} + 2(m-1)v\langle\nabla(\frac{v_t}{v}), \nabla \log v\rangle. \end{aligned} \tag{2.8}$$

These give that

$$\begin{aligned} L(F_\alpha) =& 2(m-1)v\langle\nabla F_\alpha, \nabla \log v\rangle + 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j + \alpha\frac{v_t}{v}F_1 \\ & - 2\frac{|\nabla v|^2}{v}F_1 - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} + \frac{|\nabla v|^4}{v^2} + \frac{2}{v}\langle\nabla v, \alpha\nabla v_t - \nabla(|\nabla v|^2)\rangle \\ & - 2\frac{\alpha-1}{v}h^{ij}v_iv_j + \alpha(m-1)(-2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div} h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle). \end{aligned} \tag{2.9}$$

Using the formula

$$\langle\nabla v, \nabla(vF_\alpha)\rangle = v\langle\nabla v, \nabla F_\alpha\rangle + F_\alpha|\nabla v|^2,$$

we have

$$\frac{2}{v}\langle\nabla v, \alpha\nabla v_t - \nabla(|\nabla v|^2)\rangle = 2\langle\nabla v, \nabla F_\alpha\rangle + 2F_\alpha\frac{|\nabla v|^2}{v}.$$

Hence we obtain

$$\begin{aligned} L(F_\alpha) =& 2mv\langle\nabla F_\alpha, \nabla \log v\rangle + 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j + \alpha\frac{v_t}{v}F_1 \\ & - 2\frac{|\nabla v|^2}{v}F_1 - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} + \frac{|\nabla v|^4}{v^2} + 2F_\alpha\frac{|\nabla v|^2}{v} \\ & - 2\frac{\alpha-1}{v}h^{ij}v_iv_j + \alpha(m-1)(-2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div} h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle). \end{aligned}$$

Note the fourth to eighth terms in the above equation, they can be rewritten as

$$\alpha z(z-y) - 2y(z-y) - \alpha zy + y^2 + 2(\alpha z - y)y = (\alpha-1)z^2 + (z-y)^2.$$

This completes the proof.

Then, we prove a local estimate for PME on complete manifolds under general geometric flow. We use the technique of Li and Yau [11] and some calculation of Lu, Ni, Vázquez and Villani in [3]. Denote by $B(\mathcal{O}, R)$ the ball of radius $R > 0$ and centered $\mathcal{O}$ in $(M^n, g(t))$, and denote by $r(x,t)$ the distance function from $\mathcal{O}$ to x with metric $g(t)$.

Proof of the Main Theorem 1

(i) Since bounded h tensor implies that $g(t)$ is uniform equivalence to the initial metric $g(0)$[[12], Corollary 6.11], that is $e^{-2K_1T}g(0) \le g(t) \le e^{2K_2T}g(0)$. By definition, we know that $(M, g(t))$ is also complete for $t \in [0,T]$. Inspired by the choice of cutoff function in the proof of Theorem3.1 in [10], we let $\eta(x,t) := \theta(r(x,t)/R)$ be cutoff function, where $\theta(s)$ is a smooth monotone function which satisfies $\theta(s) \equiv 1$ for $0 \le s \le 1/2$, $\theta(s) \equiv 0$ for $s \ge 1$, and $(\theta')^2/\theta \le 40$, and $\theta'' \ge -40\theta \ge -40$. On $B(\mathcal{O}, R)$, using the Laplacian comparison theorem, one can easily get

$$\frac{|\nabla\eta|^2}{\eta} \le \frac{40}{R^2},$$

and

$$\Delta\eta \ge -\frac{40(n-1)((1+K_0R)+1)}{R^2}, \quad \text{if} \quad Ric \ge -(n-1)K_0^2. \tag{2.10}$$

(ii) After the preparation in (i), now we apply L to $t\eta(-F_\alpha)$ and use the maximum principle to obtain our estimates. If $t\eta(-F_\alpha) \le 0$ on Q, then **the main theorem 1** is trivial. So we assume $\max_{(x,t)\in Q} t\eta(-F_\alpha) > 0$. Suppose $t\eta(-F_\alpha)$ achieves its maximum at (x_0, t_0). Then we have $t_0 > 0$ and

$$\eta\nabla F_\alpha = -\nabla\eta F_\alpha, \quad L(t\eta(-F_\alpha)) \ge 0,$$

at (x_0, t_0). From now on, all calculations are at (x_0, t_0).

By the evolution formula of geodesic length under geometric flow(see [13]), we calculate

$$\begin{aligned} \eta_t(-F_\alpha) &= \theta'(\frac{r}{R})\frac{1}{R}\frac{dr}{dt}(-F_\alpha) = -\theta'(\frac{r}{R})\frac{1}{R}\int_{\gamma_{t_0}} -h(S,S)ds(-F_\alpha) \le -\theta'(\frac{r}{R})\frac{1}{R}(n-1)K_1^2 r(-F_\alpha) \\ &\le -\theta'(\frac{r}{R})(n-1)K_1^2(-F_\alpha) \le \sqrt{40}(n-1)K_1^2(-F_\alpha), \end{aligned} \tag{2.11}$$

where γ_{t_0} is the geodesic connecting x and $\mathcal{O}$ under the metric $g(t_0)$, S is the unit tangent vector to γ_{t_0} and ds is the element of arc length.

Denote $C_1 := 40((n-1)(1+K_0R)+1)$, $C_2 := \sqrt{40}(n-1)K_1^2$, $\widetilde{y} := \eta y = \eta|\nabla v|^2/v$ and $\widetilde{z} := \eta z = \eta\frac{v_t}{v}$. Combining (2.5) with the above estimates of η, we have

$$\begin{aligned} 0 \le& \eta L(t\eta(-F_\alpha)) \\ =& -t\eta^2 L(F_\alpha) + t\eta(-F_\alpha)L(\eta) + 2(m-1)t\eta v\langle\nabla\eta, \nabla F_\alpha\rangle + \eta^2(-F_\alpha)L(t) \\ =& -t\eta^2\Big(2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j + 2m\langle\nabla F_\alpha, \nabla v\rangle) + (\alpha-1)(\frac{v_t}{v})^2 \\ &+ F_1^2 - \frac{2(\alpha-1)}{v}h^{ij}v_iv_j + \alpha(m-1)(-2\langle h, \nabla^2 v\rangle - 2\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle)\Big) \\ &+ t\eta\eta_t(-F_\alpha) - (m-1)t\eta(-F_\alpha)v\Delta\eta + 2(m-1)t\eta v\langle\nabla\eta, -\frac{\nabla\eta}{\eta}F_\alpha\rangle + \eta^2(-F_\alpha). \end{aligned}$$

Here, we have used Proposition 2.1. Using the following inequality, (2.10) and (2.11)

$$|\mathrm{div}h - \frac{1}{2}\nabla(tr_g h)| = |g^{ij}\nabla_i h_{jl} - \frac{1}{2}g^{ij}\nabla_l h_{ij}| \le \frac{3}{2}|g||\nabla h| \le \frac{3}{2}\sqrt{n}K_3,$$

we have

$$\begin{aligned}\eta L(t\eta(-F_\alpha)) \leq & - t\eta^2\Big(2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j\Big) + 2mt\eta^2\langle\nabla(-F_\alpha),\nabla v\rangle \\ & - (\alpha-1)t\eta^2z^2 - t\eta^2F_1 + t\eta^2(2\frac{\alpha-1}{v}h^{ij}v_iv_j + 2(m-1)\alpha h^{ij}v_{ij}) \\ & + 3t\eta^2\alpha(m-1)\sqrt{n}K_3|\nabla v| + C_2t\eta(-F_\alpha) + \frac{C_1}{R^2}(m-1)tv\eta(-F_\alpha) \\ & + 2t(m-1)v\eta\frac{|\nabla\eta|^2}{\eta}(-F_\alpha) + \eta^2(-F_\alpha).\end{aligned}$$

Using Young's inequality,

$$2\alpha(m-1)h^{ij}v_{ij} = \sqrt{2(m-1)}v_{ij} * \sqrt{2(m-1)}\alpha h^{ij} \leq (m-1)v_{ij}^2 + \alpha^2(m-1)|h|^2,$$

and attention our assumption of h,

$$-(n-1)K_1^2 \leq h \leq (n-1)K_2^2, \quad |h|^2 \leq ((n-1)(K_1^2+K_2^2))^2.$$

On the other hand,

$$v_{ij}^2 \geq \frac{(\Delta v)^2}{n}, \quad (m-1)\Delta v = z - y = F_1.$$

We get

$$\begin{aligned}\eta L(t\eta(-F_\alpha)) \leq & - \frac{n(m-1)+1}{n(m-1)}t(\widetilde{y}-\widetilde{z})^2 + 2(n-1)(m-1)K_0^2t\widetilde{y}\eta v \\ & + 2mt(\widetilde{y}-\alpha\widetilde{z})|\nabla\eta||\nabla v| - (\alpha-1)t\widetilde{z}^2 \\ & + 2(n-1)(\alpha-1)t\eta\widetilde{y}K_2^2 + t\eta^2\alpha^2(m-1)((n-1)(K_1^2+K_2^2))^2 \\ & + (\widetilde{y}-\alpha\widetilde{z})((m-1)\frac{80+C_1}{R^2}tv + 1 + C_2t) + 3\alpha(m-1)\sqrt{n}t\eta^2K_3|\nabla v|.\end{aligned}$$

Now write $C_3 := 80 + C_1$ and

$$(\widetilde{y}-\widetilde{z})^2 = \frac{1}{\alpha^2}(\widetilde{y}-\alpha\widetilde{z})^2 + 2\frac{\alpha-1}{\alpha^2}(\widetilde{y}-\alpha\widetilde{z})\widetilde{y} + (\frac{\alpha-1}{\alpha})^2\widetilde{y}^2.$$

Also note that

$$\begin{aligned} 2mt(\widetilde{y}-\alpha\widetilde{z})|\nabla\eta||\nabla v| & \leq \frac{40}{R}mt(\widetilde{y}-\alpha\widetilde{z})\widetilde{y}^{1/2}v^{1/2}, \\ 3\alpha(m-1)\sqrt{n}t\eta^2K_3|\nabla v| & \leq 3K_3\sqrt{n}\alpha(m-1)tv^{1/2}\widetilde{y}^{1/2} \leq \frac{9}{2}nK_3^2\alpha^2(m-1)^2t + \frac{1}{2}tv\widetilde{y},\end{aligned}$$

and write $a := \frac{n(m-1)}{1+n(m-1)}$, we get

$$\begin{aligned} 0 \leq & -\frac{t}{a\alpha^2}(\widetilde{y}-\alpha\widetilde{z})^2 + t(\widetilde{y}-\alpha\widetilde{z})(-\frac{2(\alpha-1)}{a\alpha^2}\widetilde{y} + \frac{40m}{R}\widetilde{y}^{\frac{1}{2}}v^{\frac{1}{2}} + C_2 + (m-1)\frac{C_3}{R^2}v + \frac{1}{t}) \\ & + 2(n-1)(m-1)tK_0^2\widetilde{y}v\eta + 2(n-1)(\alpha-1)K_2^2\widetilde{y}\eta t - (\alpha-1)t\widetilde{z}^2 \qquad (2.12) \\ & + t\eta^2\alpha^2(m-1)((n-1)(K_1^2+K_2^2))^2 + \frac{9}{2}nK_3^2\alpha^2(m-1)^2t + \frac{1}{2}tv\widetilde{y} - \frac{1}{a}(\frac{\alpha-1}{\alpha})^2t\widetilde{y}^2.\end{aligned}$$

Using

$$
\begin{aligned}
-\frac{2(\alpha-1)}{a\alpha^2}\widetilde{y}+\frac{40m}{R}\widetilde{y}^{\frac{1}{2}}v^{\frac{1}{2}} &\le \frac{200a\alpha^2m^2}{\alpha-1}\frac{v}{R^2}, \qquad C_4: &&= \frac{200a\alpha^2m^2}{\alpha-1}+(m-1)C_3,\\
-\frac{1}{3a}(\frac{\alpha-1}{\alpha})^2t\widetilde{y}^2+2(m-1)(n-1)tK_0^2v\eta\widetilde{y} &\le C_5tv^2, \qquad C_5: &&= \frac{3a\alpha^2(m-1)^2(n-1)^2K_0^4}{(\alpha-1)^2},\\
-\frac{1}{3a}(\frac{\alpha-1}{\alpha})^2t\widetilde{y}^2+\frac{1}{2}tv\widetilde{y} &\le C_6tv^2, \qquad C_6: &&= \frac{3a\alpha^2}{16(\alpha-1)^2},\\
-\frac{1}{3a}(\frac{\alpha-1}{\alpha})^2t\widetilde{y}^2+2(n-1)(\alpha-1)tK_2^2\eta\widetilde{y} &\le C_7t, \qquad C_7: &&= 3a\alpha^2(n-1)^2K_2^4,
\end{aligned}
$$

then the above quadratical inequality (2.12) on $(\widetilde{y}-\alpha\widetilde{z})$ reduces to

$$0\le -\frac{t}{a\alpha^2}(\widetilde{y}-\alpha\widetilde{z})^2+(C_4\frac{tv}{R^2}+C_2t+1)(\widetilde{y}-\alpha\widetilde{z})+(C_5+C_6)tv^2+(C_7+C_8)t.$$

Here, $C_8:=\alpha^2(m-1)((n-1)(K_1^2+K_2^2))^2+\frac{9}{2}nK_3^2\alpha^2(m-1)^2$. This implies that

$$
\begin{aligned}
\widetilde{y}-\alpha\widetilde{z} &\le \frac{a\alpha^2}{2}\Big(C_4\frac{v}{R^2}+C_2+\frac{1}{t}+\sqrt{(C_4\frac{v}{R^2}+C_2+\frac{1}{t})^2+4\frac{(C_5+C_6)v^2+(C_7+C_8)}{a\alpha^2}}\Big)\\
&\le a\alpha^2\Big(C_4\frac{v_{\max}^{R,T}}{R^2}+C_2+\frac{1}{t}\Big)+\alpha\sqrt{a(C_5+C_6)}v_{\max}^{R,T}+\alpha\sqrt{a(C_7+C_8)}.
\end{aligned}
$$

Proof of the Corollary 1.1 Direct calculation implies

$$\log\frac{v(x_2,t_2)}{v(x_1,t_1)}=\int_{t_1}^{t_2}(\frac{v_t}{v}+\langle\frac{\nabla v}{v},\gamma'(s)\rangle)ds\ge\int_{t_1}^{t_2}(\frac{v_t}{v}-\frac{|\nabla v|^2}{\alpha v}-\frac{\alpha|\gamma'(s)|_s^2}{4v})ds.$$

The result follows from the observation that $\gamma(s)$ lies completely inside $B(\mathcal{O},\frac{R}{2})$ at any time in $[0,T]$ (since bounded tensor h implies that $g(t)$ is uniform equivalence to the initial metric $g(0)$) and the estimate in the main theorem 1.

To prove the main theorem 2, we need some lemmas as well. Note

$$u_t=\Delta_f u^m \tag{2.13}$$

Suppose that u is a positive solution to (2.13). Let $v=\frac{m}{m-1}u^{m-1}$. Direct calculation shows that

$$v_t=mu^{m-2}\Delta_f u^m=(m-1)v\Delta_f v+|\nabla v|^2. \tag{2.14}$$

Since $v\neq 0$, then (2.14) is equivalent to

$$\frac{v_t}{v}=(m-1)\Delta_f v+\frac{|\nabla v|^2}{v}. \tag{2.15}$$

Let $L=\partial_t-(m-1)v\Delta_f$ and $F=\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}$. We have the following lemmas.

Lemma 2.3 Suppose that u is a poistive solution to (2.13). Then

$$
\begin{aligned}
L(\frac{v_t}{v}) &= (m-1)\frac{v_t}{v}\Delta_f v+\frac{v_t}{v}\frac{|\nabla v|^2}{v}+2m\nabla v\nabla(\frac{v_t}{v})+2(m-1)h^{ij}v_if_j\\
&\quad -2(m-1)\langle h,\nabla^2 v\rangle-2(m-1)\langle \mathrm{div}h-\frac{1}{2}\nabla(tr_g h),\nabla v\rangle-\frac{2}{v}h^{ij}v_iv_j,
\end{aligned}
\tag{2.16}
$$

$$\begin{aligned}L(\frac{|\nabla v|^2}{v}) \le 2(m-1)\frac{|\nabla v|^2}{v}\Delta_f v - \frac{2(m-1)}{N+n}|\Delta_f v|^2 - 2(m-1)Ric_f^N\langle\nabla v, \nabla v\rangle \\ + 2m\nabla v\nabla(\frac{|\nabla v|^2}{v}) + \frac{|\nabla v|^4}{v^2} - \frac{2}{v}h^{ij}v_i v_j.\end{aligned} \tag{2.17}$$

Proof Direct calculation shows that

$$\nabla(\frac{v_t}{v}) = \frac{\nabla v_t}{v} - \frac{v_t\nabla v}{v^2}. \tag{2.18}$$

Therefore, we get

$$\Delta(\frac{v_t}{v}) = \frac{1}{v}\Delta v_t - \frac{v_t}{v^2}\Delta v - \frac{2}{v^2}\nabla v\nabla v_t + \frac{2v_t}{v^3}|\nabla v|^2, \tag{2.19}$$

and

$$(m-1)v\langle\nabla f, \nabla(\frac{v_t}{v})\rangle = (m-1)\langle\nabla f, \nabla v_t\rangle - (m-1)\frac{v_t}{v}\langle\nabla f, \nabla v\rangle. \tag{2.20}$$

By (2.15) and Lemma 2.1, we have

$$\begin{aligned}\partial_t(\frac{v_t}{v}) = (m-1)\partial_t(\Delta_f v) + \partial_t(\frac{|\nabla v|^2}{v}) = (m-1)\Delta_f v_t + 2(m-1)h^{ij}v_i f_j \\ - 2(m-1)\langle h, \nabla^2 v\rangle - 2(m-1)\langle \mathrm{div} h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle - \frac{2}{v}h^{ij}v_i v_j \\ + \frac{2}{v}\nabla v\nabla v_t - \frac{v_t}{v}\frac{|\nabla v|^2}{v}.\end{aligned} \tag{2.21}$$

According to (2.18), (2.19), (2.20), and (2.21). We conclude that

$$\begin{aligned}L(\frac{v_t}{v}) &= \partial_t(\frac{v_t}{v}) - (m-1)v\Delta_f(\frac{v_t}{v}) \\ &= (m-1)\frac{v_t}{v}\Delta_f v + \frac{v_t}{v}\frac{|\nabla v|^2}{v} + 2m\nabla v\nabla(\frac{v_t}{v}) + 2(m-1)h^{ij}v_i f_j \\ &\quad - 2(m-1)\langle h, \nabla^2 v\rangle - 2(m-1)\langle \mathrm{div} h - \frac{1}{2}\nabla(tr_g h), \nabla v\rangle - \frac{2}{v}h^{ij}v_i v_j.\end{aligned} \tag{2.22}$$

On the other hand, by (2.14) we get

$$\begin{aligned}\partial_t(\frac{|\nabla v|^2}{v}) &= \frac{2v\nabla v\nabla v_t - |\nabla v|^2 v_t}{v^2} - \frac{2}{v}h^{ij}v_i v_j \\ &= \frac{2\nabla v}{v}\nabla((m-1)v\Delta_f v + |\nabla v|^2) - \frac{|\nabla v|^2}{v^2}((m-1)v\Delta_f v + |\nabla v|^2) - \frac{2}{v}h^{ij}v_i v_j \\ &= (m-1)\frac{|\nabla v|^2}{v}\Delta_f v + 2(m-1)\nabla v\nabla\Delta_f v + 2\frac{\nabla v}{v}\nabla|\nabla v|^2 - \frac{|\nabla v|^4}{v^2} - \frac{2}{v}h^{ij}v_i v_j.\end{aligned} \tag{2.23}$$

Direct calculation shows that

$$\begin{aligned}\Delta_f\frac{|\nabla v|^2}{v} &= \Delta\frac{|\nabla v|^2}{v} - \langle\nabla f, \nabla\frac{|\nabla v|^2}{v}\rangle \\ &= \frac{1}{v}\Delta|\nabla v|^2 - \frac{|\nabla v|^2}{v^2}\Delta v - \frac{2}{v^2}\nabla v\nabla|\nabla v|^2 \\ &\quad + \frac{2}{v^3}|\nabla v|^4 - \frac{1}{v}\langle\nabla f, \nabla|\nabla v|^2\rangle + \frac{|\nabla v|^2}{v^2}\langle\nabla f, \nabla v\rangle \\ &= \frac{1}{v}\Delta_f|\nabla v|^2 - \frac{|\nabla v|^2}{v^2}\Delta_f v - \frac{2}{v^2}\nabla v\nabla|\nabla v|^2 + \frac{2}{v^3}|\nabla v|^4 \\ &= \frac{1}{v}\Delta_f|\nabla v|^2 - \frac{|\nabla v|^2}{v^2}\Delta_f v - \frac{2\nabla v}{v}\nabla(\frac{|\nabla v|^2}{v}).\end{aligned} \tag{2.24}$$

According to (2.23) and (2.24), we obtain

$$\begin{aligned} L(\frac{|\nabla v|^2}{v}) &= \partial_t(\frac{|\nabla v|^2}{v}) - (m-1)v\Delta_f\frac{|\nabla v|^2}{v} \\ &= 2(m-1)(\frac{|\nabla v|^2}{v}\Delta_f v + \nabla v\nabla\Delta_f v - \frac{1}{2}\Delta_f|\nabla v|^2) \\ &\quad + 2m\nabla v\nabla(\frac{|\nabla v|^2}{v}) + \frac{|\nabla v|^4}{v^2} - \frac{2}{v}h^{ij}v_iv_j. \end{aligned} \tag{2.25}$$

According to [[14], [15]], we have

$$\frac{1}{2}\Delta_f|\nabla v|^2 \geq \frac{1}{N+n}|\Delta_f v|^2 + \nabla v\nabla\Delta_f v + Ric_f^N(\nabla v,\nabla v). \tag{2.26}$$

By (2.25) and (2.26), we conclude that (2.17) is true.

Lemma 2.4 The function F satisfies the following equation:

$$\begin{aligned} L(F) \leq &-\frac{2(m-1)}{N+n}|\Delta_f v|^2 - 2(m-1)Ric_f^N\langle\nabla v,\nabla v\rangle + 2m\nabla v\nabla F - ((m-1)\Delta_f v)^2 \\ &+ (1-\alpha)(\frac{v_t}{v})^2 - \alpha'\frac{v_t}{v} + \frac{2(\alpha-1)}{v}h^{ij}v_iv_j - 2\alpha(m-1)h^{ij}v_if_j \\ &+ 2\alpha(m-1)\langle h,\nabla^2 v\rangle + 2\alpha(m-1)\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h),\nabla v\rangle. \end{aligned} \tag{2.27}$$

Proof For the reader's convenience, we give the details of the proof of Lemma 2.4. By (2.16) and (2.17), we have

$$\begin{aligned} L(F) =&L(\frac{|\nabla v|^2}{v}) - \alpha L(\frac{v_t}{v}) - \alpha'\frac{v_t}{v} \\ \leq&2(m-1)\frac{|\nabla v|^2}{v}\Delta_f v - 2\frac{(m-1)}{N+n}|\Delta_f v|^2 - 2(m-1)Ric_f^N\langle\nabla v,\nabla v\rangle + 2m\nabla v\nabla(\frac{|\nabla v|^2}{v}) \\ &+ \frac{|\nabla v|^4}{v^2} + \frac{2(\alpha-1)}{v}h^{ij}v_iv_j - \alpha(m-1)\frac{v_t}{v}\Delta_f v - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} - 2\alpha m\nabla v\nabla(\frac{v_t}{v}) \\ &- 2\alpha(m-1)h^{ij}v_if_j + 2\alpha(m-1)\langle h,\nabla^2 v\rangle + 2\alpha(m-1)\langle \mathrm{div}h - \frac{1}{2}\nabla(tr_g h),\nabla v\rangle - \alpha'\frac{v_t}{v}. \end{aligned} \tag{2.28}$$

By the definition of F, we have

$$2m\nabla v\nabla(\frac{|\nabla v|^2}{v}) - 2\alpha m\nabla v\nabla(\frac{v_t}{v}) = 2m\nabla v\nabla(\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v}) = 2m\nabla v\nabla F. \tag{2.29}$$

According to (2.15) we get

$$\frac{|\nabla v|^4}{v^2} - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} = (1-\alpha)\frac{|\nabla v|^2}{v}\frac{v_t}{v} - (m-1)\frac{|\nabla v|^2}{v}\Delta_f v. \tag{2.30}$$

Using (2.15) again, we arrive at

$$\begin{aligned}
&(m-1)\frac{|\nabla v|^2}{v}\Delta_f v-\alpha(m-1)\frac{v_t}{v}\Delta_f v+(1-\alpha)\frac{|\nabla v|^2}{v}\frac{v_t}{v}\\
=&\frac{|\nabla v|^2}{v}(\frac{v_t}{v}-\frac{|\nabla v|^2}{v})-\alpha\frac{v_t}{v}(\frac{v_t}{v}-\frac{|\nabla v|^2}{v})+(1-\alpha)\frac{|\nabla v|^2}{v}\frac{v_t}{v}\\
=&2\frac{|\nabla v|^2}{v}\frac{v_t}{v}-\frac{|\nabla v|^4}{v^2}-\alpha(\frac{v_t}{v})^2\\
=&-(\frac{v_t}{v}-\frac{|\nabla v|^2}{v})^2+(1-\alpha)(\frac{v_t}{v})^2\\
=&-((m-1)\Delta_f v)^2+(1-\alpha)(\frac{v_t}{v})^2.
\end{aligned}\tag{2.31}$$

Putting (2.29), (2.30) and (2.31) into (2.28), we conclude that (2.27) is true.

Proof of the Main Theorem 2 We consider F in the geodesic ball $B_p(2R)$, which is centered at p with radius $2R$, where $\alpha=e^{2Hk_0t}$. Since $Ric_f^N(B_p(2R))\geq -k_0$, by (2.27) and definition of H and $a:=\frac{(n+N)(m-1)}{(n+N)(m-1)+2}$, we have

$$\begin{aligned}
L(F)\leq&\frac{-2}{(m-1)(N+n)}((m-1)\Delta_f v)^2+2vk_0\frac{|\nabla v|^2}{v}+2m\nabla v\nabla F-((m-1)\Delta_f v)^2\\
&+(1-\alpha)(\frac{v_t}{v})^2-\alpha'\frac{v_t}{v}+\frac{2(\alpha-1)}{v}h^{ij}v_iv_j-2\alpha(m-1)h^{ij}v_if_j+2\alpha(m-1)\langle h,\nabla^2v\rangle\\
&+2\alpha(m-1)\langle \mathrm{div}h-\frac{1}{2}\nabla(tr_gh),\nabla v\rangle\\
\leq&-\frac{1}{a}((m-1)\Delta_f v)^2+2Hk_0\frac{|\nabla v|^2}{v}+2m\nabla v\nabla F+(1-\alpha)(\frac{v_t}{v})^2\\
&+\frac{2(\alpha-1)}{v}h^{ij}v_iv_j-2\alpha(m-1)h^{ij}v_if_j+2\alpha(m-1)\langle h,\nabla^2v\rangle\\
&+2\alpha(m-1)\langle \mathrm{div}h-\frac{1}{2}\nabla(tr_gh),\nabla v\rangle-\alpha'\frac{v_t}{v}.
\end{aligned}$$

Since $L(\alpha^{-1}F)=(\alpha^{-1})'F+\alpha^{-1}L(F)$ and $\alpha'=2Hk_0\alpha$, then

$$\begin{aligned}
L(\alpha^{-1}F)\leq&-\frac{1}{a\alpha}((m-1)\Delta_f v)^2+2m\alpha^{-1}\nabla v\nabla F+(1-\alpha)\alpha^{-1}(\frac{v_t}{v})^2+\frac{2(\alpha-1)}{\alpha v}h^{ij}v_iv_j\\
&-2(m-1)h^{ij}v_if_j+2(m-1)\langle h,\nabla^2v\rangle+2(m-1)\langle \mathrm{div}h-\frac{1}{2}\nabla(tr_gh),\nabla v\rangle.
\end{aligned}\tag{2.32}$$

By (2.15) and definition of F, we get

$$(m-1)\Delta_f v=(\alpha^{-1}-1)\frac{|\nabla v|^2}{v}-\alpha^{-1}F,\quad \alpha\frac{v_t}{v}=\frac{|\nabla v|^2}{v}-F.\tag{2.33}$$

Putting (2.33) into (2.32), we obtain

$$\begin{aligned}
L(\alpha^{-1}F)\leq&-\frac{(1-\alpha)^2}{a\alpha^3}\frac{|\nabla v|^4}{v^2}-\frac{1}{a\alpha^3}F^2+\frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}F+\frac{2m}{\alpha}\nabla v\nabla F+\frac{1-\alpha}{\alpha^3}\frac{|\nabla v|^4}{v^2}\\
&+\frac{1-\alpha}{\alpha^3}F^2-\frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}F+\frac{2(\alpha-1)}{\alpha v}h^{ij}v_iv_j-2(m-1)h^{ij}v_if_j\\
&+2(m-1)\langle h,\nabla^2v\rangle+2(m-1)\langle \mathrm{div}h-\frac{1}{2}\nabla(tr_gh),\nabla v\rangle.
\end{aligned}\tag{2.34}$$

Attention the definition of $h, |\nabla f|$, the following reality and the Young's inequality,

$$\begin{aligned}
&|h| \leq k_1+k_2, |\mathrm{div} h-\frac{1}{2}\nabla(tr_g h)| = |g^{ij}\nabla_i h_{jl}-\frac{1}{2}g^{ij}\nabla_l h_{ij}| \leq \frac{3}{2}|g||\nabla h| \leq \frac{3}{2}\sqrt{n}k_3,\\
&3\sqrt{n}(m-1)k_3 v^{1/2}\frac{|\nabla v|}{v^{1/2}} \leq \frac{9}{4}n(m-1)^2k_3^2 v+\frac{|\nabla v|^2}{v} \leq \frac{9}{4}n(m-1)k_3^2 H+\frac{|\nabla v|^2}{v},\\
&|2(m-1)h^{ij}v_i f_j| \leq 2(m-1)|h||\nabla f|v^{1/2}\frac{|\nabla v|}{v^{1/2}} \leq (m-1)c_0(k_1+k_2)H+\frac{|\nabla v|^2}{v},
\end{aligned}$$

we have,

$$\begin{aligned}
L(\alpha^{-1}F) \leq\ & (-\frac{(\alpha-1)((\alpha-1)+a)}{a\alpha^3}\frac{|\nabla v|^4}{v^2}+\frac{2(\alpha-1)(k_1+k_2)+2\alpha}{\alpha}\frac{|\nabla v|^2}{v})-\frac{1}{a\alpha^3}F^2\\
&+\frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}F+\frac{2m}{\alpha}\nabla v\nabla F+\frac{1-\alpha}{\alpha^3}F^2-\frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}F\\
&+(m-1)c_0(k_1+k_2)H+2(m-1)\langle h,\nabla^2 v\rangle+\frac{9}{4}n(m-1)k_3^2 H\\
\leq & -\frac{1}{a\alpha^3}F^2+\frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}F+\frac{2m}{\alpha}\nabla v\nabla F+\frac{1-\alpha}{\alpha^3}F^2\\
&-\frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}F+(m-1)c_0(k_1+k_2)H+\frac{9}{4}n(m-1)k_3^2 H\\
&+\frac{a\alpha(2(\alpha-1)(k_1+k_2)+2\alpha)^2}{4(\alpha-1)((\alpha-1)+a)}+2(m-1)\langle h,\nabla^2 v\rangle,
\end{aligned}$$

where we have used $-ax^2+bx \leq b^2/4a$ to the first term in the second inequality.

Write $c_2 := (m-1)c_0(k_1+k_2)H+\frac{9}{4}n(m-1)k_3^2H+\frac{a\alpha(2(\alpha-1)(k_1+k_2)+2\alpha)^2}{4(\alpha-1)((\alpha-1)+a)}+2(m-1)(k_1+k_2)c_1$, we have

$$L(\alpha^{-1}F) \leq -\frac{1}{a\alpha^3}F^2+\frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}F+\frac{2m}{\alpha}\nabla v\nabla F+\frac{1-\alpha}{\alpha^3}F^2-\frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}F+c_2.$$

According to (2.4) and (2.5) in [7], we can construct a cut-off function ϕ such that $0 \leq \phi \leq 1$, $\sup(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$ and

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{C}{R^2}, \quad -\Delta_f\phi \leq \frac{C}{R^2}(1+R\sqrt{k_0}), \tag{2.35}$$

where C is a constant depending only on n. Set $G = t\phi\alpha^{-1}F$. Assume that G achieves its maximum at the point $(x_0, s) \in B_p(2R)\times[0,T]$ and assume $G(x_0,s) \geq 0$. By the maximum principle, we have

$$\nabla G = 0, \quad L(G) \geq 0, \quad \nabla(\alpha^{-1}F) = -\frac{\alpha^{-1}F}{\phi}\nabla\phi$$

at the point (x_0, s), and

$$\begin{aligned}
0 \le L(G) &= s\phi L(\alpha^{-1}F) - (m-1)v\frac{\Delta_f\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
&\le s\phi\{-\frac{1}{a\alpha^3}F^2 + \frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}F + \frac{2m}{\alpha}\nabla v\nabla F + \frac{1-\alpha}{\alpha^3}F^2 - \frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}F + c_2\} \\
&\quad - (m-1)v\frac{\Delta_f\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
&= -\frac{1}{a\alpha s\phi}G^2 + \frac{2(1-\alpha)}{a\alpha^2}\frac{|\nabla v|^2}{v}G - 2m\nabla v\frac{\nabla\phi}{\phi}G + \frac{1-\alpha}{\alpha s\phi}G^2 - \frac{2(1-\alpha)}{\alpha^2}\frac{|\nabla v|^2}{v}G \\
&\quad - (m-1)v\frac{\Delta_f\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} + s\phi c_2 \\
&\le -\frac{1+a(\alpha-1)}{a\alpha s\phi}G^2 + \frac{2(1-\alpha)}{a\alpha^2}\frac{|\nabla v|^2}{v}G + 2\frac{mH^{1/2}}{(m-1)^{1/2}}\frac{|\nabla\phi|}{\phi}\frac{|\nabla v|}{v^{1/2}}G \\
&\quad - \frac{2(1-\alpha)}{\alpha^2}\frac{|\nabla v|^2}{v}G - H\frac{\Delta_f\phi}{\phi}G + 2H\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} + s\phi c_2 \\
&\le -\frac{1+a(\alpha-1)}{a\alpha s\phi}G^2 + (-\frac{2(\alpha-1)(1-a)}{a\alpha^2}\frac{|\nabla v|^2}{v}G + 2\frac{mH^{1/2}}{(m-1)^{1/2}}\frac{|\nabla\phi|}{\phi}\frac{|\nabla v|}{v^{1/2}}G) \\
&\quad + H\frac{C}{R^2\phi}(1+R\sqrt{k_0}) + 2H\frac{C}{R^2\phi}G + \frac{G}{s} + s\phi c_2.
\end{aligned} \tag{2.36}$$

Multiplying both sides of (2.36) by $s\phi$, and using $-ax^2 + bx \le b^2/4a$, we get

$$\begin{aligned}
L(G) \le &-\frac{1+a(\alpha-1)}{a\alpha}G^2 + \frac{a\alpha^2m^2H}{2(m-1)(\alpha-1)(1-a)}\frac{C}{R^2}TG \\
&+ H\frac{C}{R^2}(1+R\sqrt{k_0})TG + 2H\frac{C}{R^2}TG + G + c_2T^2\phi^2.
\end{aligned}$$

Write $c_3 := \frac{1+a(\alpha-1)}{a\alpha}$, $\quad c_4 := \frac{a\alpha^2m^2H}{2(m-1)(\alpha-1)(1-a)}\frac{C}{R^2} + H\frac{C}{R^2}(1+R\sqrt{k_0}) + 2H\frac{C}{R^2} + \frac{1}{T}$. The above inequality becomes

$$0 \le L(G) \le -c_3G^2 + c_4TG + c_2T^2\phi^2.$$

We can get

$$G(x,T) \le G(x_0,s) \le \frac{c_4T}{c_3} + \sqrt{\frac{c_2}{c_3}}T\phi.$$

Hence, for all $x \in B_p(R)$, it holds that

$$F(x,T) \le \frac{\alpha c_4}{c_3} + \alpha\sqrt{\frac{c_2}{c_3}}.$$

Thus, the proof of the main theorem 2 is completed. Letting $R \to \infty$, $c_4 \to \frac{1}{T}$, we get the result of Corollary 1.2.

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多孔介质方程在一般几何流演化下的梯度估计

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摘要: 本文研究了多孔介质方程在一般几何流下的梯度估计. 通过Aroson和Bénilan对多孔介质方程的研究结果以及运用Li-Yau梯度估计的方法, 获得了对多孔介质方程的正解对于Laplace算子以及drifing Laplace 算子在一般几何流演化下的一些梯度估计, 推广了Zhu Xiao-bao和Deng Yi-hua的结果.

关键词: 梯度估计; 几何流; 多孔介质方程; 哈拿克不等式

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