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DIFFERENCE HARNACK ESTIMATES FOR WEIGHTED NONLINEAR REACTION-DIFFUSION EQUATIONS ON WEIGHTED RIEMANNIAN MANIFOLDS

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Abstract: In this paper, we study the problem of difference Harnack estimate on Riemannian manifolds. By using maximum principle and weighted p-Bochner formula, we derive the Li-Yau type difference Harnack estimate and Hamilton type estimate for the positive solutions to weighted nonlinear reaction-diffusion equation on compact weighted Riemannian manifold with curvature dimension condition $CD(0, N)$, which generalizes the non-weighted case under the condition of nonnegative Ricci curvature.

Keywords: weighted nonlinear reaction diffusion equation; Li-yau type difference Harnack estimate; hamilton type difference Harnack estimate; curvature dimension condition; weighted p-Bochner formula

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1 Introduction

Let M be an n-dimensional compact Riemannian manifold with curvature dimension condition $CD(0, N)$. In this paper, we consider a weighted nonlinear reaction-diffusion equation(WNRDE)

$$u_t = \Delta_{p,f} u^\gamma + cu^q, \tag{1.1}$$

on M, where $\gamma > 0$, $p > 1$, $q > 0$, $\Delta_{p,f} u = e^f \operatorname{div}(e^{-f} |\nabla u|^{p-2} \nabla u)$ is the weighted p-Laplacian of u, and f is a smooth function.

Gradient estimate or differential Harnack estimate is an important tool in geometric analysis. In 1986, Li and Yau [1] first proved the sharp gradient estimate for positive solutions to heat equation on Riemannian manifolds. Since then, gradient estimate has been studied extensively by many scholars. Particularly in the last decade, more attention has been paid to the study of nonlinear equations. Kotschwar and Ni [2] established gradient estimates

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for p-harmonic functions and parabolic p-Lapalacian equation on Riemannian manifolds. In [3, 4], the first author and coauthor improved Li-Yau type gradient estimates for the positive solutions to the weighted nonlinear p-heat equation on Riemannian manifolds with $CD(-K,m)$ condition. In [5], the authors proved the Li-Yau type estimate for the porous medium equation and fast diffusion equation. In [6], the first author and coauthor got sharp global Li-Yau type gradient estimates for positive solutions to doubly nonlinear diffusion equation on compact Riemannian manifolds with nonnegative Ricci curvature.

In [7], we derived the global Li-Yau type and Hamilton type gradient estimate for positive solutions to the nonlinear reaction-diffusion equation. The purpose of this paper is to extend the work in [7], that is to prove the gradient estimates for weighted nonlinear reaction-diffusion equation (1.1) on Riemannian manifolds.

To show our results, we recall some necessary notations. Let $(M,g,d\mu)(d\mu=e^{-f}dV)$ be an n-dimensional compact weighted Riemannian manifold, dV be the Riemannian volume measure, $f\in C^\infty(M)$. Define a diffusion operator $L\doteqdot\Delta_f=\Delta-\nabla f\cdot\nabla$, and N-Bakry-Émery Ricci curvature tensor

$$Ric_f^N(L)\doteqdot \mathrm{Ric}+\nabla\nabla f-\frac{\nabla f\otimes\nabla f}{N-n}.$$

If $N=\infty$, then Bakry-Émery Ricci curvature $\mathrm{Ric}_f^\infty(L)\doteqdot\mathrm{Ric}_f=\mathrm{Ric}+\nabla\nabla f$, which firstly studied by Bakry and Émery [8]. If L satisfies the curvature dimension condition $CD(K,N)$ if

$$\mathit{\Gamma}_2(u,u)\geq\frac{1}{N}(Lu)^2+K|\nabla u|^2,$$

where $\mathit{\Gamma}_2(u,u)=\frac{1}{2}L|\nabla u|^2-\langle\nabla Lu,\nabla u\rangle$. Furthermore, by the weighted Bochner formula, we know that curvature dimension condition $CD(K,N)$ is equivalent to $\mathrm{Ric}_f^N\geq Kg$ [9].

Now we give the global Li-Yau type difference Harnack estimate for WNRDE (1.1) and its applications in Harnack inequalities.

Theorem 1.1 Let M be an n-dimensional compact weighted Riemannian manifold with the $CD(0,N)$ condition. Assume that u is a smooth nonnegetive solution to (1.1), and $v=\frac{\gamma}{b}u^b$ satisfy equation (2.1) on M. Then for any $b>0$, $\bar a>0$ and $c(q-1)(q-1+b)\geq 0$, we have

$$\frac{|\nabla v|^p}{v}-\frac{v_t}{v}+\kappa v^m\leq\frac{\bar a}{t}+(p-1)(m+1)\kappa\bar a v_{min}^m, \tag{1.2}$$

where $b=\gamma-\frac{1}{p-1}$, $\bar a=\frac{Nb}{p+(p-1)Nb}$, $m=\frac{q-1}{b}$, $\kappa=cb^{m+1}\gamma^{-m}$ and $v_{min}\doteqdot\min_M v$.

Remark 1.2 When $c=0$ and $f=const.$, the estimate (1.2) reduces the Li-Yau type estimate of weighted doubly nonlinear diffusion equation in [6].

On the other hand, Hamilton [10] improved the elliptic type gradient estimate on a compact manifold. Yan and Wang [11] established elliptic type gradient estimates for positive solutions to the doubly nonlinear diffusion equation on Riemannian manifolds. Recently, the authors [7] derived Hamilton type gradient estimates for nonlinear reaction-diffusion equation on compact Riemanian manifold with nonnegative Ricci curvature. In this paper, we can prove Hamilton type estimate for WNRDE (1.1) on n-dimensional compact weighted Riemannian manifold with $CD(0,N)$ condition.

Theorem 1.3 Let M be an n-dimensional compact weighted Riemannian manifold with the $CD(0,N)$ condition. Suppose that u is a smooth positive solution to (1.1) and v satisfy equation (2.1) on M. Then for any $p>1$, $-\frac{p}{(p-1)N}<b<0$ and $\kappa(p(m+1)-1)>0$,

$$F\frac{|\nabla v|^p}{-v} \leq \frac{p}{(p-1)(Nb(p-1)+p)}\left(\frac{1}{t}+\kappa(p(m+1)-1)v_{Max}\right), \tag{1.3}$$

where $v_{Max} \doteqdot \max_M v^m$.

As applications of two estimates in Theorem 1.1 and 1.3, by integrating along minimizing geodesic paths, we can derive the corresponding Harnack inequalities.

Corollary 1.4 Let M be an n-dimensional compact weighted Riemannian manifold with the $CD(0,N)$ condition, u be a positive solution to (1.1) and v satisfy the equation (2.1). Given any $x_1, x_2 \in M$, $0 \leq t_1 < t_2 < T$ and $c>0$, we have:

1. $q>1$, $v_{max} \doteqdot \max_M v < \infty$, then

$$\begin{aligned} &v(x_2,t_2)-v(x_1,t_1) \\ \geq& -\bar{a}v_{max}\log\left(\frac{t_2}{t_1}\right)-\kappa v_{min}^m((p-1)(m+1)\bar{a}v_{max}-v_{min})(t_2-t_1)-\frac{d(x_1,x_2)^{p^*}}{p^*(p(t_2-t_1))^{1/(p-1)}}, \end{aligned} \tag{1.4}$$

where $p^*=\frac{p}{p-1}$, $d(x_1,x_2)$ denotes the geodestic distance between x_1 and x_2.

2. $q<1-b$, $v_{min} \doteqdot \min_M v < \infty$, then

$$\begin{aligned} &v(x_2,t_2)-v(x_1,t_1) \\ \geq& -\bar{a}v_{max}\log\left(\frac{t_2}{t_1}\right)-\kappa v_{min}^{m+1}((p-1)(m+1)\bar{a}-1)(t_2-t_1)-\frac{d(x_1,x_2)^{p^*}}{p^*(p(t_2-t_1))^{1/(p-1)}}. \end{aligned} \tag{1.5}$$

Corollary 1.5 Let M be an n-dimensional compact weighted Riemannian manifold with the $CD(0,N)$ condition, u be a positive solution to (1.1) and v satisfy the equation (2.1). Given any $x_1, x_2 \in M$, we have:

$$\log\frac{v(x_1,t)}{v(x_2,t)} \leq \frac{1}{(p-1)(Nb(p-1)+p)}\left(\frac{1}{t}+\kappa(p(m+1)-1)v_{max}^m\right)-\frac{d^{p^*}(x_1,x_2)}{p^*v_{max}}, \tag{1.6}$$

where $p^*=\frac{p}{p-1}$, $v_{max}=\max_M v$, $v_{Max}=\max_M v^m$ and $d(x_1,x_2)$ denotes the geodestic distance between x_1 and x_2.

The organization of this paper is as follows. In Section 2, using the weighted p-Bochner formula, we will give the proof of Li-Yau type difference Harnack estimate (1.2). In section 3, we will prove Hamilton type estimate (1.3). In Section 4, two Harnack inequalities are derived as applications of two type estimates.

2 Global Li-Yau Type Difference Harnack Estimate

In this paper, let ∇ and div be the gradient operator and divergence operator on M. Assume that u is a positive solution to (1.1), the pressure transform introduced by the first author in [6],

$$v=\frac{\gamma}{b}u^b,\ b=\gamma-\frac{1}{p-1}.$$

The WNRDE can be rewritten as

$$u_t=e^f\operatorname{div}(e^{-f}u|\nabla v|^{p-2}\nabla v)+cu^q,$$

and corresponding pressure equation for v satisfies

$$v_t=bv\Delta_{p,f}v+|\nabla v|^p+\kappa v^{m+1},\tag{2.1}$$

where $m=\frac{q-1}{b}$ and $\kappa=cb^{m+1}\gamma^{-m}$. The linearized operator of weighted p-Laplacian is defined by

$$\mathcal{L}_f(\psi)\doteqdot e^f\operatorname{div}\Big(e^{-f}w^{\frac{p}{2}-1}A(\nabla\psi)\Big),\tag{2.2}$$

and its parabolic operator is $\Box_f\doteqdot\frac{\partial}{\partial t}-bv\mathcal{L}_f$, where $w=|\nabla v|^2>0$, and $A=\mathrm{id}+(p-2)\frac{\nabla v\otimes\nabla v}{w}$.

Lemma 2.6 Let

$$F\doteqdot\frac{v_t}{v}-\frac{|\nabla v|^p}{v}-\kappa v^m=b\Delta_{p,f}v.$$

Then

$$\Box_f v=w^{\frac{p}{2}}-(p-2)vF+\kappa v^{m+1},\tag{2.3}$$

$$\Box_f v^m=mv^{m-1}v_t-(p-1)mv^mF-(p-1)(m-1)mbv^{m-1}w^{\frac{p}{2}},\tag{2.4}$$

$$\Box_f v_t=Fv_t+pw^{\frac{p}{2}-1}\langle\nabla v,\nabla v_t\rangle+(m+1)\kappa v^m v_t.\tag{2.5}$$

$$\begin{aligned}\Box_f w=&pw^{\frac{p}{2}-1}\langle\nabla v,\nabla w\rangle-\Big(\frac{p}{2}-1\Big)bvw^{\frac{p}{2}-2}|\nabla w|^2-2bvw^{\frac{p}{2}-1}\Big(|\nabla\nabla v|^2+\mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\&+2\big(F+(m+1)\kappa v^m\big)w,\end{aligned}\tag{2.6}$$

$$\Box_f w^{\frac{p}{2}}=pw^{\frac{p}{2}-1}\langle\nabla v,\nabla w^{\frac{p}{2}}\rangle-pbvw^{p-2}\Big(|\nabla\nabla v|_A^2+\mathrm{Ric}_f(\nabla v,\nabla v)\Big)+pw^{\frac{p}{2}}\big(F+(m+1)\kappa v^m\big),\tag{2.7}$$

where $|\nabla\nabla v|_A^2=|\nabla\nabla v|^2+\frac{p-2}{2}\frac{|\nabla w|^2}{w}+\frac{(p-2)^2}{4}\frac{|\nabla v\cdot\nabla w|^2}{w^2}$.

Proof For a constant β, combining the equation (2.1) and the definition of $\mathcal{L}$ in (2.2), we have

$$\begin{aligned}\Box_f v^\beta&=\beta v^{\beta-1}v_t-bve^f\operatorname{div}\Big(e^{-f}w^{\frac{p}{2}-1}A\nabla(v^\beta)\Big)\\&=\beta v^{\beta-1}v_t-\beta bv^\beta e^f\operatorname{div}(e^{-f}w^{\frac{p}{2}-1}A\nabla v)-(p-1)(\beta-1)\beta bv^{\beta-1}w^{\frac{p}{2}}\\&=(2-p)b\beta v^\beta\Delta_{p,f}v+\beta[1-(p-1)(\beta-1)b]v^{\beta-1}w^{\frac{p}{2}}+\beta\kappa v^{\beta+m}.\end{aligned}\tag{2.8}$$

Set $\beta = 1$ and $\beta = m$ in the equation (2.8), then we get (2.3) and (2.4). We can directly deduce

$$\frac{\partial}{\partial t}(\Delta_{p,f}v) = e^f \mathrm{div}\left(e^{-f}w^{\frac{p}{2}-1}\Big(\nabla v_t + (p-2)\frac{\langle\nabla v, \nabla v_t\rangle}{w}\nabla v\Big)\right) = \mathcal{L}_f(v_t).$$

Then

$$\begin{aligned}\Box_f v_t &= \partial_t v_t - bv\mathcal{L}_f(v_t)\\ &= bv_t\Delta_{p,f}v + bv\partial_t(\Delta_{p,f}v) + pw^{\frac{p}{2}-1}\langle\nabla v, \nabla v_t\rangle + (m+1)\kappa v^m v_t - bv\mathcal{L}_f(v_t)\\ &= bv_t\Delta_{p,f}v + pw^{\frac{p}{2}-1}\langle\nabla v, \nabla v_t\rangle + (m+1)\kappa v^m v_t.\end{aligned}$$

According to the weighted nonlinear Bochner formula(See [3]),

$$\mathcal{L}_f(w) = 2w^{\frac{p}{2}-1}\Big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big) + 2\langle\nabla v, \nabla\Delta_{p,f}v\rangle + \Big(\frac{p}{2}-1\Big)w^{\frac{p}{2}-2}|\nabla w|^2,$$

we obtain

$$\begin{aligned}\Box_f w =&\partial_t w - bv\mathcal{L}_f(w)\\ =&2\langle\nabla v, \nabla v_t\rangle - 2bvw^{\frac{p}{2}-1}\Big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v) + w^{1-\frac{p}{2}}\langle\nabla v, \nabla\Delta_{p,f}v\rangle\Big)\\ &-\Big(\frac{p}{2}-1\Big)bvw^{\frac{p}{2}-2}|\nabla w|^2\\ =&pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w\rangle - \Big(\frac{p}{2}-1\Big)bvw^{\frac{p}{2}-2}|\nabla w|^2 - 2bvw^{\frac{p}{2}-1}\Big(|\nabla\nabla v|^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big)\\ &+ 2(b\Delta_{p,f}v + (m+1)\kappa v^m)w,\end{aligned}$$

and

$$\begin{aligned}\Box_f w^{\frac{p}{2}} &= \partial_t w^{\frac{p}{2}} - bv\mathcal{L}_f(w^{\frac{p}{2}})\\ &= \frac{p}{2}w^{\frac{p}{2}-1}\Box_f w - \frac{p}{2}\Big(\frac{p}{2}-1\Big)bvw^{p-3}\langle\nabla w, (A\nabla w)\rangle\\ &= pw^{\frac{p}{2}-1}\langle\nabla v, \nabla w^{\frac{p}{2}}\rangle - pbvw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big) + pw^{\frac{p}{2}}\big(b\Delta_{p,f}v + (m+1)\kappa v^m\big),\end{aligned}$$

where $|\nabla\nabla v|_A^2 = |\nabla\nabla v|^2 + \frac{p-2}{2}\frac{|\nabla w|^2}{w} + \frac{(p-2)^2}{4}\frac{|\nabla v\cdot\nabla w|^2}{w^2}$ and $F = b\Delta_{p,f}v$.

Proposition 2.7 Let u and v be as same in Lemma 2.6, we have

$$\begin{aligned}\Box_f F =&\lambda w^{\frac{p}{2}-1}\big\langle\nabla F, \nabla v\big\rangle + pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\Big) + (p-1)F^2\\ &+ (p-1)F\kappa v^m + 2bm(p-1)\kappa v^{m-1}w^{\frac{p}{2}} + \kappa bv\mathcal{L}_f(v^m),\end{aligned} \tag{2.9}$$

where $\lambda = 2\gamma(p-1) + (p-2)$.

Proof For any smooth functions h, g, calculations are based on the following formula on parabolic operator $\Box_f$,

$$\Box_f\left(\frac{h}{g}\right) = \frac{\Box_f h}{g} - \frac{h\Box_f g}{g^2} + 2bvw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla\Big(\frac{h}{g}\Big)\Big), \nabla\log g\Big\rangle. \tag{2.10}$$

Then we have

$$\begin{aligned}\Box_f\Big(\frac{v_t}{v}\Big)=&\frac{\Box_f v_t}{v}-\frac{v_t\Box_f v}{v^2}+2bvw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla\Big(\frac{v_t}{v}\Big)\Big),\nabla\log v\Big\rangle\\ =&\frac{1}{v}\Big(pw^{\frac{p}{2}-1}\langle\nabla v,\nabla v_t\rangle+(p-1)v_tF\Big)-\frac{v_t}{v}\frac{w^{\frac{p}{2}}}{v}+2bw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla\Big(\frac{v_t}{v}\Big)\Big),\nabla v\Big\rangle+\kappa mv^{m-1}v_t,\end{aligned}$$

and

$$\begin{aligned}\Box_f\Big(\frac{w^{\frac{p}{2}}}{v}\Big)=&\frac{\Box_f w^{\frac{p}{2}}}{v}-\frac{w^{\frac{p}{2}}\Box_f v}{v^2}+2bvw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla\Big(\frac{w^{\frac{p}{2}}}{v}\Big)\Big),\nabla\log v\Big\rangle\\ =&\frac{1}{v}\left(pw^{\frac{p}{2}-1}\langle\nabla v,\nabla w^{\frac{p}{2}}\rangle+2(p-1)w^{\frac{p}{2}}F\right)-\frac{w^p}{v^2}+2bw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla\Big(\frac{w^{\frac{p}{2}}}{v}\Big)\Big),\nabla v\Big\rangle\\ &-pbw^{p-2}\Big(|\nabla\nabla v|_A^2+\mathrm{Ric}_f(\nabla v,\nabla v)\Big)+\Big(p(m+1)-1\Big)\kappa v^{m-1}w^{\frac{p}{2}}.\end{aligned}\tag{2.11}$$

Note that $v_t-w^{\frac{p}{2}}=vF+\kappa v^{m+1}$, and $\langle\nabla\log v,\nabla(vF)\rangle=\langle\nabla v,\nabla F\rangle+F\frac{w}{v}$. Thus

$$\begin{aligned}\Box_f F=&\Box_f\Big(\frac{v_t}{v}\Big)-\Box_f\Big(\frac{|\nabla v|^p}{v}\Big)-\kappa\Box_f v^m\\ =&pw^{\frac{p}{2}-1}\Big\langle\nabla(v_t-w^{\frac{p}{2}}),\nabla\log v\Big\rangle+2bw^{\frac{p}{2}-1}\left\langle A\nabla\left(\frac{v_t}{v}-\frac{w^{\frac{p}{2}}}{v}\right),\nabla v\right\rangle\\ &+(p-1)F\left(\frac{v_t}{v}-2\frac{w^{\frac{p}{2}}}{v}\right)-\left(\frac{v_t}{v}-\frac{w^{\frac{p}{2}}}{v}\right)\frac{w^{\frac{p}{2}}}{v}+pbw^{p-2}\Big(|\nabla\nabla v|_A^2+\mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\ &+\kappa mv^{m-1}v_t-\kappa\Box_f v^m-\Big(p(m+1)-1\Big)\kappa v^{m-1}w^{\frac{p}{2}}\\ =&\lambda w^{\frac{p}{2}-1}\Big\langle\nabla F,\nabla v\Big\rangle+pbw^{p-2}\Big(|\nabla\nabla v|_A^2+\mathrm{Ric}_f(\nabla v,\nabla v)\Big)+(p-1)F^2+(p-1)F\kappa v^m\\ &+\kappa bv\mathcal{L}_f(v^m)+2bm(p-1)\kappa v^{m-1}w^{\frac{p}{2}}.\end{aligned}$$

Proof of Theorem 1.1 For fixed $T>0$, we assume that $\max\limits_{(x,t)\in M\times(0,T)}(t(-F))>0$. Let (x_0,t_0) be a point where the function $t(-F)$ achieves a positive maximum. Then at (x_0,t_0), $\Box_f(t(-F))\geq 0$. Note that

$$F=b\Delta_{p,f}v=bw^{\frac{p}{2}-1}A^{ij}\nabla_i\nabla_j v-bw^{\frac{p}{2}-1}\nabla f\cdot\nabla v\doteqdot bw^{\frac{p}{2}-1}\mathrm{Tr}_A(\nabla\nabla v)-bw^{\frac{p}{2}-1}\nabla f\cdot\nabla v.$$

The Cauchy-Schwartz inequality yields

$$w^{p-2}|\nabla\nabla v|_A^2\geq\frac{1}{n}\Big(w^{\frac{p}{2}-1}\mathrm{Tr}_A(\nabla\nabla v)\Big)^2=\frac{1}{n}\left(\frac{F}{b}+w^{\frac{p}{2}-1}\nabla f\cdot\nabla v\right)^2.\tag{2.12}$$

Plugging inequality (2.12) and $\mathrm{Ric}_f^N\geq 0$ into equation (2.9), we have

$$\begin{aligned}&pbw^{p-2}\Big(|\nabla\nabla v|_A^2+\mathrm{Ric}_f(\nabla v,\nabla v)\Big)\\ \geq&\frac{pb}{n}\left(\frac{F}{b}+w^{\frac{p}{2}-1}\nabla f\cdot\nabla v\right)^2+pbw^{p-2}\left(\mathrm{Ric}_f^N(\nabla v,\nabla v)+\frac{|\nabla f\cdot\nabla v|^2}{N-n}\right)\\ \geq&\frac{pb}{N}F^2-pbw^{p-2}\frac{|\nabla f\cdot\nabla v|^2}{N-n}+pbw^{p-2}\frac{|\nabla f\cdot\nabla v|^2}{N-n}=\frac{pb}{N}F^2,\end{aligned}\tag{2.13}$$

where we use the inequality $(a-b)^2 \geq \frac{a^2}{1+\delta} - \frac{b^2}{\delta}$, $\delta = \frac{N-n}{n} > 0$. Then,

$$\begin{aligned}
0 \leq & \Box_f(t(-F)) = -t\Box_f F - F \\
= & -t\Big[pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big) + (p-1)F^2 \\
& + \kappa bv\mathcal{L}_f(v^m) + \kappa(p-1)Fv^m + 2bm(p-1)\kappa v^{m-1}w^{\frac{p}{2}}\Big] - F \\
\leq & -\Big(p-1+\frac{p}{Nb}\Big)tF^2 - F - \kappa t(m+1)(p-1)v^m F - \kappa tbm(m+1)(p-1)v^{m-1}w^{\frac{p}{2}}.
\end{aligned}$$

It is easy to see that for any $\gamma > 0, p > 1$, $b > 0$, and $c(q-1)(q-1+b) > 0$,

$$\kappa bm(m+1)(p-1) = cb^m\gamma^{-m}(p-1)(q-1)(q-1+b) \geq 0.$$

Thus,

$$0 \leq -\frac{t}{\bar{a}}F^2 - F - \kappa t(p-1)(m+1)v^m F,$$

where $\bar{a} = p - 1 + \frac{p}{Nb}$ and $v_{min} = \min_M v$. This inequality is equivalent to (1.2).

3 Global Hamilton Type Difference Harnack Estimate

In this section, we establish a Hamilton type difference Harnack estimate for positive solutions to WNRDE (1.1) on weighted Riemannian manifolds.

Proposition 3.8 Let $w = |\nabla v|^2$ and v be as same in Lemma 2.6, we define

$$G \doteqdot \frac{w^{\frac{p}{2}}}{-v},$$

then

$$\begin{aligned}
\Box_f G = & pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big) + 2bw^{\frac{p}{2}-1}\Big\langle A\Big(\nabla G\Big),\nabla v\Big\rangle + pw^{\frac{p}{2}-1}\langle\nabla G,\nabla v\rangle \\
& - \frac{(p-1)w^p}{v^2} - \frac{2(p-1)bw^{\frac{p}{2}}}{v}\Delta_{p,f}v - \Big(p(m+1)-1\Big)\kappa v^{m-1}w^{\frac{p}{2}}. \qquad (3.1)
\end{aligned}$$

Proof The proof is a direct result by the simplification of (2.12).

Proof of Theorem 1.3 For some fixed $T > 0$, we assume that $\max\limits_{(x,t)\in M\times(0,T)} G > 0$. Let (x_0, t_0) be a point where the function G achieves a positive maximum. Obviously, $t_0 > 0$,$\nabla v(x_0,t_0) \neq 0$ and

$$\Box_f G \geq 0, \quad \nabla G = 0. \qquad (3.2)$$

This proof will all be at point (x_0, t_0). According to the Cauchy-Schwartz inequality (2.12),

$$w^{p-2}|\nabla\nabla v|_A^2 \geq \frac{1}{n}\Big(\mathrm{Tr}_A(w^{\frac{p}{2}}\nabla\nabla v)\Big)^2 = \frac{1}{n}(\Delta_{p,f}v + w^{\frac{p}{2}-1}\nabla f\cdot\nabla v)^2,$$

and $\mathrm{Ric}_f^N \geq 0$, we have

$$\begin{aligned}
& pbw^{p-2}\Big(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v,\nabla v)\Big) - \frac{2(p-1)bw^{\frac{p}{2}}}{v}\Delta_{p,f}v \\
\leq & \frac{pb}{N}(\Delta_{p,f}v)^2 - \frac{2(p-1)bw^{\frac{p}{2}}}{v}\Delta_{p,f}v \leq -\frac{Nb(p-1)^2}{p}\frac{w^p}{v^2}. \qquad (3.3)
\end{aligned}$$

Together with (3.1), one has

$$\begin{aligned}0 \leq & \Box_f(tG) = t\Box_f G + G \\ \leq & -\frac{tNb(p-1)^2}{p}\frac{w^p}{v^2} - \frac{(p-1)w^p}{v^2} - \Big(p(m+1)-1\Big)\kappa v^{m-1} w^{\frac{p}{2}} + G. \end{aligned} \tag{3.4}$$

When $0 \geq b \geq -\frac{p}{N(p-1)}$, $v = \frac{\gamma}{b}u^b < 0$, $G = \frac{w^{\frac{p}{2}}}{-v} > 0$, then we get the Hamilton type estimate from (3.4),

$$G \leq \frac{p}{(p-1)(Nb(p-1)+p)}\left(\frac{1}{t} + \kappa(p(m+1)-1)v_{Max}\right),$$

where $v_{Max} = \max_M v^m$.

4 Applications of Difference Harnack Estimates

Proof of Corollary 1.4 Let $\sigma(t)$ be a constant speed geodesic with $\sigma(t_1) = x_1$, $\sigma(t_2) = x_2$, and $p^* = \frac{p}{p-1}$. Combining Theorem 1.1 and the Young inequality, we obtain

$$\begin{aligned} v(x_2,t_2) - v(x_1,t_1) = & \int_{t_1}^{t_2} v_t + \langle \nabla v, \dot{\sigma}\rangle dt \\ \geq & \int_{t_1}^{t_2} \kappa v^{m+1} - \left(\frac{\bar{a}}{t} + (p-1)(m+1)\kappa \bar{a} v_{min}^m\right) v - \frac{1}{p^{\frac{1}{p-1}}p^*}\left(\frac{d(x_1,x_2)}{t_2-t_1}\right)^{p^*} dt. \end{aligned}$$

When $c > 0$, $q > 1$, we can deduce $m > 0$, then

$$\begin{aligned} & \int_{t_1}^{t_2} \kappa v^{m+1} - \left(\frac{\bar{a}}{t} + (p-1)(m+1)\kappa \bar{a} v_{min}^m\right) v - \frac{1}{p^{\frac{1}{p-1}}p^*}\left(\frac{d(x_1,x_2)}{t_2-t_1}\right)^{p^*} dt \\ \geq & -\bar{a} v_{max} \log\left(\frac{t_2}{t_1}\right) - \kappa v_{min}^m ((p-1)(m+1)\bar{a} v_{max} - v_{min})(t_2-t_1) - \frac{d(x_1,x_2)^{p^*}}{p^{\frac{1}{p-1}}p^*(t_2-t_1)^{\frac{1}{p-1}}}. \end{aligned}$$

When $c > 0$, $q < 1-b$, we can deduce $m+1 < 0$, then

$$\begin{aligned} & \int_{t_1}^{t_2} \kappa v^{m+1} - \left(\frac{\bar{a}}{t} + (p-1)(m+1)\kappa \bar{a} v_{min}^m\right) v - \frac{1}{p^{\frac{1}{p-1}}p^*}\left(\frac{d(x_1,x_2)}{t_2-t_1}\right)^{p^*} dt \\ \geq & -\bar{a} v_{max} \log\left(\frac{t_2}{t_1}\right) - \kappa v_{min}^{m+1} ((p-1)(m+1)\bar{a} - 1)(t_2-t_1) - \frac{d(x_1,x_2)^{p^*}}{p^{\frac{1}{p-1}}p^*(t_2-t_1)^{\frac{1}{p-1}}}. \end{aligned}$$

Proof of Corollary 1.5 Let $\sigma^*(t)$ be a shortest speed geodesic with $\sigma^*(0) = x_1$, $\sigma^*(1) = x_2$, such that $|\dot{\sigma^*}| = d(x_2,x_1)$, $p^* = \frac{p}{p-1}$. Combining Theorem 1.3 and the Young inequality, we have

$$\begin{aligned} \log\frac{v(x_1,t)}{v(x_2,t)} \leq & \int_0^1 \frac{|\nabla v| \cdot |\dot{\sigma^*}|}{-v(\sigma^*(s),t)} ds \\ \leq & \frac{1}{(p-1)(Nb(p-1)+p)}\left(\frac{1}{t} + \kappa(p(m+1)-1)v_{Max}\right) - \frac{d^{p^*}(x_1,x_2)}{p^* v_{max}}. \end{aligned}$$

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加权黎曼流形上加权非线性反应扩散方程的微分Harnack估计

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摘要: 本文研究了黎曼流形上的微分Harnack估计问题. 利用最大值原理和加权的p-Bochner公式的方法, 在$CD(0,N)$条件下, 获得了加权黎曼流形上加权非线性反应扩散方程的Li-Yau型和Hamilton型微分Harnack估计, 推广了作者在不加权时非负Ricci曲率条件下成立的结果.

关键词: 加权反应扩散方程; Li–Yau估计; Hamilton 估计; 曲率维数条件; Bochner公式

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