THE LEAST LAPLACIAN EIGENVALUE OF UNBALANCED SIGNED GRAPHS OF FIXED ORDER AND GIVEN NUMBER OF PENDANT VERTICES

WANG Sai [1,2], WONG Dein [2], TIAN Feng-lei [3]

$\big($ 1. Xuhai College, China University of Mining and Technology, Xuzhou 221116, China $\big)$

$\big($ 2.School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China $\big)$

$\big($ 3.School of Management, Qufu Normal University, Rizhao 276826, China $\big)$

Abstract: Signed graphs are graphs whose edges get signs ± 1 and, as for unsigned graphs, they can be studied by means of graph matrices. For a signed graph Γ we consider the Laplacian matrix defined as $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the matrix of vertex degrees of G and $A(\Gamma)$ is the signed adjacency matrix. It is well known that a connected graph Γ is balanced if and only if the least Laplacian eigenvalues $\lambda_n = 0$. Therefore, if a connected graph Γ is not balanced, then $\lambda_n > 0$. In this paper, we investigate how the least eigenvalue of the Laplacian of a signed graph changes by relocating a tree branch from one vertex to another. As an application, we determine the graph whose least laplacian eigenvalue attains the minimum among all connected unbalanced signed graphs of fixed order and given number of pendant vertices.

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1 Introduction

A signed graph Γ is a pair (G, σ), where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is a sign function on the edges of G. The unsigned graph G of Γ is called the underlying graph. Most of concepts defined for (unsigned) graphs are directly extendable to signed graphs. For example, the degree of a vertex u in G, denoted by d_u, is also its degree in Γ. A signed subgraph of (G, σ) is a subgraph of G with signature induced by σ. Thus if $u \in V(G)$, then $\Gamma - u$ denotes the signed subgraph having $G - u$ as the underlying graph, while its signature is the restriction form $E(G)$ of σ to $E(G - u)$. If $U \subset V(G)$ then $\Gamma[U]$ denotes the signed induced subgraph arising from U, while $\Gamma - U = \Gamma[V(G) \setminus U]$. We also write $\Gamma - \Gamma[U]$ instead of $\Gamma - U$. The order of Γ is the order of G and it is denoted by $|\Gamma|$. The cycle, the path, and the complete graph all on n vertices are denoted by C_n, P_n and

Biography: Wang Sai (1980–), male, born at Anqing, Anhui, lecturer, major in linear and multi-linear algebra. E-mail:27624181@qq.com

K_n, respectively. The graph G which contains only one vertex is called trivial. Otherwise, it is called nontrivial. The girth g of G the length of a shortest cycle contained in the graph G. A pendant vertex of G is a vertex of degree 1. A path $P : v_0v_1 \ldots v_{t-1}v_t$ in G is called a pendant path if $d(v_i) = 2, i = 1, 2, \ldots, t-1$ and $d(v_t) = 1$. If $t = 1$, then v_0v_1 is a pendant edge of G. If $d(v_0) \geq 3$, we say that P is a maximal pendant path.

Let C be a cycle in Γ, the sign of C is given by $\sigma(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (resp. -1) is called positive (resp. negative). Clearly, a cycle is positive if and only if it contains an even number of negative edges. A signed graph is balanced if all cycles are positive, otherwise it is unbalanced. If all edges in Γ are positive (negative), then Γ is denoted by $(G, +)$ (resp. $(G, -)$), and we say that such a signature is all-positive (resp., all-negative). For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G)\backslash U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G)\backslash U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) switching equivalent to Γ, and this is denoted by $\Gamma^U \sim \Gamma$. Switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent.

Signed graphs, like unsigned ones, can be studied by means of the eigenvalues of several matrices associated to graphs. The adjacency matirx $A(\Gamma) = (a_{ij})$ of Γ is naturally defined similarly to that of unsigned graphs by putting 1 or -1, whenever the corresponding edge is either positive or negative, respectively. The Laplacian matrix L is defined as $L : L(\Gamma) = D(G) - A(\Gamma)$, where D is the diagonal matrix of vertex degrees. It is worth mentioning that switching equivalent signed graphs have similar adjacency and Laplacian matrix. In fact, let $S_U = diag(s_1, s_2, \cdots, s_n)$ with $s_i = 1$ for each $i \in U \subset V(G)$, and $s_i = -1$ otherwise. We have

$$A(\Gamma) = S_U A(\Gamma^U) S_U \quad \text{and} \quad L(\Gamma) = S_U L(\Gamma^U) S_U.$$

Conversely, if $L(\Gamma_1)$ and $L(\Gamma_2)$ are similar by a diagonal matrix with its entries are 1 or -1, then Γ_1 and Γ_2 are switching equivalent. The characteristic polynomial $\phi_L(\Gamma, x) := \phi(\Gamma, x) = \det(xI - L(\Gamma))$ is called the Laplacian characteristic polynomial of Γ. The Laplacian eigenvalues $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) \geq 0$ of Γ of order n are all real numbers, because $L(\Gamma)$ is a real semi-defined symmetric matrix.

The least Laplacian eigenvalue has a special role in the spectral theory of signed Graphs. It is well-known that Γ is balanced, that is, $\Gamma = (G, \sigma)$ is switching equivalent to $(G, +)$, and $L(\Gamma)$ is similar to $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of the unsigned graph G, if and only if $\lambda(\Gamma) : \lambda_n(\Gamma) = 0$. Therefore, if Γ is not balanced, then $\lambda_n > 0$ and λ has been shown to be a very good measure of the graph frustration, that is the smallest number of vertices to be deleted form Γ in order to get a balanced signed graph(see [1,2]). The study on the signed graph has attracted a lot of attention. One can refer to [3] for basic results on graph spectra , to [4–7] for basic results on the spectra of signed graphs, to [8] for a possibly complete bibliography on signed graphs, and to [9] for a glossary of terms related to signed graphs. Recently, Francesco Belardo and others studied the least Laplacian

eigenvalue of signed graphs. He showed that among unbalanced connected signed graphs of given order the least eigenvalue is minimal for an unbalanced triangle with a hanging path, while the least eigenvalue is maximal for the complete graph with the all-negative sign function, and among the unbalanced bicyclic signed graphs of given order $n \geq 5$ the least laplacian eigenvalue is minimal for signed graphs consisting of two triangles, only one of which is unbalanced, connected by a path, in [10] and [11] respectively.

In this paper, we investigate how the least eigenvalue of the Laplacian of a signed graph changes by relocating a tree branch from one vertex to another vertex. As an application, we determine the graph whose least laplacian eigenvalue attains the minimum among all connected unbalanced signed graphs of fixed order and given number of pendant vertices.

2　Main Results

We begin with some fundamental results which will be used later. For a vector $x = (x_1, x_2, \cdots, x_n)^T$ of $L(\Gamma)$, we denote by x_u the entry of x at $u \in V(G)$. If x is an eigenvector of $L(\Gamma) = L$ with respect to an eigenvalue λ, then we have eigenvalue equation

$$\lambda x_u = d_u x_u - \sum_{v \sim u} \sigma(uv) x_v \ \ \text{for} \ \ u \in V(G),$$

where $v \sim u$ means that v is adjacent to u.

The quadratic form $\langle L(\Gamma), x \rangle$ can be respected as

$$\langle L(\Gamma), x \rangle = x^T L(\Gamma) x = \sum_{uv \in E(G)} (x_u - \sigma(uv) x_v)^2.$$

Since $L(\Gamma)$ is a real symmetric matrix, by Rayleigh principle, it follows that

$$\lambda(\Gamma) = \lambda_n(\Gamma) := \lambda_n(L) = \min_{\|x\|=1} \langle L(\Gamma), x \rangle.$$

We need to introduce an operation on graphs as the following.

Coalescence　Let Γ_1, Γ_2 be two vertex-disjoint signed graphs, and let $v_1 \in V(\Gamma_1)$, $v_2 \in V(\Gamma_2)$. The coalescence of Γ_1, Γ_2, denoted by $\Gamma_1(v_1) \circ \Gamma_2(v_2)$, is obtained from Γ_1, Γ_2 by identifying v_1 with v_2 and forming a new vertex u. The graph $\Gamma_1(v_1) \circ \Gamma_2(v_2)$ is also written as $\Gamma_1(u) \circ \Gamma_2(u)$. Suppose Γ is a connected graph and can be expressed in the form $\Gamma = \Gamma_1(v_1) \circ \Gamma_2(v_2)$, where Γ_1 and Γ_2 are both nontrivial and connected, then Γ_1 is called a branch of Γ with root u. Let x be a vector defined on Γ. A branch of Γ is called zero branch with respect to x if $x_v = 0$ for all v in it.

Lemma 2.1　Let $\lambda(\Gamma)$ be the least laplacian eigenvalue of $\Gamma(G, \sigma)$ with corresponding unit eigenvector $x = (x_1, x_2, \cdots, x_n)^T$. If uv is a bridge, then $\sigma(uv) x_u x_v \geq 0$.

Proof　Suppose uv is a bridge, then G is a connected sum of two graphs G_1 and G_2 where $V(G) = V(G_1) \cup V(G_2)$ and $E(G)$ differs from $E(G) = E(G_1) \cup E(G_2)$ by addition of a single edge joining the vertex u of G_1 to the vertex v of G_2. If $\sigma(uv) = 1$, suppose

$x_u x_v < 0$, let y be a valuation of Γ by

$$
\begin{cases}
y_w = x_w, & w \in V(G_1); \\
y_w = -x_w, & w \in V(G_2).
\end{cases}
$$

We have

$$
\begin{aligned}
\langle L(\Gamma), y \rangle &= (x_u + x_v)^2 + \sum_{ws \in E(G_1)} (x_w - \sigma(ws) x_s)^2 + \sum_{ws \in E(G_2)} [-x_w - \sigma(ws)(-x_s)]^2 \\
&< (|x_u| + |x_v|)^2 + \sum_{ws \in E(G_1)} (x_w - \sigma(ws) x_s)^2 + \sum_{ws \in E(G_2)} (x_w - \sigma(ws) x_s)^2 \\
&= (|x_u| + |x_v|)^2 + \sum_{ws \in E(G) \backslash uv} (x_w - \sigma(ws) x_s)^2 \\
&= \lambda,
\end{aligned}
$$

contradiction. Proof of the case $\sigma(uv) = -1$ is similar, here omitted.

Next, we will give some properties of the eigenvector corresponding to the least laplacian eigenvalue of signed graphs. Note that the signature is not relevant in trees and in general for edges which are bridges, for the sake of simplicity, we will put a positive sign on bridges, unless otherwise stated.

Lemma 2.2 Let Γ be a signed graph of order n with minimum degree δ, then $\lambda(\Gamma) < \delta$. If G contains a pendant vertex, then $\lambda(\Gamma) < 1$.

Proof Let $x = (x_1, x_2, \ldots, x_n)^T$ be an any nonzero unit vector in $\mathbb{R}^n$. We have

$$
\lambda(\Gamma) \leq \langle L(\Gamma), x \rangle = \sum_{uv \in E(G)} (x_u - \sigma(uv) x_v)^2.
$$

Putting $x = (0, 0, \ldots, 1)^T$, we get that $\lambda \leq \delta$. We can easily see $x = (0, 0, \ldots, 1)^T$ is not an eigenvector corresponding to $\lambda(\Gamma)$, the inequality is strict.

Lemma 2.3 Let Γ be a signed graph which contains a tree branch T with root u. Let x be a unit eigenvector corresponding to the least laplacian eigenvalue $\lambda(\Gamma)$.

(i) $x_p x_q \geq 0$, for every $p, q \in T$.

(ii) Assume that there is a vertex $v \in T$ such that $x_v = 0$. Then $x_w = 0$ for every $w \in T$.

(iii) Let Γ be an unbalanced signed graph and $x_u \neq 0$, and let $P = v_0(u) v_1 \ldots v_k$ be a path on T, then $|x_{v_0}| = |x_u| < |x_{v_1}| < \cdots < |x_{v_k}|$.

Proof Note that all edges in T are bridges, by lemma 2.1 , (i) is proved. To prove the result (2), we consider two following cases.

Case 1 $v \in T - u$. Considering the entry of $L(\Gamma) x$ at v, we have $\lambda x_v = d_v x_v - \sum_{w \sim v} x_w$. By (i), the result holds.

Case 2 $v = u$. Let y be a valuation of Γ by

$$
\begin{cases}
y_w = x_w, & w \in \Gamma \backslash T; \\
y_w = -x_w, & w \in T.
\end{cases}
$$

Note that $\|y\| = 1$, and $\langle L(\Gamma), y \rangle = \langle L(\Gamma), x \rangle$. So y is also a eigenvector corresponding to $\lambda(\Gamma)$. Considering the entry of $L(\Gamma)x$ and $L(\Gamma)y$ at u, respectively. We have

$$\sum_{w \sim u, w \in \Gamma \setminus T} \sigma(uw) x_w + \sum_{w \in T-u} x_w = 0, \qquad \sum_{w \sim u, w \in \Gamma \setminus T} \sigma(uw) x_w + \sum_{w \in T-u} (-x_w) = 0.$$

Thus, we have $\sum\limits_{w \in T-u} x_w = 0$, and hence $x_w = 0$ for for every $w \in T$.

Next, we give the proof of (iii). By Lemma 2.3, for a connected unbalanced signed graph Γ containing a pendant vertex, we have $0 < \lambda(\Gamma) < 1$. Assume that v_k is a pendant vertex. Since $x_u \neq 0$, by (ii) we drive that all $x_{v_i} \neq 0$. Considering the eigenvector equation of x at v_k, we have

$$(\lambda(\Gamma) - 1) x_{v_k} = -x_{v_{k-1}}.$$

The result holds. Now suppose v_i is not a pendant vertex. By the eigenvector of x at v_i,

$$\lambda x_{v_i} = d_{v_i} x_{v_i} - \sum_{w \sim v_i} x_w,$$

by (i), we have

$$d_{v_i} |x_{v_i}| = \lambda(\Gamma) |x_{v_i}| + |x_{v_{i-1}}| + \sum_{w \sim v_i, w \neq v_{i-1}} |x_w|,$$

which leads to, by induction,

$$d_{v_i} |x_{v_i}| > |x_{v_{i-1}}| + \sum_{w \sim v_i, w \neq v_{i-1}} |x_w| > |x_{v_{i-1}}| + (d_{v_i} - 1) |x_{v_i}|.$$

Hence $|x_{v_i}| > |x_{v_{i-1}}|$.

Next, we will discuss how the least laplacian eigenvalue changes when relocating a T branch for one vertex to another.
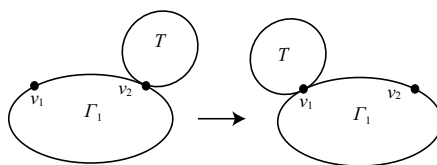

Figure 1   Relocating T from v_2 to v_1.

Lemma2.4   Let $\Gamma = \Gamma_1(v_2) \circ T(u)$ and $\Gamma^\dagger = \Gamma_1(v_1) \circ T(u)$ be two graphs as depicted in Fig.1, where v_1, v_2 are two distinct vertices of Γ_1 and u is a vertex of T. Let x be a unit eigenvector corresponding to the least laplacian eigenvalue $\lambda(\Gamma)$. If $|x_{v_1}| \geq |x_{v_2}|$, then

$$\lambda(\Gamma^\dagger) \leq \lambda(\Gamma). \tag{2.1}$$

Furthermore, if $|x_{v_1}| > |x_{v_2}|$ or $|x_{v_1}| = |x_{v_2}| > 0$, the inequality is strict.

Proof Assume that $x_{v_1} \geq 0$. Let y be a valuation of $\Gamma^{\dagger}$ by

$$
\begin{cases}
y_w = x_w, & w \in \Gamma_1; \\
y_w = |x_w| + x_{v_1} - |x_{v_2}|, & w \in T - u.
\end{cases}
$$

Then for each edge $wv_2 \in E(\Gamma)$ contained in T, we have

$$
(x_w - x_{v_2})^2 = (|x_w| - |x_{v_2}|)^2 = (|x_w| + x_{v_1} - |x_{v_2}| - x_{v_1})^2 = (y_w - y_{v_1})^2.
$$

For each edge $vw \in E(T - u)$ and $vw \in E(\Gamma_1)$, we have

$$
(x_w - \sigma(wv)x_v)^2 = (y_w - \sigma(wv)x_v)^2.
$$

Hence we have

$$
\langle L(\Gamma^{\dagger}), y \rangle = \sum_{vw \in E(\Gamma^{\dagger})} (y_w - \sigma(wv)y_v)^2 = \sum_{vw \in E(\Gamma)} (y_w - \sigma(wv)y_v)^2 = \langle L(\Gamma), x \rangle.
$$

By $|x_{v_1}| \geq |x_{v_2}|$, we have

$$
\|y\|^2 = \sum_{v \in V(\Gamma^{\dagger})} y_v^2 \geq \sum_{v \in V(\Gamma)} x_v^2 = 1.
$$

All results lead to

$$
\lambda(\Gamma^{\dagger}) \leq \frac{\langle L(\Gamma^{\dagger}), y \rangle}{\|y\|^2} \leq \langle L(\Gamma^{\dagger}), y \rangle = \langle L(\Gamma), x \rangle = \lambda(\Gamma).
$$

Suppose that the equality holds, then $|x_{v_1}| = |x_{v_2}|$ and y is an eigenvector of $\lambda(\Gamma^{\dagger})$. Considering the eigenvector equations of x and y, respectively, both at v_2, we have

$$
d_u^T x_u = \sum_{v \sim u, v \in T - u} x_v, \tag{2.2}
$$

where $d_u^T = |\{v : v \sim u, v \in T - u\}|$.

If $|x_{v_1}| > |x_{v_2}|$, the inequality (2.1) is strict. So we suppose $|x_{v_1}| = |x_{v_2}| > 0$. If $\lambda(\Gamma^{\dagger}) = \lambda(\Gamma)$, by (2.2) and lemma 2.3, we will have a contradiction. So inequality is also strict. The proof is completed.

We denote the class of unbalanced unicyclic signed graphs of order n with girth g and $k \geq 1$ pendant vertices by $\mathcal{U}_n^k(g, \overline{\sigma})$. In order to simplify the notation, we choose to represent any graph Γ in $\mathcal{U}_n^k(g, \overline{\sigma})$ with all positive edges other than the edge pq which is lying on the unique cycle of Γ. Denote by $U_n^k(g, \overline{\sigma}, l; l_1, l_2, \cdots, l_k) \in \mathcal{U}_n^k(g, \overline{\sigma})$ the graph of order n obtained by coalescing P_l with a cycle C_g by identifying one of its end vertices with some vertex of C_g, and also coalescing this P_l with each of paths $P_{l_i}(i = 1, 2, \ldots, k)$ by identifying its other end vertex with one of the vertices of P_{l_i}, where $l \geq 1, l_i \geq 2(i = 1, 2, \ldots, k)$. If $l_1 = l_2 = \cdots = l_k = 2$, $U_n^k(g, \overline{\sigma}, l; l_1, l_2, \cdots, l_k)$ is denoted by $U_n^k(g)$ (up to switching
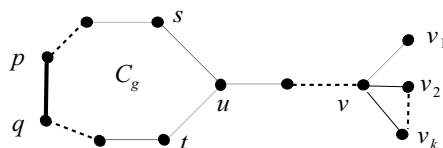
Figure 2 $U_n^k(g)$.

equivalence, the graph is unique, see Fig 2, thin and thick edges represent positive and negative edges, respectively).

To achieve the goal, some useful results will be introduced. For more details, one can refer to [10].

Lemma 2.5 (Theorem 2.2, [10]) Let $\Gamma = (G, \sigma)$ be a signed graph, and let Γ' be obtained from Γ by deleting the edge $rs \in E(G)$ with sign $\sigma(rs)$ and by adding the edge $rt \in E(G)$ with sign $\sigma(rt)$. Let $x = (x_1, x_2, \cdots, x_n)^T$ be an eigenvector related to $\lambda(\Gamma)$. If

$$(\sigma(rt) x_t - \sigma(rs) x_s)(\sigma(rt) x_t + \sigma(rs) x_s - 2x_r) \leq 0,$$

then $\lambda(\Gamma') \leq \lambda(\Gamma)$. Furthermore, $\lambda(\Gamma') = \lambda(\Gamma)$ if and only if $x_r = \sigma(rs) x_s = \sigma(rt) x_t$.

Proposition 2.6 (Proposition 2.3, [10]) Let Γ be a signed graph and Γ' switching equivalent via the state matrix S. If x is an eigenvector of λ for Γ, then Sx is the corresponding eigenvector of λ for Γ'.

Lemma 2.7 (Lemma 4.4, [10]) Let $\Gamma \in \mathcal{U}_n^k(g, \overline{\sigma})$ and $x = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to $\lambda(\Gamma)$. If σ is taken such that all edges are positive with the exception of the pq which minimizes $|x_p x_q|$, then x can be chosen such that

(1) $x_v \geq 0$ for all $v \in \Gamma$.

(2) if $x_p x_q = 0$ then either x_p or x_q is non-zero.

(3) if $x_p x_q > 0$ then $x_v > 0$ for all $v \in \Gamma$.

In view of the above results and the proof Lemma 4.2 in [10], it deserves to be mentioned that state matrix is a diagonal matrix with values in $\{1, -1\}$, then the eigenvectors after switching just change in the signs of the components corresponding to the switched vertices. The moduli of the eigenvector components are invariant with respect to switching. Let pq be the edge which minimizes $|x_p x_q|$, then the corresponding eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ to $\lambda(\Gamma)$, for any $\Gamma \in \mathcal{U}_n^k(g, \overline{\sigma})$, satisfied

(1) there exists at most one component of x on the cycle is zero.

(2) if $|x_p x_q| > 0$. then $|x_v| > 0$ for all $v \in \Gamma$.

Theorem 2.8   Among all graphs in $\mathcal{U}_n^k(g, \overline{\sigma})$, the unique, up to switching equivalence, which minimizes the least Laplacian eigenvalue is $U_n^k(g)$.

Proof   Let Γ be the signed graph which minimizes the least Laplacian eigenvalue in $\mathcal{U}_n^k(g, \overline{\sigma})$, x be a unit eigenvector corresponding to $\lambda(\Gamma)$ and C_g be the unique cycle of Γ. Γ can be considered from C_g by identifying each $w \in C_g$ with one vertex of some tree T_w of order n_w, where $\sum\limits_{w \in C_g} n_w = n$. Note that if T_w is trivial, $n_w = 1$.

We assert that each nontrivial tree T_w is a nonzero branch with respect to x.

Otherwise, by lemma 2.3 (ii), there exists a nontrivial tree T_w attached at w, $w \in C_g$, such that $x_w = 0$. By lemma 2.4, relocating the tree T_w from w to v for which $|x_v| \neq 0$, we obtain a signed graph in $\mathcal{U}_n^k(g, \overline{\sigma})$ with smaller least Laplacian eigenvalue.

T_p and T_q are trivial.

If $x_p = 0$, $x_q \neq 0$, as above the proof T_p is trivial. Let s be another vertex adjacent to p, considering the the eigenvector equation at p, we have $\lambda x_p = 2x_p + x_q - x_s$, which leads to $x_q = x_s$. If T_q is nontrivial, relocating the tree T_q from q to s for which $|x_q| = |x_s| > 0$, we obtain a signed graph in $\mathcal{U}_n^k(g, \overline{\sigma})$ with smaller least Laplacian eigenvalue, contradiction. If $|x_p x_q| > 0$, by its minimum variance, there exits a vertex w on the cycle satisfied $|x_w| \geq \max\{|x_p|, |x_q|\}$, say $|x_p|$. Relocating the tree T_p from p to w, we obtain a signed graph in $\mathcal{U}_n^k(g, \overline{\sigma})$ with smaller least Laplacian eigenvalue, contradiction.

All maximal pendant paths are attached at the same vertex u.

Otherwise, there exist two maximal pendant paths, say P and P', attached at p and p', respectively. Since P' is maximal pendant path, we have $d(p') \geq 3$. Assume that $|x_p| \geq |x_{p'}| > 0$. Then, by lemma 2.4, we will arrive at a new graph still in $\mathcal{U}_n^k(g, \overline{\sigma})$ but with smaller least Laplacian eigenvalue by relocating P' from p' to p.

Therefore, Γ, up to switching equivalence, is obtained from a unbalanced signed C_g by attaching one path at some vertex of C_g, or $\Gamma = U_n^k(g, \overline{\sigma}, l; l_1, l_2, \cdots, l_k)$. Next we only consider the case $k \geq 2$ and show that $l_1 = l_2 = \cdots = l_k = 2$. Suppose that $l_i \geq 3$, for some i. Let $P_{l_i} = (u)v_1 \ldots v_{l_i}$, by lemma 2.3 and the above discussion, $0 < |x_{v_1}| < |x_{v_{l_i-1}}|$. Relocating some P_{l_j} other than P_{l_i} from v_1 to v_{l_i-1}, by lemma 2.4, a new graph with smaller least Laplacian eigenvalue in $\mathcal{U}_n^k(g, \overline{\sigma})$ is obtained, contradiction.

In the sequel，the signature such that the sole negative edge corresponds to the cycle edge pq minimizing the product of the least eigenvector components (on the cycle) is denoted by $U_n^k(\overline{g})$. Furthermore, $\lambda(U_n^k(\overline{g}))$ has a nonnegative eigenvector. Without loss of generality, we take $x_p \leq x_q$, so that $x_v > 0$ for all $v \in U_n^k(\overline{g}) - p$.

Lemma 2.9 With respect to $g \geq 3$ and $k \geq 1$, we have

$$\lambda\left(U_n^{k-1}(g)\right) < \lambda\left(U_n^k(g)\right), \quad \lambda\left(U_n^k(g-1)\right) < \lambda\left(U_n^k(g)\right).$$

Proof The graph $U_n^k(\overline{g})$ the sole negative edge corresponds to the cycle edge pq minimizing the product of the least eigenvector components is depicted in Fig 2. From the proof of Theorem 2.8, we have T_p and T_q are trivial.

Let x be a unit eigenvalue corresponding to $\lambda\left(U_n^k(\overline{g})\right)$. Replacing the edge vv_1 by v_1v_2, we arrive at a new graph $\widetilde{\Gamma} \in \mathcal{U}_n^k(g, \overline{\sigma})$, and by lemma 2.4, (note that $|x_v| < |x_{v_1}|$), we have $\lambda\left(\widetilde{\Gamma}\right) < \lambda\left(U_n^k(\overline{g})\right)$. So, by Theorem 2.8, we have

$$\lambda\left(U_n^{k-1}(g)\right) \leq \lambda\left(\widetilde{\Gamma}\right) < \lambda\left(U_n^k(\overline{g})\right) = \lambda\left(U_n^k(g)\right).$$

For the second result, $g \geq 4$, we assert that $x_u > x_s \geq x_t$ or $x_u > x_t \geq x_s$. Otherwise, relocating T_u from u to s or t, we arrive at a switching equivalent graph with smaller the least

Laplacian eigenvalue, contradiction. We consider the case $x_u > x_t \geq x_s$. Deleting the edge su and adding the edge st, we arrive at a new graph $U_n^k(g-1)$, and $(x_t - x_u)(x_t + x_u - 2x_s) < 0$. So we have

$$\lambda\left(U_n^k(g-1)\right) < \lambda\left(U_n^k(\overline{g})\right) = \lambda\left(U_n^k(g)\right).$$

Theorem 2.10　(Interlacing [14]) Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - e$ be the signed graph obtained from Γ by deleting the edge e. Then

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma - e) \geq \lambda_2(\Gamma - e) \geq \cdots \geq \lambda_n(\Gamma) \geq \lambda_n(\Gamma - e).$$

The main result of this section is as the following.

Theorem 2.11　Among all unbalanced signed connected graphs of order n and $k \geq 1$ pendant vertices, the unique, up to switching equivalence, which minimizes the least Laplacian eigenvalue is $U_n^k(3)$.

Proof　Let Γ be a graph which minimizes the least Laplacian eigenvalue. Since Γ is unbalanced, then Γ contains at least an induced negative cycle, say C_g. Let Γ' be a connected unicyclic spanning subgraph of Γ, which contains the unique cycle C_g and all pendant edges of Γ. The signed graph $\widehat{\Gamma} \in \mathcal{U}_n^{k'}(g, \overline{\sigma})$ is switching equivalent to Γ', $k' \geq k$. By lemmas above, we have

$$\lambda\left(U_n^k(3)\right) \leq \lambda\left(U_n^k(g)\right) \leq \lambda\left(U_n^{k'}(g)\right) \leq \lambda(\widehat{\Gamma}) = \lambda(\Gamma') \leq \lambda(\Gamma). \tag{2.3}$$

So $U_n^k(3)$ is a signed graph, which minimizes the the least Laplacian eigenvalue.

Next, we will prove that the minimizer among connected unbalanced graphs of order n and $k \geq 1$ pendant vertices is just $U_n^k(3)$. By lemma 2.9 and Theorem 2.8, $\lambda(\Gamma) = \lambda(\Gamma') = \lambda\left(\widehat{\Gamma}\right) = \lambda\left(U_n^k(3)\right)$ implies $k = k'$, $g = 3$ and Γ' is switching equivalent to $U_n^k(3)$, and denoted by $U_n^k(3')$. So Γ can be obtained form $U_n^k(3')$ by adding edges. By the definition of Γ', it suffices to derive a contradiction when $\Gamma = U_n^k(3') + uv$, where uv is an edge joining a vertex of C_3 and a vertex of P_l, or two vertices of P_l. Let $x = (x_1, x_2, \cdots, x_n)^T$ be a unit eigenvector corresponding to $\lambda(\Gamma)$. Then

$$\begin{aligned} \lambda(\Gamma) = \langle L(\Gamma), x \rangle &= \langle L\left(U_n^k(3')\right), x \rangle + (x_u - \sigma(uv) x_v)^2 \\ &\geq \lambda\left(U_n^k(3')\right) + (x_u - \sigma(uv) x_v)^2 \ \geq \lambda\left(U_n^k(3')\right). \end{aligned}$$

Since $\lambda(\Gamma) = \lambda\left(U_n^k(3')\right)$, and x is an eigenvector for $\lambda\left(U_n^k(3')\right)$, we have that $x_u - \sigma(uv) x_v = 0$, namely $|x_u| = |x_v|$. By Proposition 3.5, there exists a state matrix S satisfied the vector $x' = Sx = (x_1', x_2', \cdots, x_n')^T$ belonging to $\lambda\left(U_n^k(3)\right)$. Hence $|x_u'| = |x_v'|$. By Lemma 2.3, Lemma 2.7 and the proof of Lemma 2.9, contradiction, and we are done.

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给定悬挂点的非平衡符号图的最小拉普拉斯特征值

汪赛[1,2], 王登银[2], 田凤雷[3]

(1.中国矿业大学徐海学院, 江苏 徐州 221116)

(2.中国矿业大学数学学院, 江苏 徐州 221116)

(3.曲阜师范大学管理学院, 山东 日照 276826)

摘要: 符号图是边赋值为±1 的一类图. 设符号图Γ的拉普拉斯矩阵为$L(\Gamma) = D(G) - A(\Gamma)$，这里$D(G)$表示度矩阵, $A(\Gamma)$ 表示符号图的邻接矩阵. Γ是平衡的当且仅当最小拉普拉斯特征值$\lambda_n = 0$. 因此当Γ非平衡时$\lambda_n > 0$. 本文研究了非平衡符号图的最小拉普拉斯特征值问题. 利用图特征值的嫁接方法, 获得了给定悬挂点非平衡符号图的最小拉普拉斯特征值, 并且刻画了达到最小特征值的极图.

关键词: 符号图; 拉普拉斯; 最小特征值

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