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SYNCHRONIZATION OF COMPLEX NETWORKS VIA A SIMPLE AND ECONOMICAL FIXED-TIME CONTROLLER

LI Na[1], WU Xiao-Qun[2]

(*1. School of Mathematics and Statistics, Henan University, Kaifeng 475004, China*)

(*2. School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*)

1 Introduction and Main Results

Consider the following differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)),\ \mathbf{x}(0) = \mathbf{x}_0, \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ denotes the state variable of system (1.1), $\mathbf{f}:\ \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field and $\mathbf{x}_0$ is the initial value of the system.

Definition 1.1 [1] The origin of system (1.1) is said to be globally finite-time stable if for any of its solution $\mathbf{x}(t, \mathbf{x}_0)$, the following statements hold:
(i) Lyapunov stability: for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\|\mathbf{x}(t, \mathbf{x}_0)\| < \varepsilon$ for any $\|\mathbf{x}_0\| \le \delta$ and $t \ge 0$.
(ii) Finite-time convergence: there exists a function $T:\ \mathbb{R}^n\backslash\{0\} \to (0, +\infty)$, called the settling time function, such that $\lim_{t\to T(\mathbf{x}_0)} \mathbf{x}(t, \mathbf{x}_0) = 0$ and $\mathbf{x}(t, \mathbf{x}_0) \equiv 0$ for all $t \ge T(\mathbf{x}_0)$.

Definition 1.2 [2] The origin of system (1.1) is said to be fixed-time stable if it is globally finite-time stable and the settling time function $T(\mathbf{x}_0)$ is bounded for any $\mathbf{x}_0 \in \mathbb{R}^n$, i.e., there exists $T_{\max}$ such that $T(\mathbf{x}_0) \le T_{\max}$ for all $\mathbf{x}_0 \in \mathbb{R}^n$.

Lemma 1.3 [2] If there exists a continuous, positive definite and radically unbounded function $V(\mathbf{x}(t)):\ \mathbb{R}^n \to \mathbb{R}$ such that any solution $\mathbf{x}(t)$ of system (1.1) satisfies the inequality

$$\frac{d}{dt}V(\mathbf{x}(t)) \le -\Big(aV^{\delta}(\mathbf{x}(t)) + bV^{\theta}(\mathbf{x}(t))\Big)^k$$

for $a,\ b,\ \delta,\ \theta,\ k > 0$ and $\delta k > 1,\ \theta k < 1$, then the origin of system (1.1) is fixed-time stable, and the settling time $T(\mathbf{x}_0)$ is upper bounded and satisfies

$$T(\mathbf{x}_0) \le T_1 = \frac{1}{a^k(\delta k - 1)} + \frac{1}{b^k(1 - \theta k)}. \tag{1.2}$$

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Biography: Li Na(1990–), female, born at sanmenxia, Henan, major in synchronization and control of complex networks. E-mail:lina19900305@163.com.

Lemma 1.4 If there exists a continuous, positive definite and radically unbounded function $V(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}$ such that any solution $\mathbf{x}(t)$ of system (1.1) satisfies the inequality

$$\frac{d}{dt}V(\mathbf{x}(t)) \leq -\mu(V(\mathbf{x}(t)))^{\gamma+\mathrm{sign}(V(\mathbf{x}(t))-1)}, \tag{1.3}$$

in which $\mu > 0,\ 1 \leq \gamma < 2$, then the origin of system (1.1) is globally fixed-time stable. In addition, for any initial state $\mathbf{x}_0$ of system (1.1), the settling time is described as

$$T_3 = \frac{1}{\mu(2-\gamma)} + \frac{1}{\mu\gamma}. \tag{1.4}$$

Remark 1 There have been plentiful literature investigating fixed-time synchronization in recent years. Lemma 1.3 is the most common one to prove synchronization within a settling time [2]–[3]. However, the special case of $k = 1$ in lemma 1.3 is commonly applied to simplify the controller to be designed. In this situation, the form of Lemma 1.4 is much simpler than that in Lemma 1.3. Furthermore, the estimated settling time is more accurate than those in existing literature.

Consider a nonlinearly coupled complex network consisting of N nodes described by

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + c\sum_{j=1}^{N} a_{ij}(\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{g}(\mathbf{x}_i(t))),\ i = 1, \ldots, N,$$

where $\mathbf{x}_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^\top$ denotes the state vector of node i, nonlinear function $\mathbf{f}(\mathbf{x}_i(t)) = (f_1(\mathbf{x}_i(t)), \ldots, f_n(\mathbf{x}_i(t)))^\top$ represents the dynamical behavior of the i-th node, $c > 0$ is the coupling gain. $\mathbf{g}(\mathbf{x}_j(t)) = (g_1(x_{j1}(t)), \ldots, g_n(x_{jn}(t)))^\top \in \mathbb{R}^n$ is the nonlinear coupling function. Besides, $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ denotes the outer coupling matrix, in which, $a_{ij} > 0$ if the i-th node can receive the information from node j; otherwise, $a_{ij} = 0$. In addition, $a_{ii} = 0$. The Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ is defined as $l_{ii} = \sum\limits_{j=1,j\neq i}^{N} a_{ij}$, and the off-diagonal elements $l_{ij} = -a_{ij}$. Therefore, the controlled complex network can be written as

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) - c\sum_{j=1}^{N} l_{ij}\mathbf{g}(\mathbf{x}_j(t)) + \mathbf{u}_i(t),\ i = 1, \ldots, N. \tag{1.5}$$

In this paper, complex network (1.5) is assumed to be symmetrical, the initial value is $\mathbf{x}_i(0) = \mathbf{x}_{i0},\ i = 1, \ldots, N$, and complex network (1.5) is supposed to synchronize to the same state $\mathbf{s}(t)$ satisfying

$$\dot{\mathbf{s}}(t) = \mathbf{f}(\mathbf{s}(t)), \tag{1.6}$$

in which $\mathbf{s}(t) = (s_1(t), \ldots, s_n(t))^\top$ with initial value $\mathbf{s}(0) = \mathbf{s}_0$.

Let $\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$ be the error state, then one can get

$$\dot{\mathbf{e}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{s}(t)) - c\sum_{j=1}^{N} l_{ij}(\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{g}(\mathbf{s}(t))) + \mathbf{u}_i(t),\ i = 1, \ldots, N. \tag{1.7}$$

In order to proceed further analysis, the controller is designed as

$$\mathbf{u}_i(t) = -\,k\mathbf{e}_i(t) - \mu\text{sgn}(\mathbf{e}_i(t))|\mathbf{e}_i(t)|^{2\gamma+2\text{sign}(\|\mathbf{e}(t)\|_2^2-1)-1},\ i = 1,\ldots,N, \tag{1.8}$$

where the notation $\mathbf{e}(t) = (\mathbf{e}_1^\top(t),\ldots,\mathbf{e}_N^\top(t))^\top$, $\text{sgn}(\mathbf{e}_i(t)) = \text{diag}\{\text{sign}(e_{i1}(t)),\ldots,\text{sign}(e_{in}(t))\}$, and $|\mathbf{e}_i(t)|^\varsigma = (|e_{i1}(t)|^\varsigma,\ldots,|e_{in}(t)|^\varsigma)^\top$, $\varsigma = 2\gamma + 2\text{sign}(\|\mathbf{e}(t)\|_2^2 - 1) - 1$. Here $\gamma \geq \frac{3}{2}$ is required to guarantee that controller (1.8) is meaningful, thus $\frac{3}{2} \leq \gamma < 2$. Besides, $\mu > 0$, the feedback gains $k > 0$ can be determined later.

Remark 2 In existing literature, the controller is usually designed as "$\mathbf{u}_i(t) = -k\mathbf{e}_i(t) - b\text{sgn}(\mathbf{e}_i(t))|\mathbf{e}_i(t)|^\alpha - c\text{sgn}(\mathbf{e}_i(t))|\mathbf{e}_i(t)|^\beta$", which consists of three terms: the first term is the linear feedback term, the index of the second term satisfies $0 < \alpha < 1$, and that of the last one satisfies $\beta > 1$. Controller (1.8) proposed here only contains two terms, which is obviously more economical and practical.

(H_1) The dynamical function $\mathbf{f}(\cdot)$ of complex network (1.5) satisfies the usual Lipschitz condition. That is, for $\forall \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$, there exists a positive constant ρ such that

$$(\mathbf{u}-\mathbf{v})^\top(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})) \leq \rho(\mathbf{u}-\mathbf{v})^\top(\mathbf{u}-\mathbf{v}).$$

(H_2) As for the nonlinear function $g_k(\cdot)$, $k = 1,\ldots,n$, we assume that there exist positive constants ϱ_k, such that for all $x,\ y \in \mathbb{R}$, we have $\dfrac{g_k(x) - g_k(y)}{x-y} \geq \varrho_k$.

Theorem 1.5 Suppose that $(H_1) - (H_2)$ holds. Then complex network (1.5) and target system (1.6) will reach fixed-time synchronization with controller (1.8), if there exist positive constants k such that

$$(\rho - k)I_{Nn} - c(L \otimes \Psi) < 0, \tag{1.9}$$

where $\Psi = \text{diag}\{\varrho_1,\ldots,\varrho_n\}$, ρ and ϱ_k, $k = 1,\ldots,n$ presented in (H_1) and (H_2), respectively. Furthermore, the settling time is estimated as $T_5 = \dfrac{1}{2\mu(2-\gamma)} + \dfrac{1}{2\mu(Nn)^{-\gamma}\gamma}$.

Remark 3 From Theorem 1.5, condition (1.9) can be guaranteed if $k \geq \rho$ based on the non-negative eigenvalues of matrix L.

2 Numerical Example

Example 1 Theorem 1.5 is verified in this section. Consider a network consisting of 10 nodes, with the dynamics of each node being described by a three-dimensional complex network [4], where the inner coupling matrix $\Gamma = I_3$, dynamical function $\mathbf{f}(\mathbf{x}_i(t)) = -C\mathbf{x}_i(t) + M\mathbf{h}(\mathbf{x}_i(t))$, $\mathbf{h}(\mathbf{x}_i(t)) = 0.5(|x_{i1}+1| - |x_{i1}-1|, |x_{i2}+1| - |x_{i2}-1|, |x_{i3}+1| - |x_{i3}-1|)^\top$, and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1.0 \end{pmatrix},$$

$$L=\begin{pmatrix} 5 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 5 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 6 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 6 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 6 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 5 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 5 \end{pmatrix},$$

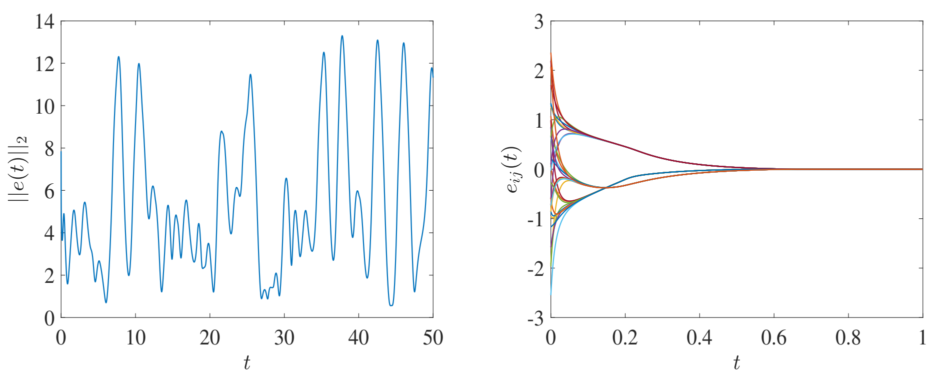
nonlinearly coupled function is defined as

$$\tilde{g}(\mathbf{v}) = \begin{cases} 2v_1 + 0.2\sin v_1, \\ 3v_2 + 0.5\cos v_2, \\ 2v_3 + 0.3\sin v_3. \end{cases}$$

Based on $\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$, the controlled error system is

$$\dot{\mathbf{e}}_i(t) = -C\mathbf{e}_i(t) + M(\mathbf{h}(\mathbf{x}_i(t)) - \mathbf{h}(\mathbf{s}(t))) - c\sum_{j=1}^{N} l_{ij}\Gamma(\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{g}(\mathbf{s}(t))) + \mathbf{u}_i(t). \qquad (2.1)$$

The evolution of 2-norm of the error states is shown in Panel (a) of Fig. 1. Obviously, the error systems will not stabilize to the origin if no controllers are exerted. In the following, we consider the controlled network (1.5).



(a) (b)

Figure 1 (a) Evolution of the 2-norm of the error states $\|\mathbf{e}(t)\|_2 = \sqrt{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{n}|e_{ij}(t)|^2}$ without controllers; (b) Evolution of error system (2.1).

According to the definitions of $\mathbf{f}(\mathbf{x}_i(t))$ and $\mathbf{g}(\mathbf{x}_i(t))$, taking $\rho = 7.7$, $\Psi = \mathrm{diag}\{1.8, 2.5, 1.7\}$ to satisfy (H_1) and (H_2), respectively. Therefore, $k = 7.7$ is chosen to make Eq. (1.9) hold. The evolutions of the state variables with controller (1.8) are shown in Panel (b) with $\gamma = 1.5$ and $\mu = 0.1$.

Panel (b) in Fig. 1 shows that stability of error network (2.1) can be realized within $t = 0.7$. Actually, $T \approx 548$ according to the settling time estimated in Theorem 1, which is larger than the real synchronization time, but is smaller than the conventional estimations [2]–[3].

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