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关于一类新型 Dedekind 和的混合均值

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摘 要: 本文研究了一类新型 Dedekind 和的混合均值问题. 利用 Dirichlet L- 函数的均值性质, 给出了新型 Dedekind 和的混合和均值的渐近公式, 推广和发展了关于 Dedekind 和问题的已有结果, 促进了相关领域的研究.

关键词: 新型 Dedekind; Dirichlet L- 函数; 混合均值

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1 引言与结论

设 k 为正整数, h 为任意整数. 经典 Dedekind 和的定义如下

$$s(h,k)=\sum_{j=1}^{k}((\frac{j}{k}))((\frac{hj}{k})),$$

此处

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{如果 } x \text{ 不为整数},\\ 0, & \text{如果 } x \text{ 为整数},\end{cases}$$

其中 $[y]$ 表示不超过 y 的最大整数.

关于 Dedekind 和的最重要的性质, 是互反公式. 即对于互素的正整数 h 和 k, 有

$$s(h,k)+s(k,h)=\frac{1}{12}(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk})-\frac{1}{4}. \tag{1.1}$$

Dedekind[1] 基于 $\log\eta(\tau)$ 的变换公式, 给出了上式的第一个证明. Rademacher[2], Berndt[3−5] 和 Dieter[6] 都分别给出了这个互反公式的不同证明.

张文鹏等人研究了 Dedekind 和的混合均值, 并得到了较强的渐近公式如下.

命题 1.1 [7] 对于整数 $q\geq 3$, 有

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\frac{1}{ab}S(a\bar{b},q)=\frac{5\pi^2}{144}q\cdot\prod_{p|q}\frac{(p^2-1)^2}{p^2(p^2+1)}+O(\exp(\frac{7\ln q}{\ln\ln q})),$$

其中 $\sum\limits_{a=1}^{q}{}'$ 表示所有与 p 互素, 且 $1\leq a\leq q$ 的整数 a 求和, $\prod\limits_{p|q}$ 表示对 q 的所有素数 p 求乘积, $\exp(y)=\mathrm{e}^y$.

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命题 1.2 [8] 设整数 $n \geq 1$, p 为素数, $k = p^n$. 则有

$$\sum_{h=1}^{k}{}' \frac{S(h,k)}{h} = \frac{\pi^2}{72}k(1-\frac{1}{p^2}) + O(\sqrt{k}).$$

设 f 为正整数, χ 为模 f 的 Dirichlet 特征. 定义新型 Dedekind 和如下

$$s(h,k;\chi) = \sum_{a=0}^{fk-1} \overline{B}_{1,\chi}(\frac{ha}{k})\overline{B}_1(\frac{a}{fk}).$$

本文将给出 $s(h,k;\chi)$ 的混合均值的渐近公式. 主要结论如下.

定理 1.1 设 f 为正整数, $p > 2$ 为素数, 满足 $(f,p)=1$. 设 χ 为模 f 的 Dirichlet 特征, 满足 $\chi(-1)=1$. 则有

$$\begin{aligned}\sum_{a=1}^{fp}{}' \sum_{b=1}^{fp}{}' \frac{1}{ab} S(a\bar{b},p;\chi) \quad = \quad & \frac{\pi^2}{72} \frac{fp\,\tau(\overline{\chi})L^2(2,\chi)}{L(4,\chi)} \prod_{q|fp} \frac{(1-\frac{1}{q^2})^2(1-\frac{\chi(q)}{q^2})^2}{1-\frac{\chi(q)}{q^4}} \\ & + O(f^{\frac{3}{2}}p^{\frac{1}{2}}\log^2(fp)),\end{aligned}$$

其中 $\sum\limits_{a=1}^{fp}{}'$ 表示所有与 fp 互素且 $1 \leq a \leq fp$ 的整数 a 求和, $\bar{b}$ 表示同余方程 $x \cdot b \equiv 1(\bmod\ fp)$ 的解.

定理 1.2. 设 f 为正整数, $p > 2$ 为素数, 满足 $(f,p)=1$. 设 χ 为模 f 的 Dirichlet 特征, 满足 $\chi(-1)=1$. 则有

$$\sum_{h=1}^{fp}{}' \frac{S(h,p;\chi)}{h} = \frac{fp}{2\pi^2}\tau(\overline{\chi})L^2(2,\chi)\prod_{q|fp}(1-\frac{\chi(q)}{q^2})^2 + O(f^{\frac{3}{2}}p^{\frac{1}{2}}\log^2(fp)).$$

2 一些引理

引理 2.1 设 f,k 为正整数, h 为任意整数, 满足 $(fk,h)=1$ 与 $(f,k)=1$. 设 χ 为模 f 的 Dirichlet 原特征. 则有

$$S(h,k;\chi) = \begin{cases} \dfrac{f}{\pi^2 k\phi(f)}\tau(\overline{\chi})\sum\limits_{d|k}\dfrac{d^2}{\phi(d)}\chi(\dfrac{k}{d})\sum\limits_{\substack{\psi \bmod fd \\ \psi(-1)=-1}}\psi(h)L(1,\chi\overline{\psi})L(1,\psi), & \text{如果 } \chi(-1)=1, \\ 0, & \text{如果 } \chi(-1)=-1, \end{cases}$$

其中 $\tau(\overline{\chi})$ 为 Gauss 和, $\sum\limits_{\substack{\psi \bmod fd \\ \psi(-1)=-1}}$ 表示对模 fd 的所有奇特征求和.

证 对任意实数 y, 由文献 [9] 中的定理 3.1 可知

$$\overline{B}_{n,\chi}(y) = f^{n-1}\sum_{b=0}^{f-1}\overline{\chi}(b)\overline{B}_n(\frac{b+y}{f}).$$

此外当 $0 \le x < 1$ 时, 有 $B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{e(rx)}{r^n}$, 其中 $e(y) = \mathrm{e}^{2\pi i y}$. 因此

$$\begin{aligned}
s(h,k;\chi) &= \sum_{a=0}^{fk-1}\overline{B}_{1,\chi}(\frac{ha}{k})\overline{B}_1(\frac{a}{fk}) = \sum_{a=0}^{fk-1}\sum_{b=0}^{f-1}\overline{\chi}(b)\overline{B}_1(\frac{ha+kb}{fk})\overline{B}_1(\frac{a}{fk})\\
&= \sum_{\substack{a=1\\ fk\nmid ha+kb}}^{fk-1}\sum_{b=0}^{f-1}\overline{\chi}(b)B_1(\frac{ha+kb}{fk})B_1(\frac{a}{fk})\\
&= -\frac{1}{4\pi^2}\sum_{\substack{a=1\\ fk\nmid ha+kb}}^{fk-1}\sum_{b=0}^{f-1}\overline{\chi}(b)\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{rs}e(\frac{(rh+s)a+krb}{fk})\\
&= -\frac{1}{4\pi^2}\sum_{a=0}^{fk-1}\sum_{b=0}^{f-1}\overline{\chi}(b)\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{rs}\,e(\frac{(rh+s)a+krb}{fk})\\
&\quad+\frac{1}{4\pi^2}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{rs}\sum_{b=0}^{f-1}\overline{\chi}(b)e(\frac{rb}{f})\\
&\quad+\frac{1}{4\pi^2}\sum_{\substack{a=1\\ fk\mid ha+kb}}^{fk-1}\sum_{b=0}^{f-1}\overline{\chi}(b)\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{rs}e(\frac{(rh+s)a+krb}{fk})\\
&= -\frac{1}{4\pi^2}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{rs}\sum_{a=0}^{fk-1}e^{2\pi i\frac{(rh+s)a}{fk}}\sum_{b=0}^{f-1}\overline{\chi}(b)e^{2\pi i\frac{rb}{f}}.
\end{aligned}$$

注意到 χ 为模 f 的原特征, 以及 $(fk,h)=1$, $(f,k)=1$. 可得

$$\begin{aligned}
S(h,k;\chi) &= -\frac{fk}{4\pi^2}\tau(\overline{\chi})\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0\\ rh+s\equiv 0(\bmod fk)}}^{+\infty}\frac{\chi(r)}{rs} = -\frac{fk}{4\pi^2}\tau(\overline{\chi})\sum_{d\mid fk}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0\\ rh+s\equiv 0\left(\bmod \frac{fk}{d}\right)\\ \left(s,\frac{fk}{d}\right)=1}}^{+\infty}\frac{\chi(rd)}{rdsd}\\
&= -\frac{fk}{4\pi^2}\tau(\overline{\chi})\sum_{d\mid fk}\frac{1}{d^2}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{\chi(rd)}{rs}\sum_{\substack{s\equiv(-rh)(\bmod \frac{fk}{d})\\ (s,\frac{fk}{d})=1}}1.
\end{aligned}$$

再由特征和的正交性质, 有

$$\begin{aligned}
S(h,k;\chi) &= -\frac{fk}{4\pi^2}\tau(\overline{\chi})\sum_{d\mid fk}\frac{d^{-2}}{\phi(\frac{fk}{d})}\sum_{\psi \bmod \frac{fk}{d}}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{\chi(rd)}{rs}\psi(s)\overline{\psi}(-rh)\\
&= -\frac{fk}{4\pi^2}\tau(\overline{\chi})\sum_{d\mid fk}\frac{d^{-2}}{\phi(\frac{fk}{d})}\,\chi(d)\sum_{\psi \bmod \frac{fk}{d}}\overline{\psi}(-h)\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\chi(r)\overline{\psi}(r)}{r}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{\psi(s)}{s}
\end{aligned}$$

$$=\begin{cases}\dfrac{fk}{\pi^2}\tau(\overline{\chi})\displaystyle\sum_{d|fk}\dfrac{d^{-2}}{\phi(\frac{fk}{d})}\chi(d)\sum_{\substack{\psi \bmod \frac{fk}{d}\\ \psi(-1)=-1}}\overline{\psi}(h)L(1,\chi\overline{\psi})L(1,\psi), & \text{当 } \chi(-1)=1,\\ 0, & \text{当 } \chi(-1)=-1,\end{cases}$$

$$=\begin{cases}\dfrac{1}{\pi^2 fk}\tau(\overline{\chi})\displaystyle\sum_{d|fk}\dfrac{d^{2}}{\phi(d)}\chi\left(\dfrac{fk}{d}\right)\sum_{\substack{\psi \bmod d\\ \psi(-1)=-1}}\psi(h)L(1,\chi\overline{\psi})L(1,\psi), & \text{当 } \chi(-1)=1,\\ 0, & \text{当 } \chi(-1)=-1,\end{cases}$$

$$=\begin{cases}\dfrac{f}{\pi^2 k\phi(f)}\tau(\overline{\chi})\displaystyle\sum_{d|k}\dfrac{d^{2}}{\phi(d)}\chi\left(\dfrac{k}{d}\right)\sum_{\substack{\psi \bmod fd\\ \psi(-1)=-1}}\psi(h)L(1,\chi\overline{\psi})L(1,\psi), & \text{如果 } \chi(-1)=1,\\ 0, & \text{如果 } \chi(-1)=-1.\end{cases}$$

引理 2.1 证毕.

3 定理 1.1 的证明

由引理 2.1 有

$$S(h,p;\chi)=\frac{fp}{\pi^2\phi(p)\phi(f)}\tau(\overline{\chi})\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}\psi(h)L(1,\chi\overline{\psi})L(1,\psi)+O\left(\frac{f^{\frac{3}{2}}}{p}\right).$$

因此

$$\begin{aligned}&\sum_{a=1}^{fp}{}'\sum_{b=1}^{fp}{}'\frac{1}{ab}S(a\bar{b},p;\chi)\\ =\ &\frac{fp}{\pi^2\phi(p)\phi(f)}\tau(\overline{\chi})\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}\sum_{a=1}^{fp}{}'\frac{\psi(a)}{a}\sum_{b=1}^{fp}{}'\frac{\overline{\psi}(b)}{b}L(1,\chi\overline{\psi})L(1,\psi)+O\left(\frac{f^{\frac{3}{2}}\log^2(fp)}{p}\right).\end{aligned}$$

容易证明

$$L(1,\psi)=\sum_{n=1}^{\infty}\frac{\psi(n)}{n}=\sum_{n=1}^{fp}\frac{\psi(n)}{n}+\int_{fp}^{\infty}\frac{\sum_{fp<n\le y}\psi(n)}{y^2}dy=\sum_{n=1}^{fp}\frac{\psi(n)}{n}+O\left(\frac{\log(fp)}{(fp)^{\frac{1}{2}}}\right),$$

从而

$$\begin{aligned}&\sum_{a=1}^{fp}{}'\sum_{b=1}^{fp}{}'\frac{1}{ab}S(a\bar{b},p;\chi)\\ =\ &\frac{fp}{\pi^2\phi(p)\phi(f)}\tau(\overline{\chi})\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}L(1,\psi)^2L(1,\chi\overline{\psi})L(1,\overline{\psi})+O\left(f^{\frac{3}{2}}p^{\frac{1}{2}}\log^2(fp)\right).\end{aligned}$$

取参数 N 满足 $fp\le N\le (fp)^3$, 有

$$L^2(1,\psi)=\sum_{n=1}^{\infty}\frac{\psi(n)d(n)}{n}=\sum_{1\le n\le N}\frac{\psi(n)d(n)}{n}+\int_{N}^{\infty}\frac{\sum_{N<n\le y}\psi(n)d(n)}{y^2}dy,$$

$$L(1,\chi\overline{\psi})L(1,\overline{\psi})=\sum_{n=1}^{\infty}\frac{\overline{\psi}(n)r(n)}{n}=\sum_{1\le n\le N}\frac{\overline{\psi}(n)r(n)}{n}+\int_{N}^{\infty}\frac{\sum_{N<n\le y}\overline{\psi}(n)r(n)}{y^2}dy,$$

其中 $d(n)=\sum\limits_{t|n}1$ 为除数函数, $r(n)=\sum\limits_{t|n}\chi(t)$. 再由 Pólya-Vinogradov 不等式可得

$$\begin{aligned}\sum_{N<n\leq y}\overline{\psi}(n)r(n) &= 2\sum_{n\leq\sqrt{y}}\overline{\psi}(n)\sum_{m\leq\frac{y}{n}}\chi(m)\overline{\psi}(m)-(\sum_{n\leq\sqrt{y}}\overline{\psi}(n))^2\\&\quad -2\sum_{n\leq\sqrt{N}}\overline{\psi}(n)\sum_{m\leq\frac{y}{n}}\chi(m)\overline{\psi}(m)+(\sum_{n\leq\sqrt{N}}\overline{\psi}(n))^2\\&\ll y^{\frac{1}{2}}(fp)^{\frac{1}{2}}(\log(fp))^2,\end{aligned}$$

以及

$$\begin{aligned}\sum_{N<n\leq y}\psi(n)d(n) &= 2\sum_{n\leq\sqrt{y}}\psi(n)\sum_{m\leq\frac{y}{n}}\psi(m)-(\sum_{n\leq\sqrt{y}}\psi(n))^2\\&\quad -2\sum_{n\leq\sqrt{N}}\psi(n)\sum_{m\leq\frac{N}{n}}\psi(m)+(\sum_{n\leq\sqrt{N}}\psi(n))^2\\&\ll y^{\frac{1}{2}}(fp)^{\frac{1}{2}}(\log(fp))^2.\end{aligned}$$

则有

$$\begin{aligned}\sideset{}{'}\sum_{a=1}^{fp}\sideset{}{'}\sum_{b=1}^{fp}\frac{1}{ab}S(a\bar{b},p;\chi) &= \frac{fp}{\pi^2\phi(p)\phi(f)}\tau(\overline{\chi})\sum_{n=1}^{N}\frac{d(n)}{n}\sum_{m=1}^{N}\frac{r(m)}{m}\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}\psi(n)\overline{\psi}(m)\\&\quad +O(\frac{f^{\frac{5}{2}}p^{\frac{3}{2}}\log^2(fp)}{N^{\frac{1}{2}}})+O(f^{\frac{3}{2}}p^{\frac{1}{2}}\log^2(fp)).\end{aligned}\tag{3.1}$$

由特征的正交关系可得

$$\begin{aligned}&\sum_{n=1}^{N}\frac{d(n)}{n}\sum_{m=1}^{N}\frac{r(m)}{m}\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}\psi(n)\overline{\psi}(m)\\&= \frac{1}{2}\sum_{n=1}^{N}\frac{d(n)}{n}\sum_{m=1}^{N}\frac{r(m)}{m}\sum_{\psi \bmod fp}(1-\psi(-1))\psi(n)\overline{\psi}(m)\\&= \frac{\phi(fp)}{2}\mathop{\sum_{\substack{n=1\\(n,fp)=1}}^{N}\sum_{\substack{m=1\\(m,fp)=1}}^{N}}_{n\equiv m(\bmod fp)}\frac{d(n)r(m)}{nm}-\frac{\phi(fp)}{2}\mathop{\sum_{\substack{n=1\\(n,fp)=1}}^{N}\sum_{\substack{m=1\\(m,fp)=1}}^{N}}_{n\equiv -m(\bmod fp)}\frac{d(n)r(m)}{nm}\\&:= \frac{\phi(fp)}{2}M_1-\frac{\phi(fp)}{2}M_2.\end{aligned}\tag{3.2}$$

首先考虑 M_1, 有

$$\begin{aligned}M_1 &= \mathop{\sum_{\substack{1\leq n\leq fp-1\\(n,fp)=1}}\sum_{\substack{1\leq m\leq fp-1\\(m,fp)=1}}}_{n\equiv m(\bmod fp)}\frac{d(n)r(m)}{nm}+\mathop{\sum_{\substack{1\leq n\leq fp-1\\(n,fp)=1}}\sum_{\substack{fp\leq m\leq N\\(m,fp)=1}}}_{n\equiv m(\bmod fp)}\frac{d(n)r(m)}{nm}\\&\quad +\mathop{\sum_{\substack{fp\leq n\leq N\\(n,fp)=1}}\sum_{\substack{1\leq m\leq fp-1\\(m,fp)=1}}}_{n\equiv m(\bmod fp)}\frac{d(n)r(m)}{nm}+\mathop{\sum_{\substack{fp\leq n\leq N\\(n,fp)=1}}\sum_{\substack{fp\leq m\leq N\\(m,fp)=1}}}_{n\equiv m(\bmod fp)}\frac{d(n)r(m)}{nm}\\&:= \Omega_1+\Omega_2+\Omega_3+\Omega_4.\end{aligned}\tag{3.3}$$

易证

$$\Omega_1 = \sum_{\substack{1\le n\le fp-1\\(n,fp)=1}} \frac{d(n)r(n)}{n^2} = \sum_{\substack{n=1\\(n,fp)=1}}^{\infty} \frac{d(n)r(n)}{n^2} + O((fp)^{\varepsilon-1}).$$

由欧拉乘积公式可得

$$\begin{aligned}
&\sum_{\substack{n=1\\(n,fp)=1}}^{\infty} \frac{d(n)r(n)}{n^2}\\
= &\prod_{q\nmid fp}(1+\frac{d(q)r(q)}{q^2}+\frac{d(q^2)r(q^2)}{q^4}+\cdots+\frac{d(q^k)r(q^k)}{q^{2k}}+\cdots)\\
= &\prod_{q\nmid fp}(\sum_{k=0}^{\infty}\frac{d(q^k)r(q^k)}{q^{2k}}) = \prod_{q\nmid fp}(\sum_{k=0}^{\infty}\frac{1}{q^{2k}}\frac{(k+1)(1-\chi(q)^{k+1})}{1-\chi(q)})\\
= &\prod_{q\nmid fp}\frac{1}{1-\chi(q)}\sum_{k=0}^{\infty}\frac{(k+1)(1-\chi(q)^{k+1})}{q^{2k}} = \prod_{q\nmid fp}\frac{1}{1-\chi(q)}[\sum_{k=0}^{\infty}(k+1)(\frac{1}{q^2})^k-\sum_{k=0}^{\infty}\frac{(k+1)\chi(q)^{k+1}}{q^{2k}}]\\
= &\prod_{q\nmid fp}\frac{1}{1-\chi(q)}[\sum_{k=0}^{\infty}(k+1)(\frac{1}{q^2})^k-\chi(q)\sum_{k=0}^{\infty}(k+1)(\frac{\chi(q)}{q^2})^k]\\
= &\prod_{q\nmid fp}\frac{1}{1-\chi(q)}[\frac{1}{(1-\frac{1}{q^2})^2}-\chi(q)\frac{1}{(1-\frac{\chi(q)}{q^2})^2}]\\
= &\prod_{q\nmid fp}\frac{q^4}{1-\chi(q)}[\frac{(q^2-\chi(q))^2-\chi(q)(q^2-1)^2}{(q^2-1)^2(q^2-\chi(q))^2}] = \prod_{q\nmid fp}\frac{q^4}{1-\chi(q)}[\frac{q^4-q^4\chi(q)-\chi(q)+\chi(q)^2}{(q^2-1)^2(q^2-\chi(q))^2}]\\
= &\prod_{q\nmid fp}\frac{q^4(q^4-\chi(q))}{(q^2-1)^2(q^2-\chi(q))^2} = \prod_{q\nmid fp}\frac{1-\frac{\chi(q)}{q^4}}{(1-\frac{1}{q^2})^2(1-\frac{\chi(q)}{q^2})^2}\\
= &\frac{\zeta^2(2)L^2(2,\chi)}{L(4,\chi)}\prod_{q|fp}\frac{(1-\frac{1}{q^2})^2(1-\frac{\chi(q)}{q^2})^2}{1-\frac{\chi(q)}{q^4}} = \frac{\pi^4}{36}\frac{L^2(2,\chi)}{L(4,\chi)}\prod_{q|fp}\frac{(1-\frac{1}{q^2})^2(1-\frac{\chi(q)}{q^2})^2}{1-\frac{\chi(q)}{q^4}}.
\end{aligned}$$

因此

$$\Omega_1 = \frac{\pi^4}{36}\frac{L^2(2,\chi)}{L(4,\chi)}\prod_{q|fp}\frac{\left(1-\frac{1}{q^2}\right)^2\left(1-\frac{\chi(q)}{q^2}\right)^2}{1-\frac{\chi(q)}{q^4}} + O((fp)^{\epsilon-1}). \tag{3.4}$$

类似可得

$$\begin{aligned}
\Omega_2 \ll\ &(fp)^{\epsilon}\sum_{\substack{1\le n\le fp-1\\(n,fp)=1}}\ \sum_{\substack{fp\le m\le N\\(m,fp)=1\\ n\equiv m(\bmod fp)}}\frac{1}{nm} = (fp)^{\epsilon}\sum_{\substack{1\le n\le fp-1\\(n,fp)=1}}\ \sum_{1\le l\le[\frac{N}{fp}]-1}\ \sum_{\substack{0\le r\le fp-1\\(r,fp)=1\\ n\equiv lfp+r(\bmod fp)}}\frac{1}{n(lfp+r)}\\
\ll\ &(fp)^{\epsilon}\sum_{1\le l\le\frac{N}{fp}}\ \sum_{1\le r\le fp-1}\frac{1}{r(lfp+r)} \ll (fp)^{\epsilon}\sum_{1\le l\le\frac{N}{fp}}\frac{1}{lfp}\sum_{1\le r\le fp-1}\frac{1}{r}\\
\ll\ &(fp)^{\epsilon-1},
\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Omega_3 &\ll (fp)^\epsilon \sum_{\substack{fp\le n\le N\\(n,fp)=1\\n\equiv m(\bmod fp)}}\sum_{\substack{1\le m\le fp-1\\(m,fp)=1}}\frac{1}{nm} = (fp)^\epsilon \sum_{1\le l\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r\le fp-1\\(r,fp)=1\\lfp+r\equiv m(\bmod fp)}}\sum_{\substack{1\le m\le fp-1\\(m,fp)=1}}\frac{1}{(lfp+r)m}\\
&\ll (fp)^\epsilon \sum_{1\le l\le\frac{N}{fp}}\sum_{1\le r\le fp-1}\frac{1}{(lfp+r)r} \ll (fp)^\epsilon \sum_{1\le l\le\frac{N}{fp}}\frac{1}{lfp}\sum_{1\le r\le fp-1}\frac{1}{r}\\
&\ll (fp)^{\epsilon-1}, \end{aligned}\tag{3.6}$$

以及

$$\begin{aligned}\Omega_4 &\ll (fp)^\epsilon \sum_{\substack{fp\le n\le N\\(n,fp)=1\\n\equiv m(\bmod fp)}}\sum_{\substack{fp\le m\le N\\(m,fp)=1}}\frac{1}{nm}\\
&= (fp)^\epsilon \sum_{1\le l_1\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r_1\le fp-1\\(r_1,fp)=1\\l_1fp+r_1\equiv l_2fp+r_2(\bmod fp)}}\sum_{1\le l_2\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r_2\le fp-1\\(r_2,fp)=1}}\frac{1}{(l_1fp+r_1)(l_2fp+r_2)}\\
&\ll (fp)^\epsilon \sum_{1\le l_1\le\frac{N}{fp}}\sum_{1\le r\le fp-1}\sum_{1\le l_2\le\frac{N}{fp}}\frac{1}{(l_1fp+r)(l_2fp+r)}\\
&\ll (fp)^\epsilon \sum_{1\le l_1\le\frac{N}{fp}}\sum_{1\le r\le fp-1}\frac{1}{l_1fp+r}\sum_{1\le l_2\le\frac{N}{fp}}\frac{1}{l_2fp}\\
&\ll (fp)^{\epsilon-1}.\end{aligned}\tag{3.7}$$

结合 (3.3)–(3.7) 有

$$M_1=\frac{\pi^4}{36}\frac{L^2(2,\chi)}{L(4,\chi)}\prod_{q|fp}\frac{(1-\frac{1}{q^2})^2(1-\frac{\chi(q)}{q^2})^2}{1-\frac{\chi(q)}{q^4}}+O((fp)^{\epsilon-1}).\tag{3.8}$$

接下来考虑 M_2. 可得

$$\begin{aligned}M_2 &= \sum_{\substack{n=1\\(n,fp)=1\\n\equiv -m(\bmod fp)}}^{N}\sum_{\substack{m=1\\(m,fp)=1}}^{N}\frac{r(n)\overline{r}(m)}{nm} \ll (fp)^\epsilon \sum_{\substack{n=1\\(n,fp)=1\\n\equiv -m(\bmod fp)}}^{N}\sum_{\substack{m=1\\(m,fp)=1}}^{N}\frac{1}{nm}\\
&\ll (fp)^\epsilon \sum_{0\le l_1\le\frac{N}{fp}}\sum_{\substack{1\le r_1\le fp-1\\l_1fp+r_1\equiv-(l_2fp+r_2)(\bmod fp)}}\sum_{0\le l_2\le\frac{N}{fp}}\sum_{1\le r_2\le fp-1}\frac{1}{(l_1fp+r_1)(l_2fp+r_2)}\\
&\ll (fp)^\epsilon \sum_{0\le l_1\le\frac{N}{fp}}\sum_{1\le r\le fp-1}\sum_{0\le l_2\le\frac{N}{fp}}\frac{1}{(l_1fp+r)(l_2fp+fp-r)}\end{aligned}$$

$$\begin{aligned}
&= (fp)^{\epsilon} \sum_{1\le r\le fp-1} \frac{1}{r(fp-r)} + (fp)^{\epsilon} \sum_{1\le r\le fp-1} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{r(l_2fp+fp-r)}\\
&\quad +(fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \frac{1}{(l_1fp+r)(fp-r)}\\
&\quad +(fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{(l_1fp+r)(l_2fp+fp-r)}\\
&\ll (fp)^{\epsilon-1} \sum_{1\le r\le fp-1} \left(\frac{1}{r}+\frac{1}{fp-r}\right) + (fp)^{\epsilon} \sum_{1\le r\le fp-1} \frac{1}{r} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{l_2fp}\\
&\quad +(fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \frac{1}{l_1fp} \sum_{1\le r\le fp-1} \frac{1}{fp-r} + (fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \frac{1}{l_1fp+r} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{l_2fp}\\
&\ll (fp)^{\epsilon-1}. \qquad (3.9)
\end{aligned}$$

结合 (3.1), (3.2), (3.8) 和 (3.9) 式, 并取 $N=(fp)^3$, 立即可得

$$\sum_{a=1}^{fp}{}' \sum_{b=1}^{fp}{}' \frac{1}{ab} S(a\bar{b},p;\chi) = \frac{\pi^2}{72} \frac{fp\,\tau(\overline{\chi}) L^2(2,\chi)}{L(4,\chi)} \prod_{q|fp} \frac{\left(1-\frac{1}{q^2}\right)^2 \left(1-\frac{\chi(q)}{q^2}\right)^2}{1-\frac{\chi(q)}{q^4}} + O\left(f^{\frac{3}{2}} p^{\frac{1}{2}} \log^2(fp)\right).$$

定理 1.1 证毕.

4 定理 1.2 的证明

由引理 2.1 有 $S(h,p;\chi)=\frac{fp}{\pi^2\phi(p)\phi(f)}\tau(\overline{\chi}) \sum\limits_{\substack{\psi \bmod fp\\ \psi(-1)=-1}} \psi(h) L(1,\chi\overline{\psi}) L(1,\psi) + O\left(\frac{f^{\frac{3}{2}}}{p}\right)$. 因此

$$\sum_{h=1}^{fp}{}' \frac{S(h,p;\chi)}{h} = \frac{fp}{\pi^2\phi(p)\phi(f)} \tau(\overline{\chi}) \sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}} \Big(\sum_{h=1}^{fp}{}' \frac{\psi(h)}{h}\Big) L(1,\chi\overline{\psi}) L(1,\psi) + O\Big(\frac{f^{\frac{3}{2}}\log(fp)}{p}\Big).$$

容易证明

$$L(1,\psi)=\sum_{n=1}^{\infty}\frac{\psi(n)}{n} = \sum_{n=1}^{fp}\frac{\psi(n)}{n} + \int_{fp}^{\infty} \frac{\sum_{fp<n\le y}\psi(n)}{y^2}dy = \sum_{n=1}^{fp}\frac{\psi(n)}{n} + O\left(\frac{\log(fp)}{(fp)^{\frac{1}{2}}}\right),$$

从而

$$\sum_{h=1}^{fp}{}' \frac{S(h,p;\chi)}{h} = \frac{fp}{\pi^2\phi(p)\phi(f)} \tau(\overline{\chi}) \sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}} L(1,\psi)^2 L(1,\chi\overline{\psi}) + O\left(f^{\frac{3}{2}} p^{\frac{1}{2}} \log(fp)\right).$$

取参数 N 满足 $fp\le N\le (fp)^3$, 有

$$L^2(1,\psi)=\sum_{n=1}^{\infty}\frac{\psi(n)d(n)}{n} = \sum_{1\le n\le N}\frac{\psi(n)d(n)}{n} + \int_{N}^{\infty}\frac{\sum_{N<n\le y}\psi(n)d(n)}{y^2}dy,$$

$$L(1,\chi\overline{\psi})=\sum_{n=1}^{\infty}\frac{\chi(n)\overline{\psi}(n)}{n} = \sum_{1\le n\le N}\frac{\chi(n)\overline{\psi}(n)}{n} + \int_{N}^{\infty}\frac{\sum_{N<n\le y}\chi(n)\overline{\psi}(n)}{y^2}dy,$$

其中 $d(n)=\sum_{t|n}1$ 为除数函数. 再由 Pólya-Vinogradov 不等式可得

$$\begin{aligned}\sum_{N<n\leq y}\psi(n)d(n) &= 2\sum_{n\leq\sqrt{y}}\psi(n)\sum_{m\leq\frac{y}{n}}\psi(m)-(\sum_{n\leq\sqrt{y}}\psi(n))^2\\ &\quad -2\sum_{n\leq\sqrt{N}}\psi(n)\sum_{m\leq\frac{N}{n}}\psi(m)+(\sum_{n\leq\sqrt{N}}\psi(n))^2\\ &\ll y^{\frac12}(fp)^{\frac12}(\log(fp))^2,\end{aligned}$$

以及

$$\sum_{N<n\leq y}\chi(n)\overline{\psi}(n)\ll(fp)^{\frac12}\log(fp).$$

则有

$$\begin{aligned}\sideset{}{'}\sum_{h=1}^{fp}\frac{S(h,p;\chi)}{h} &= \frac{fp}{\pi^2\phi(p)\phi(f)}\tau(\overline{\chi})\sum_{n=1}^{N}\frac{d(n)}{n}\sum_{m=1}^{N}\frac{\chi(m)}{m}\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}\psi(n)\overline{\psi}(m)\\ &\quad +O\left(\frac{f^{\frac52}p^{\frac32}\log^2(fp)}{N^{\frac12}}\right)+O\left(f^{\frac32}p^{\frac12}\log(fp)\right).\end{aligned}\tag{4.1}$$

由特征的正交关系可得

$$\begin{aligned}&\sum_{n=1}^{N}\frac{d(n)}{n}\sum_{m=1}^{N}\frac{\chi(m)}{m}\sum_{\substack{\psi \bmod fp\\ \psi(-1)=-1}}\psi(n)\overline{\psi}(m)\\ =&\ \frac12\sum_{n=1}^{N}\frac{d(n)}{n}\sum_{m=1}^{N}\frac{\chi(m)}{m}\sum_{\psi \bmod fp}(1-\psi(-1))\psi(n)\overline{\psi}(m)\\ =&\ \frac{\phi(fp)}{2}\sum_{\substack{n=1\\(n,fp)=1}}^{N}\sum_{\substack{m=1\\(m,fp)=1\\ n\equiv m(\bmod fp)}}^{N}\frac{d(n)\chi(m)}{nm}-\frac{\phi(fp)}{2}\sum_{\substack{n=1\\(n,fp)=1}}^{N}\sum_{\substack{m=1\\(m,fp)=1\\ n\equiv -m(\bmod fp)}}^{N}\frac{d(n)\chi(m)}{nm}\\ :=&\ \frac{\phi(fp)}{2}M_1-\frac{\phi(fp)}{2}M_2.\end{aligned}\tag{4.2}$$

首先考虑 M_1, 有

$$\begin{aligned}M_1 &= \sum_{\substack{1\leq n\leq fp-1\\(n,fp)=1}}\sum_{\substack{1\leq m\leq fp-1\\(m,fp)=1\\ n\equiv m(\bmod fp)}}\frac{d(n)\chi(m)}{nm}+\sum_{\substack{1\leq n\leq fp-1\\(n,fp)=1}}\sum_{\substack{fp\leq m\leq N\\(m,fp)=1\\ n\equiv m(\bmod fp)}}\frac{d(n)\chi(m)}{nm}\\ &\quad+\sum_{\substack{fp\leq n\leq N\\(n,fp)=1}}\sum_{\substack{1\leq m\leq fp-1\\(m,fp)=1\\ n\equiv m(\bmod fp)}}\frac{d(n)\chi(m)}{nm}+\sum_{\substack{fp\leq n\leq N\\(n,fp)=1}}\sum_{\substack{fp\leq m\leq N\\(m,fp)=1\\ n\equiv m(\bmod fp)}}\frac{d(n)\chi(m)}{nm}\\ &:= \Omega_1+\Omega_2+\Omega_3+\Omega_4.\end{aligned}\tag{4.3}$$

易证

$$\Omega_1=\sum_{\substack{1\le n\le fp-1\\(n,fp)=1}}\frac{d(n)\chi(n)}{n^2}=\sum_{\substack{n=1\\(n,fp)=1}}^{\infty}\frac{d(n)\chi(n)}{n^2}+O((fp)^{\varepsilon-1}).$$

由欧拉乘积公式可得

$$\begin{aligned}\sum_{\substack{n=1\\(n,fp)=1}}^{\infty}\frac{d(n)\chi(n)}{n^2}&=\prod_{q\nmid fp}(1+\frac{d(q)\chi(q)}{q^2}+\frac{d(q^2)\chi(q^2)}{q^4}+\cdots+\frac{d(q^k)\chi(q^k)}{q^{2k}}+\cdots)\\&=\prod_{q\nmid fp}(\sum_{k=0}^{\infty}\frac{d(q^k)\chi(q^k)}{q^{2k}})=\prod_{q\nmid fp}(\sum_{k=0}^{\infty}\frac{(k+1)\chi(q)^k}{q^{2k}})\\&=\prod_{q\nmid fp}\frac{1}{(1-\frac{\chi(q)}{q^2})^2}=L^2(2,\chi)\prod_{q\mid fp}(1-\frac{\chi(q)}{q^2})^2.\end{aligned}$$

因此

$$\Omega_1=L^2(2,\chi)\prod_{q\mid fp}(1-\frac{\chi(q)}{q^2})^2+O((fp)^{\varepsilon-1}).\tag{4.4}$$

类似可得

$$\begin{aligned}\Omega_2&\ll(fp)^{\epsilon}\sum_{\substack{1\le n\le fp-1\\(n,fp)=1}}\sum_{\substack{fp\le m\le N\\(m,fp)=1\\n\equiv m(\bmod fp)}}\frac{1}{nm}=(fp)^{\epsilon}\sum_{\substack{1\le n\le fp-1\\(n,fp)=1}}\sum_{1\le l\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r\le fp-1\\(r,fp)=1\\n\equiv lfp+r(\bmod fp)}}\frac{1}{n(lfp+r)}\\&\ll(fp)^{\epsilon}\sum_{1\le l\le\frac{N}{fp}}\sum_{1\le r\le fp-1}\frac{1}{r(lfp+r)}\ll(fp)^{\epsilon}\sum_{1\le l\le\frac{N}{fp}}\frac{1}{lfp}\sum_{1\le r\le fp-1}\frac{1}{r}\\&\ll(fp)^{\epsilon-1},\end{aligned}\tag{4.5}$$

$$\begin{aligned}\Omega_3&\ll(fp)^{\epsilon}\sum_{\substack{fp\le n\le N\\(n,fp)=1}}\sum_{\substack{1\le m\le fp-1\\(m,fp)=1\\n\equiv m(\bmod fp)}}\frac{1}{nm}=(fp)^{\epsilon}\sum_{1\le l\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r\le fp-1\\(r,fp)=1}}\sum_{\substack{1\le m\le fp-1\\(m,fp)=1\\lfp+r\equiv m(\bmod fp)}}\frac{1}{(lfp+r)m}\\&\ll(fp)^{\epsilon}\sum_{1\le l\le\frac{N}{fp}}\sum_{1\le r\le fp-1}\frac{1}{(lfp+r)r}\ll(fp)^{\epsilon}\sum_{1\le l\le\frac{N}{fp}}\frac{1}{lfp}\sum_{1\le r\le fp-1}\frac{1}{r}\\&\ll(fp)^{\epsilon-1},\end{aligned}\tag{4.6}$$

以及

$$\begin{aligned}\Omega_4&\ll(fp)^{\epsilon}\sum_{\substack{fp\le n\le N\\(n,fp)=1}}\sum_{\substack{fp\le m\le N\\(m,fp)=1\\n\equiv m(\bmod fp)}}\frac{1}{nm}\\&=(fp)^{\epsilon}\sum_{1\le l_1\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r_1\le fp-1\\(r_1,fp)=1}}\sum_{1\le l_2\le[\frac{N}{fp}]-1}\sum_{\substack{0\le r_2\le fp-1\\(r_2,fp)=1\\l_1fp+r_1\equiv l_2fp+r_2(\bmod fp)}}\frac{1}{(l_1fp+r_1)(l_2fp+r_2)}\end{aligned}$$

$$\begin{aligned}
&\ll (fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{(l_1fp+r)(l_2fp+r)} \\
&\ll (fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \frac{1}{l_1fp+r} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{l_2fp} \\
&\ll (fp)^{\epsilon-1}. \qquad (4.7)
\end{aligned}$$

结合 (4.3)–(4.7) 式有

$$M_1 = L^2(2,\chi)\prod_{q|fp}\left(1-\frac{\chi(q)}{q^2}\right)^2 + O((fp)^{\varepsilon-1}). \qquad (4.8)$$

接下来考虑 M_2. 可得

$$\begin{aligned}
M_2 &= \sum_{\substack{n=1\\(n,fp)=1\\ n\equiv -m(\bmod fp)}}^{N} \sum_{\substack{m=1\\(m,fp)=1}}^{N} \frac{r(n)\chi(m)}{nm} \ll (fp)^{\epsilon} \sum_{\substack{n=1\\(n,fp)=1\\ n\equiv -m(\bmod fp)}}^{N} \sum_{\substack{m=1\\(m,fp)=1}}^{N} \frac{1}{nm} \\
&\ll (fp)^{\epsilon} \sum_{0\le l_1\le \frac{N}{fp}} \sum_{1\le r_1\le fp-1} \sum_{\substack{0\le l_2\le \frac{N}{fp}\\ l_1fp+r_1\equiv -(l_2fp+r_2)(\bmod fp)}} \sum_{1\le r_2\le fp-1} \frac{1}{(l_1fp+r_1)(l_2fp+r_2)} \\
&\ll (fp)^{\epsilon} \sum_{0\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \sum_{0\le l_2\le \frac{N}{fp}} \frac{1}{(l_1fp+r)(l_2fp+fp-r)} \\
&= (fp)^{\epsilon} \sum_{1\le r\le fp-1} \frac{1}{r(fp-r)} + (fp)^{\epsilon} \sum_{1\le r\le fp-1} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{r(l_2fp+fp-r)} \\
&\quad +(fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \frac{1}{(l_1fp+r)(fp-r)} \\
&\quad +(fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{(l_1fp+r)(l_2fp+fp-r)} \\
&\ll (fp)^{\epsilon-1} \sum_{1\le r\le fp-1} \left(\frac{1}{r}+\frac{1}{fp-r}\right) + (fp)^{\epsilon} \sum_{1\le r\le fp-1} \frac{1}{r} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{l_2fp} \\
&\quad +(fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \frac{1}{l_1fp} \sum_{1\le r\le fp-1} \frac{1}{fp-r} + (fp)^{\epsilon} \sum_{1\le l_1\le \frac{N}{fp}} \sum_{1\le r\le fp-1} \frac{1}{l_1fp+r} \sum_{1\le l_2\le \frac{N}{fp}} \frac{1}{l_2fp} \\
&\ll (fp)^{\epsilon-1}. \qquad (4.9)
\end{aligned}$$

结合 (4.1), (4.2), (4.8) 和 (4.9) 式, 并取 $N=(fp)^3$, 立即可得

$$\sum_{h=1}^{fp}{}' \frac{S(h,p;\chi)}{h} = \frac{fp}{2\pi^2}\tau(\overline{\chi})L^2(2,\chi)\prod_{q|fp}\left(1-\frac{\chi(q)}{q^2}\right)^2 + O\left(f^{\frac{3}{2}}p^{\frac{1}{2}}\log^2(fp)\right).$$

定理 1.2 证毕.

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ON THE HYBRID MEAN OF NEW DEDEKIND SUMS

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Abstract: In this paper, the hybrid mean of new Dedekind sums is studied. By using the mean value theorem for Dirichlet L-function, we give asymptotic formulas for the hybrid mean of new Dedekind sums, which generalize and develop existed results on Dedekind sums.

Keywords: New Dedekind sum; Dirichlet L-function; hybrid mean

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