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BIFURCATION AND POSITIVE SOLUTIONS OF A p-LAPLACIAN PROBLEM

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Abstract: This paper studies a p-Laplacian problem with non-asymptotic nonlinearity at zero or infinity. By using the bifurcation and topological methods, the existence/nonexistence and multiplicity of positive solutions are obtained. The previous results of the existence of positive solutions are enriched and generalized.

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1 Introduction

Consider the following p-Laplacian problem

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a nonnegative parameter, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in (1, +\infty)$ and $f : [0, +\infty) \to [0, +\infty)$ is some given continuous nonlinearity. We also assume that $f(s) > 0$ for $s > 0$ and there exist $f_0,\ f_\infty \in [0, +\infty]$ such that

$$f_0 = \lim_{s\to 0^+} \frac{f(s)}{\varphi_p(s)},\ f_\infty = \lim_{s\to +\infty} \frac{f(s)}{\varphi_p(s)},$$

where $\varphi_p(s) = |s|^{p-2}s$.

If $f_0,\ f_\infty \in (0, +\infty)$ with $f_0 \neq f_\infty$, it follows from Theorem 5.1–5.2 in [1] that problem (1.1) has at least one positive solution for any $\lambda \in (\min\{\lambda_1/f_0, \lambda_1/f_\infty\}, \max\{\lambda_1/f_0, \lambda_1/f_\infty\})$, where λ_1 is the first eigenvalue of problem (1.1) with $f(s) = \varphi_p(s)$. Here we study the cases of $f_0 \notin (0, +\infty)$ or $f_\infty \notin (0, +\infty)$. If f is superlinear, we also require that f satisfies the following subcritical growth condition

$$\lim_{s\to +\infty} \frac{f(s)}{s^{q-1}} = C$$

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for some $q \in (p, p_*)$ and positive constant C, where

$$p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N \end{cases}$$

is the Serrin's exponent (see [2]).

Our main result is the following theorem.

Theorem 1.1 (a) If $f_0 \in (0, +\infty)$ and $f_\infty = 0$, problem (1.1) has at least one positive solution for every $\lambda \in (\lambda_1/f_0, +\infty)$.

(b) If $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$, problem (1.1) has at least one positive solution for every $\lambda \in (0, \lambda_1/f_0)$.

(c) If $f_0 = 0$ and $f_\infty \in (0, +\infty)$, problem (1.1) has at least one positive solution for every $\lambda \in (\lambda_1/f_\infty, +\infty)$.

(d) If $f_0 = f_\infty = 0$, there exists $\lambda_* > 0$ such that problem (1.1) has at least two positive solutions for any $\lambda \in (\lambda_*, +\infty)$.

(e) If $f_0 = 0$ and $f_\infty = +\infty$, problem (1.1) has at least one positive solution for any $\lambda \in (0, +\infty)$.

(f) If $f_0 = +\infty$ and $f_\infty = 0$, problem (1.1) has at least one positive solution for any $\lambda \in (0, +\infty)$.

(g) If $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$, problem (1.1) has at least one positive solution for any $\lambda \in (0, \lambda_1/f_\infty)$.

(h) If $f_0 = f_\infty = \infty$, there exists $\lambda^* > 0$ such that problem (1.1) has at least two positive solutions for any $\lambda \in (0, \lambda^*)$.

2 Proof of Theorem 1.1

We first have the following two nonexistence results.

Lemma 2.1 Assume that there exists a positive constant $\rho > 0$ such that

$$f(s)/\varphi_p(s) \geq \rho$$

for any $s > 0$. Then there exists $\xi_* > 0$ such that problem (1.1) has no positive solution for any $\lambda \in (\xi_*, +\infty)$.

Proof By contradiction, assume that u_n $(n = 1, 2, \cdots)$ are positive solutions of problem (1.1) with $\lambda = \lambda_n$ $(n = 1, 2, \cdots)$ such that $\lambda_n \to +\infty$ as $n \to +\infty$. Then we have that $\lambda_n f(u_n)/\varphi_p(u_n) > \lambda_1$ for n large enough. By Theorem 2.6 of [3], we know that u_n must change sign in Ω for n large enough, which is a contradiction.

Lemma 2.2 Assume that there exists a positive constant $\varrho > 0$ such that

$$f(s)/\varphi_p(s) \leq \varrho$$

for any $s > 0$. Then there exists $\eta_* > 0$ such that problem (1.1) has no positive solution for any $\lambda \in (0, \eta_*)$.

Proof Suppose, on the contrary, that there exists one positive solution u. Then we have that

$$\lambda_1 \int_\Omega |u|^p \, dx \leq \int_\Omega |\nabla u|^p \, dx = \lambda \int_\Omega \frac{f(u)}{\varphi_p(u)} u^p \, dx \leq \lambda \varrho \int_\Omega u^p \, dx,$$

which implies that $\lambda \geq \lambda_1/\varrho$.

Let

$$E = \left\{ u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \right\}$$

with the usual norm

$$\|u\| = \max_{\overline{\Omega}} |u| + \max_{\overline{\Omega}} |\nabla u| .$$

Set

$$\mathbb{P} := \left\{ u \in E : u > 0 \text{ in } \Omega \ \text{ and } \frac{\partial u}{\partial \omega} < 0 \text{ on } \partial\Omega \right\},$$

where ω is the outward pointing normal to $\partial\Omega$.

Proof of Theorem 1.1 (a) From Lemma 5.4 of [1], there exists a continuum $\mathscr{C}$ of nontrivial solutions of problem (1.1) emanating from $(\lambda_1/f_0, 0)$ such that $\mathscr{C} \subset (\mathbb{R} \times \mathbb{P}) \cup \{(\lambda_1/f_0, 0)\}$, meets ∞ in $\mathbb{R} \times E$. It suffices to show that $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to $(+\infty, +\infty)$. Lemma 2.2 implies that $\lambda > 0$ on $\mathscr{C}$ and $\lambda = 0$ is not the blow up point of $\mathscr{C}$.

We claim that $\mathscr{C}$ is unbounded in the direction of E. Suppose, by contradiction, that $\mathscr{C}$ is bounded in the direction of E. So there exist $(\lambda_n, u_n) \in \mathscr{C}$ and a positive constant M such that $\lambda_n \to +\infty$ as $n \to +\infty$ and $\|u_n\| \leq M$ for any $n \in \mathbb{N}$. It follows that $f(u_n)/u_n \geq \delta$ for some positive constant δ and all $n \in \mathbb{N}$. Lemma 2.1 implies that $u_n \equiv 0$ for n large enough, which is a contradiction. Lemma 5.1 of [4] implies that the unique blow up point of $\mathscr{C}$ is $\lambda = +\infty$. Now the desired conclusion can be got immediately from the global structure of $\mathscr{C}$.

(b) It is enough to show that $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to $(0, +\infty)$. Lemma 2.1 implies that $\mathscr{C}$ is bounded in the direction of λ. By virtue of Lemma 5.1 of [4], we know that $(0, +\infty)$ is the unique blow up point of $\mathscr{C}$.

(c) If (λ, u) is any solution of (1.1) with $\|u\| \not\equiv 0$, dividing (1.1) by $\|u\|^{2(p-1)}$ and setting $w = u/\|u\|^2$ yield

$$\begin{cases} -\mathrm{div}\left(|\nabla w|^{p-2} \nabla w\right) = \lambda \frac{f(u)}{\|u\|^{2(p-1)}} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Define

$$\widetilde{f}(w) = \begin{cases} \|w\|^{2(p-1)} f\left(\frac{w}{\|w\|^2}\right) & \text{if}\, w \neq 0, \\ 0 & \text{if } w = 0. \end{cases}$$

Then (2.1) is equivalent to

$$\begin{cases} -\mathrm{div}\left(|\nabla w|^{p-2} \nabla w\right) = \lambda \widetilde{f}(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

By doing some simple calculations, we can show that $\widetilde{f}_0 = f_\infty$ and $\widetilde{f}_\infty = f_0$. Applying the conclusion of (a) and the inversion $w \to w/\|w\|^2 = u$, we obtain the desired conclusion.

(d) Define

$$f^n(s) = \begin{cases} \varphi_p(s)/n, & s \in [0, 1/n], \\ (f(2/n) - 1/n^p)\, ns + 2/n^p - f(2/n), & s \in (1/n, 2/n), \\ f(s), & s \in [2/n, +\infty) \end{cases}$$

and consider the following problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda f^n(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that $\lim\limits_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = 1/n$ and $f_\infty^n = f_\infty = 0$. From the conclusion of (a), we obtain a sequence of unbounded continua $\mathscr{C}_n$ emanating from $(n\lambda_1, 0)$ and joining to $(+\infty, +\infty) := z_*$. Let $\mathscr{C} = \limsup\limits_{n\to+\infty} \mathscr{C}_n$. For any $(\lambda, u) \in \mathscr{C}$, the definition of superior limit (see [5]) shows that there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}_n$ such that $(\lambda_n, u_n) \to (\lambda, u)$ as $n \to +\infty$. Then a continuity argument shows that u is a solution of problem (1.1).

From Proposition 2 of [4], for each $\epsilon > 0$ there exists an N_0 such that for every $n > N_0$, $\mathscr{C}_n \subset V_\epsilon(\mathscr{C})$ with $V_\epsilon(\mathscr{C})$ denoting the ϵ-neighborhood of $\mathscr{C}$. It follows that $(n\lambda_1, +\infty) \subseteq \mathrm{Proj}(\mathscr{C}_n) \subseteq \mathrm{Proj}(V_\epsilon(\mathscr{C}))$, where $\mathrm{Proj}(\mathscr{C}_n)$ denotes the projection of $\mathscr{C}_n$ on $\mathbb{R}$. So we have that $(n\lambda_1 + \epsilon, +\infty) \subseteq \mathrm{Proj}(\mathscr{C})$ for any $n > N_0$. Hence, we have $\mathscr{C} \setminus \{\infty\} \neq \emptyset$.

Let

$$S_1 = \{(+\infty, u) : 0 < \|u\| < +\infty\}.$$

For any fixed $n \in \mathbb{N}$, we claim that $\mathscr{C}_n \cap S_1 = \emptyset$. Otherwise, there exists a sequence $(\lambda_m, u_m) \in \mathscr{C}_n$ such that $(\lambda_m, u_m) \to (+\infty, u_*) \in S_1$ with $\|u_*\| < +\infty$. It follows that $\|u_m\| \le M_n$ for some constant $M_n > 0$. It implies that $f^n(u_m)/u_m \ge \delta_n$ for some positive constant δ_n and all $m \in \mathbb{N}$. Lemma 2.1 implies that $u_m \equiv 0$ for m large enough, which contradicts the fact of $\|v_*\| > 0$. It follows that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right) \cap S_1 = \cup_{n=1}^{+\infty}(\mathscr{C}_n \cap S_1) = \emptyset$. Since $\mathscr{C} \subseteq \left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right)$, one has that $\mathscr{C} \cap S_1 = \emptyset$. Furthermore, set

$$S_2 := \{(\lambda, +\infty) : 0 \le \lambda < +\infty\}.$$

For any fixed $n \in \mathbb{N}$, by $f_\infty = 0$ and an argument similar to that of (a), we have that $\mathscr{C}_n \cap S_2 = \emptyset$. Reasoning as the above, we have that $\mathscr{C} \cap S_2 = \emptyset$. Hence, $\mathscr{C} \cap (S_1 \cup S_2) = \emptyset$. Taking $z^* = (+\infty, 0)$, we have $z^* \in \liminf\limits_{n\to+\infty} \mathscr{C}_n$ with $\|z^*\|_{\mathbb{R}\times E} = +\infty$. Therefore, we obtain that $\mathscr{C} \cap \{\infty\} = \{z_*, z^*\}$. Clearly, $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right) \cap \overline{B}_R$ is pre-compact. So Lemma 3.1 of [6] implies that $\mathscr{C}$ is connected. By an argument similar to that of Theorem 1.3 of [4], we can show that $\mathscr{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$. Now the desired conclusion can be deduced from the global structure of $\mathscr{C}$.

(e) By an argument similar to that of (d), in view of the conclusion of (b), we can get the desired conclusion.

(f) By an argument similar to that of (c) and the conclusion of (e), we can prove it.

(g) By an argument similar to that of (c) and the conclusion of (b), we can obtain it.

(h) Define

$$f^n(s)=\begin{cases} n\varphi_p(s), & s\in[0,1/n]\,,\\ n\left(f\left(2/n\right)-1/n^{p-2}\right)\left(s-1/n\right)+1/n^{p-2}, & s\in(1/n,2/n)\,,\\ f(s), & s\in[2/n,+\infty)\,. \end{cases}$$

Clearly, we have that $\lim\limits_{n\to+\infty}f^n(s)=f(s)$, $f_0^n=n$ and $f_\infty^n=f_\infty=+\infty$. The conclusion of (b) implies that there exists a sequence unbounded continua $\mathscr{C}_n$ emanating from $(\lambda_1/n,0)$ and joining to $(0,+\infty)$. Taking $z^*=(0,0)$, then $z^*\in\liminf\limits_{n\to+\infty}\mathscr{C}_n$. Lemma 2.1 of [4] implies that $\mathscr{C}=\limsup\limits_{n\to+\infty}\mathscr{C}_n$ is unbounded and connected such that $z^*\in\mathscr{C}$ and $(0,+\infty)\in\mathscr{C}$. We claim that $\mathscr{C}\cap((0,+\infty]\times\{0\})=\emptyset$. If there exists a sequence $\{(\lambda_n,u_n)\}$ with $u_n\in\mathbb{P}$ such that $\lim\limits_{n\to+\infty}\lambda_n=\mu>0$ and $\lim\limits_{n\to+\infty}\|u_n\|=0$ as $n\to+\infty$. $f_0=+\infty$ implies that $\lambda_n\frac{f(u_n)}{u_n^{p-1}}>\lambda_1$ for n large enough. By Theorem 2.6 of [3], we know that u_n must change its sign for n large enough, which is a contradiction.

References

[1] Dai G, Ma R. Unilateral global bifurcation phenomena and nodal solutions for p-Laplacian[J]. J. Differential Equations, 2012, 252: 2448–2468.

[2] Serrin J, Zou H. Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities[J]. Acta Math., 2002, 189: 79–142.

[3] Allegretto W, Huang Y X. A Picone's identity for the p-Laplacian and applications[J]. Nonlinear Anal., 1998, 32: 819–830.

[4] Dai G. Bifurcation and one-sign solutions of the p-Laplacian involving a nonlinearity with zeros[J]. Discrete Contin. Dyn. Syst., 2016, 36(10): 5323–5345.

[5] Whyburn G T. Topological analysis[M]. Princeton: Princeton University Press, 1958.

[6] Dai G. Bifurcation and nonnegative solutions for problem with mean curvature operator on general domain[J], Indiana Univ. Math. J., 2018, 67(6): 1–19.

分歧和一类p-Laplace问题的正解

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摘要: 本文研究一类在零点或无穷远点非渐近线性的p-Laplace问题. 利用分歧和拓扑方法, 获得了问题正解的存在性, 不存在性和多解性结果, 丰富和推广了正解存在性的已有结果.

关键词: 分歧; 正解; p-Laplace; 拓扑方法

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