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MODIFIED LAVRENTIEV REGULARIZATION METHOD FOR THE CAUCHY PROBLEM OF HELMHOLTZ-TYPE EQUATION

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Abstract: In this paper, a Cauchy problem of Helmholtz-type equation with nonhomogeneous Dirichlet and Neumann datum is researched. We establish the result of conditional stability under an a-priori assumption for exact solution. A modified Lavrentiev regularization method is used to overcome its ill-posedness, and under an a-priori and an a-posteriori selection rule for the regularization parameter we obtain the convergence result for the regularized solution, the corresponding results of numerical experiments verify that the proposed method is stable and workable, this work is an extension on the related research results of existing literature in the aspect of regularization theory and algorithm for Cauchy problem of Helmholtz-type equation.

Keywords: ill-posed problem; Cauchy problem; Helmholtz-type equation; modified Lavrentiev method; convergence estimate

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1 Introduction

In some practical and applied fields, such as Debye-Huckel theory, implicit marching strategies of the heat equation, the linearization of the Poisson-Boltzmann equation, Helmholtz-type equation had many important applications, see [1–4], etc. In the past century, the direct problem for it caused the extensive attention and was researched widely. However, in some science research areas, the data of the entire boundary can not be acquired, we only can measure the one on a part of the boundary or at certain internal points of one domain, which is called as the inverse problem for the Helmholtz-type equation. This paper studies the Cauchy problem of Helmholtz-type equation

$$\begin{cases} \Delta w(y,x) - k^2 w(y,x) = 0, & x \in (0,\pi),\ y \in (0,T), \\ w(0,x) = \varphi(x), & x \in [0,\pi], \\ w_y(0,x) = \psi(x), & x \in [0,\pi], \\ w(y,0) = w(y,\pi) = 0, & y \in [0,T], \end{cases} \tag{1.1}$$

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where $k > 0$ is the wave number. In view of the linear property of (1.1), it can be divided into two problems, i.e., the Cauchy problem with nonhomogeneous Dirichlet data

$$\begin{cases} \Delta u(y,x) - k^2 u(y,x) = 0, & x \in (0,\pi),\ y \in (0,T), \\ u(0,x) = \varphi(x), & x \in [0,\pi], \\ u_y(0,x) = 0, & x \in [0,\pi], \\ u(y,0) = u(y,\pi) = 0, & y \in [0,T], \end{cases} \tag{1.2}$$

and the Cauchy problem with inhomogeneous Neumann data

$$\begin{cases} \Delta v(y,x) - k^2 v(y,x) = 0, & x \in (0,\pi),\ y \in (0,T), \\ v(0,x) = 0, & x \in [0,\pi], \\ v_y(0,x) = \psi(x), & x \in [0,\pi], \\ v(y,0) = v(y,\pi) = 0, & y \in [0,T], \end{cases} \tag{1.3}$$

it is easily to be know that the solution of problem (1.1) can be expressed as $w = u + v$. Then, we only need to research problems (1.2) and (1.3), respectively.

Problems (1.2) and (1.3) are both the ill-posed problems in the sense that a small disturbance on the Cauchy datum can lead to an tremendous error in the solution [5–7], so some regularization techniques must be carried to overcome the ill-posedness and stabilize numerical calculations (see some regularization strategies in [8, 9]). In the past years, we find that many scholars have considered the Cauchy problem of Helmholtz-type equation and proposed some efficient regularized methods and numerical techniques, such as quasi-reversibility type method [10–14], filtering function method [15], iterative method [16], mollification method [17, 18], spectral method [19, 20], alternating iterative algorithm [21, 22], modified Tikhonov method [20, 23], Fourier method [12, 24], novel trefftz method [25], weighted generalized Tikhonov method [26], and so on.

In this paper, we firstly establish the conditional stabilities for problems (1.2), (1.3), and then construct a kind of modified Lavrentiev regularization method to solve these two problems. In our work, we shall derive some a-priori and a-posteriori convergence results of Hölder type for our regularization solutions, and give an a-posteriori selection rule for the regularization parameter which is relatively rare in solving the Cauchy problem of Helmholtz-type equation. The work is an extension and supplement for the existing ones.

The paper is organized as follows. In Section 2, we derive the conditional stabilities of (1.2) and (1.3). Sections 3 constructs the modified Lavrentiev regularization methods, Sections 4 states some preparation knowledge. In Section 5, the a-priori and a-posteriori convergence estimates of sharp type are established. Some numerical results are shown in Section 6. The corresponding conclusions and discussions are drawn in Section 7.

2 Conditional Stability

We know that the Cauchy problem of the Helmholtz-type equation is ill-posed in the sense of Hadamard that the solution (if it exists) discontinuity depends on the given Cauchy data. Under an additional condition, a continuous dependence of the solution on the Cauchy data can be obtained, which is so-called conditional stability [27–29]. In this section, under an a-priori bound assumption for exact solutions, we give the conditional stabilities of problems (1.2) and (1.3). For $\gamma \geq 1/2$, $q > 0$, we define

$$\mathcal{D}^{\xi}_{\gamma,q} = \{\xi \in L^2(0,\pi); \sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma} e^{2(T+q)\sqrt{n^2+k^2}} |< \xi, X_n >|^2 < +\infty\}, \tag{2.1}$$

here, $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(0,\pi)$, $X_n := X_n(x) = \sqrt{2/\pi}\sin(nx)$ is the eigenfunctions in $L^2(0,\pi)$, and the norm of $\mathcal{D}^{\xi}_{\gamma,q}$ is defined as

$$\|\xi\|_{\mathcal{D}^{\xi}_{\gamma,q}} = (\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma} e^{2(T+q)\sqrt{n^2+k^2}} |< \xi, X_n >|^2)^{1/2}. \tag{2.2}$$

Applying the method of variables separation, the solutions of (1.2) and (1.3) respectively can be expressed as

$$u(y,x) = \sum_{n=1}^{\infty} \cosh\left(\sqrt{n^2+k^2}y\right)\varphi_n X_n,\ \varphi_n = \langle\varphi, X_n\rangle, \tag{2.3}$$

$$v(y,x) = \sum_{n=1}^{\infty} \frac{\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}}\psi_n X_n,\ \psi_n = \langle\psi, X_n\rangle. \tag{2.4}$$

Theorem 2.1 Let $E > 0$, $u(T,x)$ satisfy an a-priori bound condition

$$\|u(T,x)\|_{\mathcal{D}^{u}_{\gamma,q}} \leq E, \tag{2.5}$$

then for each fixed $0 < y \leq T$, it holds that

$$\|u(y,x)\|_{L^2(0,\pi)} \leq 2^{\frac{y}{T+q}} \left(K^{\gamma} e^{\sqrt{K}T}\right)^{-\frac{y}{T+q}} E^{\frac{y}{T+q}} \|\varphi\|^{1-\frac{y}{T+q}}_{L^2(0,\pi)}, \tag{2.6}$$

where $K = 1+k^2$.

Proof Note that, for $0 < y \leq T$, $n \geq 1$, $e^{\sqrt{n^2+k^2}y}/2 \leq \cosh(\sqrt{n^2+k^2}y) \leq e^{\sqrt{n^2+k^2}y}$, $n^2+k^2 \geq 1+k^2$, then from (2.3), (2.5) and Hölder inequality, we have

$$\begin{aligned}
\|u(y,x)\|_{L^2(0,\pi)} &= \|\sum_{n=1}^{\infty}\cosh(\sqrt{n^2+k^2}y)\varphi_n X_n\|_{L^2(0,\pi)} \\
&\leq \sqrt{\sum_{n=1}^{\infty}\cosh^2(\sqrt{n^2+k^2}y)\varphi_n^2} = \sqrt{\sum_{n=1}^{\infty}\cosh^2(\sqrt{n^2+k^2}y)\varphi_n^{\frac{2y}{T+q}}\varphi_n^{2-\frac{2y}{T+q}}} \\
&\leq \sqrt{(\sum_{n=1}^{\infty}(\cosh(\sqrt{n^2+k^2}y))^{\frac{2(T+q)}{y}}\varphi_n^2)^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\varphi_n^2)^{1-\frac{y}{T+q}}} \\
&\leq \sqrt{(\sum_{n=1}^{\infty}(e^{\sqrt{n^2+k^2}y})^{\frac{2(T+q)}{y}}\varphi_n^2)^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\varphi_n^2)^{1-\frac{y}{T+q}}} \\
&= \sqrt{(\sum_{n=1}^{\infty}e^{2(T+q)\sqrt{n^2+k^2}}\varphi_n^2)^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\varphi_n^2)^{1-\frac{y}{T+q}}}
\end{aligned}$$

$$= \sqrt{(\sum_{n=1}^{\infty}\frac{(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\cosh^2(\sqrt{n^2+k^2}T)\varphi_n^2}{(n^2+k^2)^{2\gamma}\cosh^2(\sqrt{n^2+k^2}T)})^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\varphi_n^2)^{1-\frac{y}{T+q}}}$$

$$= \sqrt{(\frac{4}{K^{2\gamma}e^{2\sqrt{K}T}})^{\frac{y}{T+q}}}\times\sqrt{(\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|<u(T,x),X_n(x)>|^2)^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\varphi_n^2)^{1-\frac{y}{T+q}}}$$

$$\leq 2^{\frac{y}{T+q}}(K^{\gamma}e^{\sqrt{K}T})^{-\frac{y}{T+q}}E^{\frac{y}{T+q}}\|\varphi\|_{L^2(0,\pi)}^{1-\frac{y}{T+q}}.$$

Theorem 2.2 Suppose that $v(T,x)$ satisfies the a-priori condition

$$\|v(T,x)\|_{\mathcal{D}_{\gamma,q}^{v}}\leq E, \tag{2.7}$$

then for the fixed $0<y\leq T$, we have

$$\|v(y,x)\|_{L^2(0,\pi)} \leq 2^{\frac{y}{T+q}}\left(K^{\left(\frac{1}{2}-\gamma\right)-\frac{T+q}{2y}}\right)^{\frac{y}{T+q}}\left(e^{\sqrt{K}T}\left(1-e^{-2\sqrt{K}T}\right)\right)^{-\frac{y}{T+q}}E^{\frac{y}{T+q}}\|\psi\|_{L^2(0,\pi)}^{1-\frac{y}{T+q}}. \tag{2.8}$$

Proof For $n\geq 1$, we notice that $\sinh(\sqrt{n^2+k^2}y)\leq e^{\sqrt{n^2+k^2}y}$, and $n^2+k^2\geq 1+k^2:=K$, $\sinh(\sqrt{n^2+k^2}y)\geq e^{\sqrt{K}y}(1-e^{-2\sqrt{K}y})/2$, from (2.4), (2.7) and Hölder inequality, we have

$$\|v(y,x)\|_{L^2(0,\pi)}\leq\|\sum_{n=1}^{\infty}\frac{\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}}\psi_nX_n\|_{L^2(0,\pi)}$$

$$\leq \sqrt{\sum_{n=1}^{\infty}\frac{\sinh^2(\sqrt{n^2+k^2}y)}{(\sqrt{n^2+k^2})^2}\psi_n^2}=\sqrt{\sum_{n=1}^{\infty}\frac{\sinh^2(\sqrt{n^2+k^2}y)}{(\sqrt{n^2+k^2})^2}\psi_n^{\frac{2y}{T+q}}\psi_n^{2-\frac{2y}{T+q}}}$$

$$\leq \sqrt{(\sum_{n=1}^{\infty}(\frac{\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}})^{\frac{2(T+q)}{y}}\psi_n^2)^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\psi_n^2)^{1-\frac{y}{T+q}}}$$

$$\leq \sqrt{(\sum_{n=1}^{\infty}(\frac{e^{\sqrt{n^2+k^2}y}}{\sqrt{n^2+k^2}})^{\frac{2(T+q)}{y}}\psi_n^2)^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\psi_n^2)^{1-\frac{y}{T+q}}}$$

$$= \sqrt{(\sum_{n=1}^{\infty}e^{2(T+q)\sqrt{n^2+k^2}}\psi_n^2(\frac{1}{\sqrt{n^2+k^2}})^{\frac{2(T+q)}{y}})^{\frac{y}{T+q}}(\sum_{n=1}^{\infty}\psi_n^2)^{1-\frac{y}{T+q}}}$$

$$= \sqrt{(\sum_{n=1}^{\infty}\frac{(\sqrt{n^2+k^2})^2\cdot(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}}{(n^2+k^2)^{2\gamma}\sinh^2(\sqrt{n^2+k^2}T)}\frac{\sinh^2(\sqrt{n^2+k^2}T)}{(\sqrt{n^2+k^2})^2}\psi_n^2(\frac{1}{\sqrt{n^2+k^2}})^{\frac{2(T+q)}{y}})^{\frac{y}{T+q}}}$$

$$\times\sqrt{(\sum_{n=1}^{\infty}\psi_n^2)^{1-\frac{y}{T+q}}}$$

$$\leq \sqrt{(\sum_{n=1}^{\infty}\frac{4(\sqrt{K})^{2-\frac{2(T+q)}{y}}}{K^{2\gamma}e^{2T\sqrt{K}}(1-e^{-2T\sqrt{K}})^2}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|<v(T,x),X_n(x)>|^2)^{\frac{y}{T+q}}}$$

$$\times\sqrt{(\sum_{n=1}^{\infty}\psi_n^2)^{1-\frac{y}{T+q}}}$$

$$\begin{aligned}
&\leq \sqrt{\left(\frac{2(\sqrt{K})^{1-\frac{T+q}{y}}}{K^{\gamma}e^{T\sqrt{K}}(1-e^{-2T\sqrt{K}})}\right)^{\frac{2y}{T+q}}\left(\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|<v(T,x),X_n(x)>|^2\right)^{\frac{y}{T+q}}} \\
&\quad \times\sqrt{\left(\sum_{n=1}^{\infty}\psi_n^2\right)^{1-\frac{y}{T+q}}} \\
&\leq 2^{\frac{y}{T+q}}\left(K^{(\frac{1}{2}-\gamma)-\frac{T+q}{2y}}\right)^{\frac{y}{T+q}}\left(e^{\sqrt{K}T}(1-e^{-2\sqrt{K}T})\right)^{-\frac{y}{T+q}}E^{\frac{y}{T+q}}\|\psi\|_{L^2(0,\pi)}^{1-\frac{y}{T+q}}.
\end{aligned}$$

From the inequality above, we can derive the conditional stability result (2.8).

3 Regularization Method

From (2.3), (2.4), we know that $\cosh(\sqrt{n^2+k^2}y)$, $\frac{\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}}$ are unbounded as n tends to infinity, so problems (1.2), (1.3) are both ill-posed, i.e., the solutions do not depend continuously on the Cauchy datum φ and ψ. In order to restore the stability of solutions given by (2.3) and (2.4), we need eliminate the high frequencies of two functions to construct the regularized solutions for (1.2), (1.3).

3.1 Regularization Method for Problem (1.2)

We adopt the similar idea in [30], then problem (1.2) can be equivalently expressed as the following operator equation

$$A_1(y)u(y,x)=\varphi(x), \tag{3.1}$$

where $A_1(y)=1/\cosh(\sqrt{L_x}y)$, and $A_1(y):L^2(0,\pi)\to L^2(0,\pi)$ is a bounded linear self-adjoint compact operator with the eigenvalues $1/\cosh(\sqrt{n^2+k^2}y)$ and eigenelements X_n, $L_x:L^2(0,\pi)\to L^2(0,\pi)$ is a linear positive defined self-adjoint operator, the eigenvalues and eigenelements are n^2+k^2 and X_n, respectively.

Let us introduce the Hilbert scale $(H_\mu)_{\mu\in\mathbb{R}^+}$ according to $H_0=L^2$, $H_\mu=D(L_x^{\mu/2})$, and $\|u\|_\mu=\|L_x^{\mu/2}u\|_{L^2}$ is the norm in H_μ. For $\gamma\geq 1/2$, we construct a generalized Tikhonov regularization solution $u_\alpha^\delta(y,x)$ by solving the minimization problem

$$\min_{u\in L^2(0,\pi)}J_\alpha(u),J_\alpha(u)=\left\|A_1(y)u-\varphi^\delta(x)\right\|_{L^2(0,\pi)}^2+\alpha\left\|L_x^{\frac{\gamma}{2}}(u-u^*)\right\|_{L^2(0,\pi)}^2, \tag{3.2}$$

here, $\varphi^\delta(x)=u^\delta(0,x)$ denotes the noisy data, δ is measured error bound, and α plays the role of regularization parameter, $u^*\in L^2(0,\pi)$ is the reference element (initial guess). Hence $u_\alpha^\delta(y,x)$ is the solution of Euler equation

$$\left(\frac{1}{\cosh^2(\sqrt{L_x}y)}+\alpha L_x^\gamma\right)(u_\alpha^\delta-u^*)=\frac{1}{\cosh(\sqrt{L_x}y)}\left(\varphi^\delta(x)-\frac{1}{\cosh(\sqrt{L_x}y)}u^*\right) \tag{3.3}$$

of the functional J_α. Note that the operator $A_1(y)$ is a monotone compact operator, i.e., $\langle A_1(y)u(y,\cdot),u(y,\cdot)\rangle_{L^2(0,\pi)}\geq 0$, and $A_1(y)$ is compact with $\dim\mathcal{R}(A_1(y))=\infty$, then (3.1) is an ill-posed problem of type II in sense of Nashed [31] (also see [32]). So adopting the similar idea with [33], we can replaced (3.3) by the simpler regularized equation below

$$\left(\frac{1}{\cosh(\sqrt{L_x}y)}+\alpha L_x^\gamma\right)(u_\alpha^\delta-u^*)=\left(\varphi^\delta(x)-\frac{1}{\cosh(\sqrt{L_x}y)}u^*\right), \tag{3.4}$$

which is a Lavrentiev-type method (see [34]), i.e.,

$$u_\alpha^\delta+\alpha L_x^\gamma\cosh(\sqrt{L_x}y)(u_\alpha^\delta-u^*)=\cosh(\sqrt{L_x}y)\varphi^\delta(x). \tag{3.5}$$

We know that the ordinary Lavrentiev method [35] is characterized by (3.5), and αL_x^γ is replaced by αI.

Setting $q>0$, now we firstly replace $\cosh(\sqrt{L_x}y)$ by $\cosh(\sqrt{L_x}(T+q))$ in the left side of (3.5), and then express it a singularly perturbed form, it can be obtained a modified Lavrentiev method for solving linear ill-posed problem (3.1). The regularized equation can be written as

$$\frac{1}{\cosh(\sqrt{L_x}y)}u_\alpha^\delta+\alpha L_x^\gamma\frac{\cosh(\sqrt{L_x}(T+q))}{\cosh(\sqrt{L_x}y)}(u_\alpha^\delta-u^*)=\varphi^\delta(x). \tag{3.6}$$

We take the reference element (initial guess) $u^*\equiv 0$ and solve equation (3.6), then the regularized solution can be written as

$$u_\alpha^\delta(y,x)=\sum_{n=1}^{\infty}\frac{\cosh(\sqrt{n^2+k^2}y)\varphi_n^\delta X_n(x)}{1+\alpha\left(n^2+k^2\right)^\gamma\cosh\left(\sqrt{n^2+k^2}(T+q)\right)}, \tag{3.7}$$

here $\varphi_n^\delta=\langle\varphi^\delta,X_n\rangle_{L^2(0,\pi)}$, and the noisy data φ^δ satisfies

$$\|\varphi^\delta-\varphi\|_{L^2(0,\pi)}\le\delta. \tag{3.8}$$

3.2 Regularization Method for Problem (1.3)

As in Subsection 3.1, we also can convert (1.3) into the operator equation

$$A_2(y)v(y,x)=v_y(0,x)=\psi(x), \tag{3.9}$$

here $A_2(y)=\sqrt{L_x}/\sinh(\sqrt{L_x}y)$, $A_2(y):L^2(0,\pi)\to L^2(0,\pi)$ is a bounded linear self-adjoint compact operator with the eigenvalues $\sqrt{n^2+k^2}/\sinh(\sqrt{n^2+k^2}y)$ and eigenelements X_n.

Note that $A_2(y):L^2(0,\pi)\to L^2(0,\pi)$ also is a monotone and compact operator with $\dim\mathcal{R}(A(y))=\infty$, then (3.9) is an ill-posed problem of type II in sense of Nashed. Similar with the process in Subsection 3.1, for $\gamma\ge 1/2$, we construct a generalized Tikhonov regularization solution $u_\alpha^\delta(y,x)$ by solving the minimization problem

$$\min_{v\in L^2(0,\pi)}I_\beta(v),I_\beta(v)=\left\|A_2(y)v-\psi^\delta(x)\right\|_{L^2(0,\pi)}^2+\beta\left\|L_x^{\frac{\gamma}{2}}(v-v^*)\right\|_{L^2(0,\pi)}^2, \tag{3.10}$$

here $\psi^\delta(x) = v_y^\delta(0,x)$ denotes the noisy data, $v^* \in L^2(0,\pi)$ is the reference element (initial guess). Hence $v_\beta^\delta(y,x)$ is the solution of Euler equation

$$\left(\frac{L_x}{\sinh^2(\sqrt{L_x}y)} + \alpha L_x^\gamma\right)(v_\beta^\delta - v^*) = \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)}\left(\psi^\delta(x) - \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)}v^*\right) \tag{3.11}$$

of the functional I_β. Since the operator $A_2(y)$ is a monotone compact operator, we can replaced (3.11) by the simpler regularized equation (Lavrentiev-type method)

$$\left(\frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)} + \beta L_x^\gamma\right)(v_\beta^\delta - v^*) = \left(\psi^\delta(x) - \frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)}v^*\right), \tag{3.12}$$

i.e.,

$$v_\beta^\delta + \beta L_x^{\gamma-\frac{1}{2}}\sinh(\sqrt{L_x}y)(v_\beta^\delta - v^*) = \frac{\sinh(\sqrt{L_x}y)}{\sqrt{L_x}}\psi^\delta(x). \tag{3.13}$$

Let $q > 0$, we replace $\sinh(\sqrt{L_x}y)$ by $\sinh(\sqrt{L_x}(T+q))$ in the left side of (3.13), and express it a singularly perturbed form, we can obtain a modified Lavrentiev method for solving ill-posed problem (3.9). The regularized equation can be written as

$$\frac{\sqrt{L_x}}{\sinh(\sqrt{L_x}y)}v_\beta^\delta + \beta L_x^\gamma \frac{\sinh(\sqrt{L_x}(T+q))}{\sinh(\sqrt{L_x}y)}(v_\beta^\delta - v^*) = \psi^\delta(x). \tag{3.14}$$

We also choose the initial guess $v^* \equiv 0$ and solve equation (3.14), the regularized solution can be expressed as

$$v_\beta^\delta(y,x) = \sum_{n=1}^{\infty} \frac{\sinh(\sqrt{n^2+k^2}y)\psi_n^\delta X_n(x)}{\sqrt{n^2+k^2}\left(1 + \beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))\right)}, \tag{3.15}$$

here $\psi_n^\delta = \langle \psi^\delta, X_n\rangle_{L^2(0,\pi)}$, the noisy data ψ^δ satisfies

$$\|\psi^\delta - \psi\|_{L^2(0,\pi)} \le \delta, \tag{3.16}$$

and δ is the measured error bound, β is regularization parameter.

4 Preparation Knowledge

Let $\alpha, \beta, q, k > 0$, $\gamma \ge 1/2$, $K = 1 + k^2$, $n \ge 1$, for each fixed $0 < y \le T + q$, we define the functions

$$H_1(n) = \frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\frac{\alpha(n^2+k^2)^\gamma}{2} + e^{-(T+q)\sqrt{n^2+k^2}}} \tag{4.1}$$

and

$$H_2(n) = \frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\sqrt{K}\left(\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\left(\frac{1-e^{-2\sqrt{K}(T+q)}}{2}\right) + e^{-(T+q)\sqrt{n^2+k^2}}\right)}. \tag{4.2}$$

We also require the following Lemma 4.1 which is given and proven in the reference [36].

Lemma 4.1 If $0 \le r \le s < \infty$, $s \ne 0$, and $\nu > 0$, then

$$\frac{\nu e^{-r}}{\nu + e^{-s}} \le H\left(\frac{r}{s}\right)\nu^{\frac{r}{s}}, \tag{4.3}$$

where

$$H(\eta) = \begin{cases} \eta^{\eta}(1-\eta)^{1-\eta}, \eta \in (0,1), \\ 1, \eta = 0, 1. \end{cases} \tag{4.4}$$

Theorem 4.2 Let $\alpha > 0$, $H_1(n)$ is defined by (4.1), then for each fixed $0 < y \le T + q$, we have

$$H_1(n) \le 2\alpha^{-\frac{y}{T+q}}. \tag{4.5}$$

Proof Apply Lemma 4.1 with $\nu = \frac{\alpha(n^2+k^2)^{\gamma}}{2}$, $r = (T+q-y)\sqrt{n^2+k^2}$, $s = (T+q)\sqrt{n^2+k^2}$, and from $H(\eta) \le 1$, we have

$$\begin{aligned} H_1(n) &= \frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\frac{\alpha}{2}(n^2+k^2)^{\gamma} + e^{-(T+q)\sqrt{n^2+k^2}}} = \frac{1}{\frac{\alpha}{2}(n^2+k^2)^{\gamma}} \frac{\frac{\alpha}{2}(n^2+k^2)^{\gamma} \cdot e^{-(T+q-y)\sqrt{n^2+k^2}}}{\frac{\alpha}{2}(n^2+k^2)^{\gamma} + e^{-(T+q)\sqrt{n^2+k^2}}} \\ &\le \left(\frac{\alpha(n^2+k^2)^{\gamma}}{2}\right)^{-1} \cdot H\left(\frac{T+q-y}{T+q}\right)\left(\frac{\alpha(n^2+k^2)^{\gamma}}{2}\right)^{\frac{T+q-y}{T+q}} \\ &= \left(1 - \frac{y}{T+q}\right)^{1-\frac{y}{T+q}}\left(\frac{y}{T+q}\right)^{\frac{y}{T+q}}\left(\frac{\alpha(n^2+k^2)^{\gamma}}{2}\right)^{-\frac{y}{T+q}} \\ &= 2^{\frac{y}{T+q}}((n^2+k^2)^{\gamma})^{-\frac{y}{T+q}}\left(1-\frac{y}{T+q}\right)^{1-\frac{y}{T+q}}\left(\frac{y}{T+q}\right)^{\frac{y}{T+q}}\alpha^{-\frac{y}{T+q}} \\ &\le 2((n^2+k^2)^{\gamma})^{-\frac{y}{T+q}}\alpha^{-\frac{y}{T+q}}. \end{aligned}$$

Note that, $((n^2+k^2)^{\gamma})^{-\frac{y}{T+q}} \le (K^{\gamma})^{-\frac{y}{T+q}}$, $K = 1 + k^2 > 1$, $(K^{\gamma})^{-\frac{y}{T+q}} < 1$, thus $H_1(n) \le 2\alpha^{-\frac{y}{T+q}}$.

Theorem 4.3 Let $\beta > 0$, $H_2(n)$ is defined by (4.2), then for the fixed $0 < y \le T + q$, it holds that

$$H_2(n) \le 2C_1\beta^{-\frac{y}{T+q}},\ C_1 = K^{\frac{y}{2(T+q)}-\frac{1}{2}}\left(1 - e^{-2\sqrt{K}(T+q)}\right)^{-\frac{y}{T+q}}. \tag{4.6}$$

Proof We take $\nu = \beta(n^2+k^2)^{\gamma-\frac{1}{2}}\left(\frac{1-e^{-2\sqrt{K}(T+q)}}{2}\right)$, $r = (T+q-y)\sqrt{n^2+k^2}$, $s = (T+q)\sqrt{n^2+k^2}$ in Lemma 4.1, and from $H(\eta) \le 1$, inequality (4.6) can be derived.

5 Convergence Estimate

In this section, under the a-priori and a-posteriori selection rules for the regularization parameter we derives the convergence estimate for modified Lavrentiev regularization method.

5.1 Convergence Estimate for the Method of Problem (1.2)

5.1.1 A-Priori Convergence Estimate

Theorem 5.1 Let u be the exact solution of problem (1.2) given by (2.3), u_α^δ defined by equation (3.7) is the regularization solution, the measured data φ^δ satisfies (3.8). If the exact solution u satisfies

$$\|u(T,\cdot)\|^2_{\mathcal{D}^u_{\gamma,q}} = \sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|\langle u(T,\cdot),X_n\rangle|^2 \le E^2, \tag{5.1}$$

and the regularization parameter α is chosen as

$$\alpha = \delta/E, \tag{5.2}$$

then for fixed $0 < y \le T$, we have the convergence estimate

$$\|u_\alpha^\delta(y,\cdot) - u(y,\cdot)\| \le 4E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}. \tag{5.3}$$

Proof Denote u_α be the solution of problem (3.7) with exact data φ. We use the triangle inequalities, then

$$\|u_\alpha^\delta - u\| \le \|u_\alpha^\delta - u_\alpha\| + \|u_\alpha - u\|. \tag{5.4}$$

For $0 < y \le T+q$, as $n \ge 1$, $e^{\sqrt{n^2+k^2}y}/2 \le \cosh(\sqrt{n^2+k^2}y) \le e^{\sqrt{n^2+k^2}y}$, from (3.7), (4.5), (3.8), we note that

$$\begin{aligned}
\|u_\alpha^\delta(y,\cdot) - u_\alpha(y,\cdot)\| &\le \sqrt{\sum_{n=1}^{\infty}\left(\frac{\cosh(\sqrt{n^2+k^2}y)}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\right)^2(\varphi_n^\delta-\varphi_n)^2} \\
&\le \sqrt{\sum_{n=1}^{\infty}\left(\frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\frac{\alpha(n^2+k^2)^\gamma}{2}+e^{-(T+q)\sqrt{n^2+k^2}}}\right)^2(\varphi_n^\delta-\varphi_n)^2} \le 2\delta\alpha^{-\frac{y}{T+q}}.
\end{aligned} \tag{5.5}$$

On the other hand, by (2.3), (3.7), (4.5), (5.1), we have

$$\begin{aligned}
&\|u_\alpha(y,\cdot) - u(y,\cdot)\| \\
= &\left\|\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\cosh(\sqrt{n^2+k^2}y)\varphi_n X_n\right\| \\
\le &\sqrt{\sum_{n=1}^{\infty}\left(\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\right)^2\left(\cosh(\sqrt{n^2+k^2}T)\varphi_n\right)^2} \\
\le &\alpha\sqrt{\sum_{n=1}^{\infty}\left(\frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\frac{\alpha(n^2+k^2)^\gamma}{2}+e^{-(T+q)\sqrt{n^2+k^2}}}\right)^2(n^2+k^2)^{2\gamma}e^{2\sqrt{n^2+k^2}(T+q-y)}|\langle u(T,\cdot),X_n\rangle|^2} \\
\le &\alpha\sqrt{\sum_{n=1}^{\infty}H_1^2(n)(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|\langle u(T,\cdot),X_n\rangle|^2} \le 2\alpha^{1-\frac{y}{T+q}}E.
\end{aligned} \tag{5.6}$$

Finally, we can complete the proof by using (5.4), (5.5), (5.6) and (5.2).

5.1.2 A-Posteriori Convergence Estimate

In Theorem 5.1, we select the regularization parameter α by an a-priori rule (5.2), which needs the a-priori bound E of exact solution. However, in practice the a-priori bound E generally can be not known easily. In the following we adopt a kind of the a-posteriori rule to select α, this method need not know the a-priori bound for exact solution, and the regularization parameter α depend on the measured data φ^δ and measured error bound δ. On the reference that describes the a-posteriori rule in selecting the regularization parameter, we can see [37], etc.

We select the regularization parameter α by the following equation

$$\|u_\alpha^\delta(0,x) - \varphi^\delta(x)\| = \tau\delta, \tag{5.7}$$

here $\tau > 1$ is a constant. We need two lemmas that will be used in deriving the a-posteriori convergence estimate.

Lemma 5.2 Let $\rho(\alpha) = \|u_\alpha^\delta(0,x) - \varphi^\delta(x)\|$, then we have the following conclusions

(a) $\rho(\alpha)$ is a continuous function;

(b) $\lim\limits_{\alpha\to 0}\rho(\alpha) = 0$;

(c) $\lim\limits_{\alpha\to +\infty}\rho(\alpha) = \|\varphi^\delta\|$;

(d) For $\alpha \in (0,+\infty)$, $\rho(\alpha)$ is a strictly increasing function.

Proof It can be easily proven by setting

$$\rho(\alpha) = \left(\sum_{n=1}^{\infty}\left(\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\right)^2\left(\varphi_n^\delta\right)^2\right)^{1/2}. \tag{5.8}$$

Lemma 5.2 indicates that there exists a unique solution for (5.7) if $\|\varphi^\delta\| > \tau\delta > 0$.

Lemma 5.3 For the fixed $\tau > 1$, the regularized solution (3.7) combining with a-posteriori rule (5.7) determine that the regularization parameter $\alpha = \alpha(\delta,\varphi^\delta)$ satisfies $\alpha \geq \frac{(\tau-1)e^{\sqrt{K}T}}{2}\frac{\delta}{E}$.

Proof From (5.7), there holds

$$\begin{aligned}
\tau\delta &= \left\|\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\varphi_n^\delta X_n(x)\right\| \\
&\leq \left\|\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}(\varphi_n^\delta-\varphi_n)X_n(x)\right\| \\
&\quad + \left\|\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\varphi_n X_n(x)\right\| \\
&\leq \delta + \left\|\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh(\sqrt{n^2+k^2}(T+q))}\varphi_n X_n(x)\right\|,
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
&\left\|\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}\varphi_n X_n(x)\right\| \\
\leq\ &\left(\sum_{n=1}^{\infty}\left(\frac{\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}\right)^2\varphi_n^2\right)^{1/2} \\
\leq\ &\left(\sum_{n=1}^{\infty}\alpha^2(n^2+k^2)^{2\gamma}\cosh^2(\sqrt{n^2+k^2}(T+q))\varphi_n^2\right)^{1/2} \qquad (5.10)\\
\leq\ &\left(\sum_{n=1}^{\infty}\frac{\alpha^2}{\cosh^2(\sqrt{n^2+k^2}T)}\cdot(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\cosh^2(\sqrt{n^2+k^2}T)\varphi_n^2\right)^{1/2} \\
\leq\ &\left(\sum_{n=1}^{\infty}\frac{4\alpha^2}{e^{2\sqrt{n^2+k^2}T}}\cdot(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|\langle u(T,\cdot),X_n\rangle|^2\right)^{1/2}\leq(2/e^{\sqrt{K}T})\alpha E,
\end{aligned}$$

from (5.9), (5.10), we get that $(\tau-1)\delta\leq(2/e^{\sqrt{K}T})\alpha E$. The proof is completed.

Theorem 5.4 Let u given by (2.3) be the exact solution of problem (1.2), u_α^δ defined by (3.7) is the regularization solution, the measured data φ^δ satisfies (3.8). If the exact solution u satisfies a priori bound (5.1), the regularization parameter is chosen by a-posteriori rule (5.7), then for each fixed $0<y\leq T$, we have the convergence estimate

$$\|u_\alpha^\delta(y,\cdot)-u(y,\cdot)\|\leq CE^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}, \qquad (5.11)$$

where $C=\max\left\{2\left((\tau-1)e^{\sqrt{K}T}/2\right)^{-\frac{y}{T+q}}, 2^{\frac{y}{T+q}}\left(K^{\gamma}e^{\sqrt{K}T}\right)^{-\frac{y}{T+q}}(\tau+1)^{1-\frac{y}{T+q}}\right\}$.

Proof As in (5.4), we know that

$$\|u_\alpha^\delta(y,\cdot)-u(y,\cdot)\|\leq\|u_\alpha^\delta(y,\cdot)-u_\alpha(y,\cdot)\|+\|u_\alpha(y,\cdot)-u(y,\cdot)\|. \qquad (5.12)$$

By (5.5) and Lemma 5.3, we get

$$\|u_\alpha^\delta(y,\cdot)-u_\alpha(y,\cdot)\|\leq 2\delta\alpha^{-\frac{y}{T+q}}\leq 2\left((\tau-1)e^{\sqrt{K}T}/2\right)^{-\frac{y}{T+q}}E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}. \qquad (5.13)$$

Now we give the estimate for the second term of (5.12). For fixed $0<y\leq T$, note that

$$\begin{aligned}
&A_1(y)(u_\alpha(y,\cdot)-u(y,\cdot)) \qquad (5.14)\\
=\ &A_1(y)\sum_{n=1}^{\infty}\frac{-\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))\cosh(\sqrt{n^2+k^2}y)\varphi_n X_n(x)}{1+\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))} \\
=\ &\sum_{n=1}^{\infty}\frac{-\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}\varphi_n X_n(x) \\
=\ &\sum_{n=1}^{\infty}\frac{\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}(\varphi_n^\delta-\varphi_n)X_n(x) \\
&+\sum_{n=1}^{\infty}\frac{-\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^{\gamma}\cosh(\sqrt{n^2+k^2}(T+q))}\varphi_n^\delta X_n(x),
\end{aligned}$$

using (3.8), (5.7), (5.14), we can obtain that

$$\|A_1(y)\left(u_\alpha(y,\cdot)-u(y,\cdot)\right)\| \le \delta+\tau\delta=(\tau+1)\delta. \tag{5.15}$$

Meanwhile, according to the definition in (2.2) and a-priori condition (5.1), we have

$$\begin{aligned}
&\|u_\alpha(y,\cdot)-u(y,\cdot)\|_{\mathcal{D}^{u_\alpha-u}_{\gamma,q}}\\
=&\left(\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\left(\frac{\alpha(n^2+k^2)^\gamma\cosh^2(\sqrt{n^2+k^2}(T+q))}{1+\alpha(n^2+k^2)^\gamma\cosh^2(\sqrt{n^2+k^2}(T+q))}\right)^2\cosh^2(\sqrt{n^2+k^2}y)\varphi_n^2\right)^{\frac{1}{2}}\\
\le&\left(\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\cosh^2(\sqrt{n^2+k^2}T)\varphi_n^2\right)^{\frac{1}{2}}\le E,
\end{aligned} \tag{5.16}$$

then, by the condition stability result (2.6), it can be obtained that

$$\|u_\alpha(y,\cdot)-u(y,\cdot)\| \le 2^{\frac{y}{T+q}}\left(K^\gamma e^{\sqrt{K}T}\right)^{-\frac{y}{T+q}}(\tau+1)^{1-\frac{y}{T+q}}E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}. \tag{5.17}$$

Finally, combining (5.13) with (5.17), we can obtain the convergence estimate (5.11).

5.2 Convergence Estimate for the Method of Problem (1.3)

5.2.1 A-Priori Convergence Estimate

Theorem 5.5 Let v given by (2.4) be the exact solution of problem (1.3), v_β^δ defined by (3.15) is the regularization solution, the measured data ψ^δ satisfies (3.16). If the exact solution v satisfies

$$\|v(T,\cdot)\|^2_{\mathcal{D}^v_{\gamma,q}}=\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|\langle v(T,\cdot),X_n\rangle|^2\le E^2, \tag{5.18}$$

and the regularization parameter β is chosen as

$$\beta=\delta/E, \tag{5.19}$$

then for fixed $0<y\le T$, we have the following convergence estimate

$$\|v_\beta^\delta(y,\cdot)-v(y,\cdot)\|\le 2C_1\left(1+1/e^{\sqrt{K}y}\right)E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}, \tag{5.20}$$

where C_1 is given in Theorem 4.3.

Proof Denote v_β be the solution defined by (3.15) with exact data ψ. Using the triangle inequality, we get that

$$\|v_\beta^\delta-v\|\le\|v_\beta^\delta-v_\beta\|+\|v_\beta-v\|. \tag{5.21}$$

For $0 < y \le T+q$, as $n \ge 1$, $\sinh(\sqrt{n^2+k^2}y) \le e^{\sqrt{n^2+k^2}y}$, $\sinh(\sqrt{n^2+k^2}y) \ge e^{\sqrt{n^2+k^2}y}(1-e^{-2\sqrt{K}y})/2$, from (3.15), (4.6), (3.16), we note that

$$\begin{aligned}
&\|v_\beta^\delta(y,\cdot)-v_\beta(y,\cdot)\| \\
\le\ & \sqrt{\sum_{n=1}^{\infty}\Big(\frac{\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}(1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q)))}\Big)^2(\psi_n^\delta-\psi_n)^2} \\
\le\ & \sqrt{\sum_{n=1}^{\infty}\Big(\frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\sqrt{K}(\beta(n^2+k^2)^{\gamma-\frac{1}{2}}(\frac{1-e^{-2\sqrt{K}(T+q)}}{2})+e^{-(T+q)\sqrt{n^2+k^2}})}\Big)^2(\psi_n^\delta-\psi_n)^2} \\
\le\ & 2C_1\delta\beta^{-\frac{y}{T+q}}.
\end{aligned}\tag{5.22}$$

On the other hand, by (2.4), (3.15), (4.6), (5.18), we have

$$\begin{aligned}
&\|v_\beta(y,\cdot)-v(y,\cdot)\| \\
\le\ & \sqrt{\sum_{n=1}^{\infty}\Big(\frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\Big)^2\Big(\frac{\sinh(\sqrt{n^2+k^2}T)}{\sqrt{n^2+k^2}}\psi_n\Big)^2} \\
\le\ & \beta\sqrt{\sum_{n=1}^{\infty}\Big(\frac{e^{\sqrt{n^2+k^2}y}}{\sqrt{n^2+k^2}e^{\sqrt{n^2+k^2}y}(1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q)))}\Big)^2} \\
&\times\sqrt{(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\Big(\frac{\sinh(\sqrt{n^2+k^2}T)}{\sqrt{n^2+k^2}}\psi_n\Big)^2} \\
\le\ & \beta\sqrt{\sum_{n=1}^{\infty}\Big(\frac{e^{\sqrt{n^2+k^2}y}(n^2+k^2)^{\gamma}e^{(T+q)\sqrt{n^2+k^2}}|\langle v(T,\cdot),X_n\rangle|}{\sqrt{K}e^{\sqrt{K}y}(1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q)))}\Big)^2} \\
\le\ & \frac{\beta E}{e^{\sqrt{K}y}}\sqrt{\sum_{n=1}^{\infty}\Big(\frac{e^{-(T+q-y)\sqrt{n^2+k^2}}}{\sqrt{K}(\beta(n^2+k^2)^{\gamma-\frac{1}{2}}(\frac{1-e^{-2\sqrt{K}(T+q)}}{2})+e^{-(T+q)\sqrt{n^2+k^2}})}\Big)^2} \\
\le\ & (2/e^{\sqrt{K}y})C_1E\beta^{1-\frac{y}{T+q}}.
\end{aligned}\tag{5.23}$$

From (5.19), (5.21), (5.22), (5.23), the convergence result (5.20) can be derived.

5.2.2 A-Posteriori Convergence Estimate

We find β such that

$$\|(v_\beta^\delta)_y(0,x)-\psi^\delta(x)\| = \tau\delta, \tag{5.24}$$

here $\tau > 1$ is a constant.

Lemma 5.6 Let $\varrho(\beta) = \|(v_\beta^\delta)_y(0,x)-\psi^\delta(x)\|$, then we have the following conclusions

(a) $\varrho(\beta)$ is a continuous function;

(b) $\lim\limits_{\beta\to 0}\varrho(\beta) = 0$;

(c) $\lim\limits_{\beta\to+\infty}\varrho(\beta) = \|\psi^\delta\|$;

(d) For $\beta \in (0,+\infty)$, $\varrho(\beta)$ is a strictly increasing function.

Proof It can be easily proven by setting

$$\varrho(\beta) = \left(\sum_{n=1}^{\infty}\left(\frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\right)^2\left(\psi_n^\delta\right)^2\right)^{1/2}. \tag{5.25}$$

Lemma 5.6 means that there exists a unique solution for (5.24) if $\|\psi^\delta\| > \tau\delta > 0$.

Lemma 5.7 For the fixed $\tau > 1$, the regularization solution (3.15) combining with a-posteriori rule (5.24) determine that the regularization parameter $\beta = \beta(\delta, \psi^\delta)$ satisfies $\beta \geq \sinh(\sqrt{K}T)(\tau - 1)\frac{\delta}{E}$.

Proof From (5.24), there holds

$$\begin{aligned}
\tau\delta &= \left\|\sum_{n=1}^{\infty} \frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\psi_n^\delta X_n(x)\right\| \\
&\leq \left\|\sum_{n=1}^{\infty} \frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}(\psi_n^\delta - \psi_n) X_n(x)\right\| \\
&\quad + \left\|\sum_{n=1}^{\infty} \frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\psi_n X_n(x)\right\| \\
&\leq \delta + \left\|\sum_{n=1}^{\infty} \frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\psi_n X_n(x)\right\|,
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
&\left\|\sum_{n=1}^{\infty} \frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\psi_n X_n(x)\right\| \\
\leq &\left(\sum_{n=1}^{\infty}\left(\frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\right)^2 \psi_n^2\right)^{1/2} \\
\leq &\left(\sum_{n=1}^{\infty}\beta^2(n^2+k^2)^{2\gamma-1}\sinh^2(\sqrt{n^2+k^2}(T+q))\psi_n^2\right)^{1/2} \\
\leq &\left(\sum_{n=1}^{\infty}\frac{\beta^2(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}}{\sinh^2(\sqrt{n^2+k^2}T)}\cdot\frac{\sinh^2(\sqrt{n^2+k^2}T)}{(\sqrt{n^2+k^2})^2}\psi_n^2\right)^{1/2} \\
\leq &\left(\sum_{n=1}^{\infty}\frac{\beta^2}{\sinh^2(\sqrt{K}T)}\cdot(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}|\langle v(T,\cdot), X_n\rangle|^2\right)^{1/2} \\
\leq &\ (1/\sinh(\sqrt{K}T))\beta E,
\end{aligned} \tag{5.27}$$

combing with (5.26) and (5.27), we otain that $(\tau-1)\delta \leq (1/\sinh(\sqrt{K}T))\beta E$.

Theorem 5.8 Let v given by (2.4) be the exact solution of problem (1.3), v_β^δ defined by (3.15) is the regularization solution, the measured data φ^δ satisfies (3.16). If the exact solution v satisfies a priori bound (5.18), and the regularization parameter is chosen by a-posteriori rule (5.24), then for fixed $0 < y \leq T$, we have the convergence estimate

$$\|v_\beta^\delta(y,\cdot) - v(y,\cdot)\| \leq C_2 E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}, \tag{5.28}$$

where

$$\begin{aligned} C_2 &= \max\{2C_1\left(\sinh(\sqrt{K}T)(\tau-1)\right)^{-\frac{y}{T+q}}, \\ &\quad 2^{\frac{y}{T+q}}\left(K^{\left(\frac{1}{2}-\gamma\right)-\frac{T+q}{2y}}\right)^{\frac{y}{T+q}}\left(e^{\sqrt{K}T}\left(1-e^{-2\sqrt{K}T}\right)\right)^{-\frac{y}{T+q}}(\tau+1)^{1-\frac{y}{T+q}}\}, \end{aligned}$$

C_1 is given in Theorem 4.3.

Proof Notice that

$$\|v_\beta^\delta(y,\cdot)-v(y,\cdot)\| \le \|v_\beta^\delta(y,\cdot)-v_\beta(y,\cdot)\| + \|v_\beta(y,\cdot)-v(y,\cdot)\|. \tag{5.29}$$

By (5.22) and Lemma 5.7, we get

$$\|v_\beta^\delta(y,\cdot)-v_\beta(y,\cdot)\| \le 2C_1\delta\beta^{-\frac{y}{T+q}} \le 2C_1\left(\sinh(\sqrt{K}T)(\tau-1)\right)^{-\frac{y}{T+q}} E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}. \tag{5.30}$$

Below, we do the estimate for the second term of (5.29). For fixed $0<y\le T$, we have

$$\begin{aligned} & A_2(y)\left(v_\beta(y,\cdot)-v(y,\cdot)\right) \qquad (5.31)\\ =\; & A_2(y)\sum_{n=1}^{\infty}\frac{-\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}\left(1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))\right)}\psi_n X_n(x) \\ =\; & \sum_{n=1}^{\infty}\frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}(\psi_n^\delta-\psi_n)X_n(x) \\ & +\sum_{n=1}^{\infty}\frac{-\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\psi_n^\delta X_n(x), \end{aligned}$$

using (3.16), (5.24), (5.31), we can obtain

$$\|A_2(y)\left(v_\beta(y,\cdot)-v(y,\cdot)\right)\| \le \delta+\tau\delta = (\tau+1)\delta. \tag{5.32}$$

Meanwhile, according to the definition in (2.2) and a-priori bound condition (5.18), we have

$$\begin{aligned} & \|v_\beta(y,\cdot)-v(y,\cdot)\|_{\mathcal{D}_{\gamma,q}^{v_\beta-v}} \qquad (5.33)\\ =\; & \Big(\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\left(\frac{\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}{1+\beta(n^2+k^2)^{\gamma-\frac{1}{2}}\sinh(\sqrt{n^2+k^2}(T+q))}\right)^2 \\ & \left(\frac{\sinh(\sqrt{n^2+k^2}y)}{\sqrt{n^2+k^2}}\right)^2\psi_n^2\Big)^{\frac{1}{2}} \\ \le\; & \left(\sum_{n=1}^{\infty}(n^2+k^2)^{2\gamma}e^{2(T+q)\sqrt{n^2+k^2}}\left(\frac{\sinh(\sqrt{n^2+k^2}T)}{\sqrt{n^2+k^2}}\right)^2\psi_n^2\right)^{\frac{1}{2}} \le E, \end{aligned}$$

then, by the condition stability result (2.8), we can get that

$$\begin{aligned} & \|v_\beta(y,\cdot)-v(y,\cdot)\| \qquad (5.34)\\ \le\; & 2^{\frac{y}{T+q}}\left(K^{\left(\frac{1}{2}-\gamma\right)-\frac{T+q}{2y}}\right)^{\frac{y}{T+q}}\left(e^{\sqrt{K}T}\left(1-e^{-2\sqrt{K}T}\right)\right)^{-\frac{y}{T+q}}(\tau+1)^{1-\frac{y}{T+q}}E^{\frac{y}{T+q}}\delta^{1-\frac{y}{T+q}}. \end{aligned}$$

Finally, combining (5.30) with (5.34), we can obtain the convergence estimate (5.28).

6 Numerical Experiments

In this section, we use numerical experiment to verify the efficiency of our method. For the simplification, we only investigate the numerical efficiency of the regularization method for (1.2), which is similar with the case of inhomogeneous Neumann data (1.3).

Example We can verify that $u(y,x) = \sin(x)\cosh(\sqrt{1+k^2}y)(k > 0)$ is the exact solution of problem (1.2). We take the Cauchy data $\varphi(x) = u(0,x) = \sin(x)$. Denote $\Delta x = \frac{\pi}{N}$ as the step size for variable x, $x_\imath = \imath\Delta x$ as the nodes in $[0,\pi]$ for $\imath = 0,1,2,\cdots,N$, and choose the measured data as $\varphi^\delta = \varphi + \varepsilon\mathrm{randn}(\mathrm{size}(\varphi))$, where φ is a $(N+1)\times 1$ dimension vector, ε is the noisy level, the function randn$(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0 and standard deviation 1, randn(size(φ)) returns an array of random entries that is of the same size as φ. The bound of measured error δ is calculated in the sense of the root mean square error

$$\delta := \|\varphi^\delta - \varphi\|_{l_2} = (\frac{1}{N+1}\sum_{\imath=0}^{N}|\varphi^\delta(x_\imath) - \varphi(x_\imath)|^2)^{1/2}. \tag{6.2}$$

For each $0 < y \leq 1$, the regularization solution $u_\alpha^\delta(y,x)$ is computed by (3.7) for $n = 1,2,\cdots,M$, and the relative root mean square error is computed by

$$\epsilon(u) = \frac{\sqrt{\frac{1}{N+1}\sum_{\imath=0}^{N}\left(u(y,x_\imath) - u_\alpha^\delta(y,x_\imath)\right)^2}}{\sqrt{\frac{1}{N+1}\sum_{\imath=0}^{N}u^2(y,x_\imath)}}. \tag{6.3}$$

Since the a-priori bound E is generally difficult to be obtained in practice, we only give the numerical results by the a-posteriori selection rule (5.7) for the regularization parameter α, here α is found by the Matlab command fzero, and we take $\tau = 1.1$.

For $k = 0.5, 1.5$, $\gamma = 2$, $q = 0.5$, the relative root mean square errors for various noisy level ε are presented in Tables 6.1–6.2. For $k = 0.5, 1.5$, taking $\varepsilon = 0.01$, $q = 0.5$, we also compute the corresponding errors to investigate the influence of γ on numerical results, which are shown in Tables 6.3–6.4. For $k = 0.5, 1.5$, taking $\varepsilon = 0.01$, $\gamma = 2$, we calculate the errors to investigate the influence of q on numerical results, the results are shown in Tables 6.5–6.6.

From Tables 6.1–6.6, we observe that our method is stable and feasible. From Tables 6.1–6.2, we see that numerical results become better as ε goes to zero, which verifies the convergence of our method in practice. Tables 6.3–6.4 show that, for the same ε, q, the error decreases as γ becomes large. Tables 6.5–6.6 indicate that, for the same ε, γ, numerical results become well as q increases. Then, in order to guarantee to obtain the satisfied calculational result, we should choose the parameter γ, q as a relative large positive number, this conclusion are coincident with the expression of the regularization solution (3.7) and the convergence result (5.11).

Table 6.1 $k = 0.5$, $\gamma = 2$, $q = 0.5$, the relative root mean square errors for various noisy level ε at $y = 0.6, 1$

ε	0.001	0.005	0.01	0.05	0.1
α	6.5541e-05	3.2747e-04	6.5333e-04	0.0031	0.0058
$\epsilon_{0.6}(u)$	6.5366e-04	0.0032	0.0062	0.0272	0.0495
$\epsilon_1(u)$	8.2972e-04	0.0040	0.0077	0.0315	0.0549

Table 6.2 $k = 1.5$, $\gamma = 2$, $q = 0.5$, the relative root mean square errors for various noisy level ε at $y = 0.6, 1$

ε	0.001	0.005	0.01	0.05	0.1
α	3.5832e-06	1.7930e-05	3.5915e-05	1.8155e-04	3.6522e-04
$\epsilon_{0.6}(u)$	6.4477e-04	0.0032	0.0063	0.0304	0.0580
$\epsilon_1(u)$	7.7046e-04	0.0037	0.0074	0.0346	0.0650

Table 6.3 $k = 0.5$, $q = 0.5$, $\varepsilon = 0.01$, the relative root mean square errors for various γ at $y = 0.6, 1$

γ	1	2	3	4	5	6
α	8.2264e-04	6.5333e-04	5.0207e-04	3.4938e-04	2.1572e-04	1.2325e-04
$\epsilon_{0.6}(u)$	0.0065	0.0062	0.0057	0.0049	0.0041	0.0035
$\epsilon_1(u)$	0.0082	0.0077	0.0067	0.0054	0.0043	0.0035

Table 6.4 $k = 1.5$, $q = 0.5$, $\varepsilon = 0.01$, the relative root mean square errors for various γ at $y = 0.6, 1$

γ	1	2	3	4	5	6
α	1.1693e-04	3.5915e-05	1.1029e-05	3.3810e-06	1.0296e-06	3.0864e-07
$\epsilon_{0.6}(u)$	0.0064	0.0063	0.0063	0.0062	0.0060	0.0057
$\epsilon_1(u)$	0.0076	0.0074	0.0072	0.0071	0.0068	0.0063

7 Conclusion and Discussion

The article researches a Cauchy problem of the Helmholtz-type equation with nonhomogeneous Dirichlet and Neumann datum. For problems (1.2) and (1.3), we respectively give the conditional stability estimate under an a-priori bound assumption for exact solution. One modified Lavrentiev method is constructed to solve these two problems, and some convergence results of Hölder type for our method are derived under an a-priori and an

Table 6.5 $k = 0.5$, $\gamma = 2$, $\varepsilon = 0.01$, the relative root mean square errors for various q at $y = 0.6, 1$

q	0.1	0.5	1	1.5	2	2.5
α	9.7801e-04	6.5333e-04	3.7885e-04	2.1455e-04	1.1932e-04	6.5121e-05
$\epsilon_{0.6}(u)$	0.0063	0.0062	0.0061	0.0058	0.0056	0.0053
$\epsilon_1(u)$	0.0079	0.0077	0.0074	0.0070	0.0065	0.0060

Table 6.6 $k = 1.5$, $\gamma = 2$, $\varepsilon = 0.01$, the relative root mean square errors for various q at $y = 0.6, 1$

q	0.1	0.5	1	1.5	2	2.5
α	7.2900e-05	3.5915e-05	1.4619e-05	5.9326e-06	2.4051e-06	9.7384e-07
$\epsilon_{0.6}(u)$	0.0064	0.0063	0.0063	0.0063	0.0062	0.0061
$\epsilon_1(u)$	0.0075	0.0074	0.0073	0.0072	0.0071	0.0070

a-posteriori selection rule for the regularization parameter, respectively. We also verify the practicability of this method by making the corresponding numerical experiments.

It should be pointed out that the proposed method also can be used to solve the Cauchy problem of elliptic equation in cylindrical domain. However this method can not be applied to deal with some other problems in more general domains, which is a deficiency of this article. In addition, in the procedure of the computation, we need to choose the suitable parameters which include the regularization parameter α, positive integer N and positive numbers γ, q. We choose the parameters N, γ and q by using the a-priori method, but not to consider the a-posteriori rule for them. It is well know that the selection of the parameter is a sensitive and widespread concerned issue in the inverse problems, their values often can influence the numerical computation effect directly, so it is necessary to consider the a-posteriori selection rule for the parameters N, γ and q in future works.

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Helmholtz型方程柯西问题的修正Lavrentiev正则化方法

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摘 要: 本文研究了带非齐次Dirichlet及Neumann数据的一类Helmholtz型方程柯西问题. 文章在解的先验假设下建立问题的条件稳定性结果, 利用修正Lavrentiev正则化方法克服其不适定性, 并结合正则化参数的先验与后验选取规则获得了正则化解的收敛性结果, 相应的数值实验结果验证了所提方法是稳定可行的, 推广了已有文献在Helmholtz型方程柯西问题正则化理论与算法方面的相关研究结果.

关键词: 不适定问题; 柯西问题; Helmholtz型方程; 修正Lavrentiev正则化方法; 收敛性估计

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