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FEKETE-SZEGÖ PROBLEMS FOR SEVERAL QUASI-SUBORDINATION SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE DZIOK-SRIVASTAVA OPERATOR

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Abstract: In the article we introduce two quasi-subordination subclasses of the function class $\sum$ of analytic and bi-univalent functions associated with the Dziok-Srivastava operator, and some problems for their coefficient estimation and Fekete-Szegö functional. By using differential quasi-subordination and convolution operator theory, we obtain some results about the corresponding bound estimations of the coefficient a_2 and a_3 as well as Fekete-Szegö functional inequalities for theses subclasses, which generalize and improve some earlier known results.

Keywords: Fekete-Szegö problem; bi-univalent function; Gaussian hypergeometric function; Dziok-Srivastava operator; quasi-subordination

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1 Introduction

In the article, our aim focuses on the certain quasi-subordination subclasses of analytic and bi-univalent functions associated with the Dziok-Srivastava operator. To state our results, at first we will recall some notations and basic properties for analytic and bi-univalent functions and Dziok-Srivastava operator.

Let $\mathcal{A}$ be the class of normalized analytic function $f(z)$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} :| z |< 1\}$.

Let the subclass $\mathcal{S}$ of $\mathcal{A}$ be the set of all univalent functions in Δ. According to the Koebe one quarter theorem [1], the inverse f^{-1} of every $f \in \mathcal{S}$ satisfies

$$f^{-1}(f(z)) = z \ \ (z \in \Delta) \quad \text{and} \quad f(f^{-1}(w)) = w \ \ (w \in \Delta_\rho),$$

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where $\rho \geq \frac{1}{4}$ denotes the radius of the image $f(\Delta)$ and $\Delta_\rho = \{z \in \mathbb{C} :| z |< \rho\}$. It is recalled that

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

If both the function $f \in \mathcal{A}$ and its inverse f^{-1} are univalent in Δ, then it is bi-univalent. Denote by Σ the class of all bi-univalent functions $f \in \mathcal{A}$ in Δ.

For given $f, g \in \mathcal{A}$, define the Hadamard product or convolution $f * g$ by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \ \ (z \in \Delta),$$

where $f(z)$ is given by (1.1) and $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$. Assume that the Gaussian hypergeometric function ${}_qF_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s; z)$ is defined by

$$\begin{aligned} {}_qF_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s; z) &= \sum_{n=0}^{\infty} \frac{\Pi_{k=1}^{q}(\alpha_k)_n}{\Pi_{j=1}^{s}(\beta_j)_n} \frac{z^n}{n!} \\ &= 1 + \sum_{n=2}^{\infty} \frac{\Pi_{k=1}^{q}(\alpha_k)_{n-1}}{\Pi_{j=1}^{s}(\beta_j)_{n-1}} \frac{z^{n-1}}{(n-1)!} \ \ (z \in \Delta) \end{aligned}$$

for the complex parameters α_k and β_j with $\beta_j \neq 0, -1, -2, -3, \ldots$ $(k = 1, \cdots , q; j = 1, \cdots , s)$, where $(\ell)_n$ denotes the Pochhammer symbol or shifted factorial by

$$(\ell)_n = \frac{\Gamma(\ell + n)}{\Gamma(\ell)} = \begin{cases} 1, & \text{if } n = 0, \ell \in \mathbb{C} \setminus \{0\}, \\ \ell(\ell + 1)(\ell + 2) \ldots (\ell + n - 1), & \text{if } n \in \mathbb{N} = \{1, 2, 3, \ldots\}. \end{cases}$$

Dziok and Srivastava [2, 3] ever introduced the convolution operator ${}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s) = {}_q\mathcal{I}_s$ later named by themselves as follows

$$\begin{aligned} {}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z) &= z{}_qF_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} p_n(q, s)a_n z^n \ \ (z \in \Delta), \end{aligned} \tag{1.3}$$

where

$$p_n(q, s) = \frac{\Pi_{k=1}^{q}(\alpha_k)_{n-1}}{\Pi_{j=1}^{s}(\beta_j)_{n-1}(n - 1)!}. \tag{1.4}$$

Note that

$$\begin{aligned} & z[{}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z)]' \\ = \ & \alpha_1 {}_q\mathcal{I}_s(\alpha_1 + 1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z) - (\alpha_1 - 1){}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z) \\ = \ & \alpha_2 {}_q\mathcal{I}_s(\alpha_1, \alpha_2 + 1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z) - (\alpha_2 - 1){}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z) \\ & \vdots \\ = \ & \alpha_q {}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q + 1; \beta_1, \cdots , \beta_s)f(z) - (\alpha_q - 1){}_q\mathcal{I}_s(\alpha_1, \cdots , \alpha_q; \beta_1, \cdots , \beta_s)f(z) \ \ (z \in \Delta). \end{aligned}$$

Here we remind some reduced versions of Dziok-Srivastava operator ${}_q\mathcal{I}_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)$ for suitable parameters $\alpha_k(k = 1, \cdots, q)$ and $\beta_j(j = 1, \cdots, s)$; refer to the generalized Bernardi operator $\mathcal{J}_\eta = {}_2\mathcal{I}_1(1, 1+\eta; 2+\eta)(\Re(\eta) > -1)$ [4]; Carlson-Shaffer operator $\mathcal{L}(a, c) = {}_2\mathcal{I}_1(a, 1; c)$ [5]; Choi-Saigo-Srivastava operator $\mathcal{I}_{\lambda,\mu} = {}_2\mathcal{I}_1(\mu, 1; \lambda + 1)(\lambda > -1, \mu \geq 0)$ [6]; Hohlov operator $\mathcal{I}_c^{a,b} = {}_2\mathcal{I}_1(a, b; c)$ [7, 8]; Noor integral operator $\mathcal{I}^n = {}_2\mathcal{I}_1(2, 1; n + 1)$ [9]; Owa-Srivastava fractional differential operator $\Omega_z^\lambda = {}_2\mathcal{I}_1(2, 1; 2 - \lambda)(0 \leq \lambda < 1)$ [10, 11]; Ruscheweyh derivative operator $\mathcal{D}^\delta = {}_2\mathcal{I}_1(1 + \delta, 1; 1)$ [12].

In 1967, Lewin [13] introduced the analytic and bi-univalent function and proved that $| a_2 | < 1.51$. Moreover, Brannan and Clunie [14] conjectured that $| a_2 | \leq \sqrt{2}$, and Netanyahu [15] obtained that $\max_{f \in \sum} | a_2 | = \frac{4}{3}$. Later, Styer and Wright [17] showed that there exists function $f(z)$ so that $| a_2 | > \frac{4}{3}$. However, so far the upper bound estimate $| a_2 | < 1.485$ of coefficient for functions in $\sum$ by Tan [18] is best. Unfortunately, as for the coefficient estimate problem for every Taylor-Maclaurin coefficient $| a_n | \ (n \in \mathbb{N} \setminus \{1, 2\})$ it is probably still an open problem. Based on the works of Brannan and Taha [19] and Srivastava et al. [20], many subclasses of analytic and bi-univalent functions class $\sum$ were introduced and investigated, and the non-sharp estimates of first two Taylor-Maclaurin coefficients $| a_2 |$ and $| a_3 |$ were given; refer to Deniz [21], Frasin and Aouf [22], Hayami and Owa [23], Patil and Naik [24, 25], Srivastava et al. [26, 27], Tang et al. [28] and Xu et al. [29, 30] for more detailed information. Recently, Srivastava et al. [31, 32] gave some new subclasses of the function class $\sum$ of analytic and bi-univalent functions to unify the works of Deniz [21], Frasin [33], Keerthi and Raja [34], Srivastava et al. [35], Murugusundaramoorthy et al.[36] and Xu et al. [29], etc. Besides, we also refer to Goyal et al. [37] for the subclasses of analytic and bi-univalent associated with quasi-subordination. Since Fekete-Szegö [38] studied the determination of the sharp upper bounds for the subclass of $\mathcal{S}$, Fekete-Szegö functional problem was considered in many classes of functions; refer to Abdel-Gawad [39] for class of quasi-convex functions, Koepf [40] for class of close-to-convex functions, Orhan and Răducanu [16] for class of starlike functions, Magesh and Balaji [41] for class of convex and starlike functions, Orhan et al. [42] for the classes of bi-convex and bi-starlike type functions, Panigrahi and Raina [43] for class of quasi-subordination functions, Tang et al. [28] for classes of m-mold symmetric bi-univalent functions. In addition, Murugusundaramoorthy et al. [36, 44, 45] and Patil and Naik [46] ever introduced and investigated several new subclasses of the function class $\sum$ of analytic and bi-univalent functions involving the hohlov operator. Moreover, Al-Hawary et al. [47] studied the Fekete-Szegö functional problem for the classes of analytic functions of complex order defined by the Dziok-Srivastava operator. Motivated by the statements above, in the article we are ready to introduce and investigate two new subclasses of the function class $\sum$ of analytic and bi-univalent functions associated with the Dziok-Srivastava operator and quasi-subordination, and consider the corresponding bound estimates of the coefficient a_2 and a_3 as well as the corresponding Fekete-Szegö functional inequalities. Furthermore, the consequences and connections to some earlier known results would be pointed out.

For two analytic functions f and g, if there exist two analytic functions φ and h with $|\varphi(z)| \leq 1$, $h(0)=0$ and $|h(z)|<1$ for $z \in \Delta$ so that $f(z)=\varphi(z)g(h(z))$, then f is quasi-subordinate to g, i.e., $f \prec_{\text{quasi}} g$. Note that if $\varphi \equiv 1$, then f is subordinate to g in Δ, i.e., $f \prec g$. Further, if $h(z)=z$, then f is majorized by g in Δ, i.e. $f \leq g$. For the related work on quasi-subordination, refer to Robertson [48], and Frasin and Aouf [22]. Write

$$\varphi(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad (|\varphi(z)| \leq 1, z \in \Delta). \tag{1.5}$$

First we will introduce the following general subclasses of analytic and bi-univalent functions associated with the Dziok-Srivastava operator.

Definition 1.1 A function $f(z) \in \sum$ given by (1.1), belongs to the class $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$ if the following quasi-subordinations are satisfied

$$\left[\frac{z({}_q\mathcal{I}_s f)'(z)}{{}_q\mathcal{I}_s f(z)}\right]\left[\frac{({}_q\mathcal{I}_s f)(z)}{z}\right]^{\eta} - 1 \prec_{\text{quasi}} (\phi(z)-1) \tag{1.6}$$

and

$$\left[\frac{z({}_q\mathcal{I}_s g)'(w)}{{}_q\mathcal{I}_s g(w)}\right]\left[\frac{({}_q\mathcal{I}_s g)(w)}{w}\right]^{\eta} - 1 \prec_{\text{quasi}} (\phi(w)-1) \tag{1.7}$$

for $z, w \in \Delta$, where $\eta \geq 0$ and the function g is the inverse of f given by (1.2).

Definition 1.2 A function $f(z) \in \sum$ given by (1.1), belongs to the class $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}$ $(\tau,\mu,\lambda,\gamma;\phi)$ if the following quasi-subordinations are satisfied:

$$\frac{1}{\tau}\left[\frac{z({}_q\mathcal{I}_s f)'(z) + \mu z^2({}_q\mathcal{I}_s f)''(z)}{(1-\lambda)z + \lambda(1-\gamma)({}_q\mathcal{I}_s f)(z) + \gamma z({}_q\mathcal{I}_s f)'(z)} - \frac{1}{[1+\gamma(1-\lambda)]}\right] \prec_{\text{quasi}} (\phi(z)-1) \tag{1.8}$$

and

$$\frac{1}{\tau}\left[\frac{w({}_q\mathcal{I}_s g)'(z) + \mu w^2({}_q\mathcal{I}_s g)''(w)}{(1-\lambda)w + \lambda(1-\gamma)({}_q\mathcal{I}_s g)(w) + \gamma w({}_q\mathcal{I}_s g)'(w)} - \frac{1}{[1+\gamma(1-\lambda)]}\right] \prec_{\text{quasi}} (\phi(w)-1) \tag{1.9}$$

for $z, w \in \Delta$, where $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \mu \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \gamma \leq 1$ and the function g is the inverse of f given by (1.2).

Lemma 1.3 (see [1, 49]) Let $\mathcal{P}$ be the class of all analytic functions $q(z)$ of the following form

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \ \ (z \in \Delta)$$

satisfying $\Re q(z) > 0$ and $q(0)=1$. Then the sharp estimates $|c_n| \leq 2 (n \in \mathbb{N})$ are true. In particular, the equality holds for all n for the next function

$$q(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

2 Coefficient Bounds for the Function Class $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$

Denote the functions s and t in $\mathcal{P}$ by

$$s(z)=\frac{1+u(z)}{1-u(z)}=1+\sum_{n=1}^{\infty}c_nz^n \quad \text{and} \quad t(w)=\frac{1+v(w)}{1-v(w)}=1+\sum_{n=1}^{\infty}d_nw^n \ \ (z,w\in\Delta). \tag{2.1}$$

Equivalently, from (2.1) we know that

$$u(z)=\frac{s(z)-1}{s(z)+1}=\frac{c_1}{2}z+\frac{1}{2}(c_2-\frac{c_1^2}{2})z^2+\dots \ \ (z\in\Delta) \tag{2.2}$$

and

$$v(w)=\frac{t(w)-1}{t(w)+1}=\frac{d_1}{2}w+\frac{1}{2}(d_2-\frac{d_1^2}{2})w^2+\dots \ \ (w\in\Delta). \tag{2.3}$$

Given $\phi\in\mathcal{P}$ with $\phi'(0)>0$, let $\phi(\Delta)$ be symmetric with respect to the real axis. When the series expansion form of ϕ is denoted by

$$\phi(z)=1+\sum_{n=1}^{\infty}E_nz^n \ \ (E_1>0,z\in\Delta), \tag{2.4}$$

by (2.2)–(2.3) and (2.4) it follows that

$$\phi(u(z))=1+\frac{1}{2}E_1c_1z+[\frac{1}{2}E_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}E_2c_1^2]z^2+\dots \ \ (z\in\Delta) \tag{2.5}$$

and

$$\phi(v(w))=1+\frac{1}{2}E_1d_1w+[\frac{1}{2}E_1(d_2-\frac{d_1^2}{2})+\frac{1}{4}E_2d_1^2]w^2+\dots \ \ (w\in\Delta). \tag{2.6}$$

In the section we study the estimates for the class $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$. Now, we establish the next theorem.

Theorem 2.1 If $f(z)$ given by (1.1) belongs to the class $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$, then

$$\mid a_2\mid\le \min\{\frac{E_1}{(\eta+1)},\frac{2(|E_2-E_1|+E_1)}{\sqrt{(\eta+1)(\eta+2)}},\frac{E_1\sqrt{2E_1}}{\sqrt{|\Xi(\eta,B_0,E_1,E_2)|}}\}\frac{|B_0|}{|p_2(q,s)|} \tag{2.7}$$

and

$$\mid a_3\mid\le \frac{(|B_0+|B_1|)E_1}{(\eta+2)|p_3(q,s)|}+\min\{\frac{|B_0|E_1^2}{\eta+1},\frac{2(|E_2-E_1|+E_1)}{\eta+2}\}\frac{|B_0|}{(\eta+1)|p_3(q,s)|}, \tag{2.8}$$

where

$$\Xi(\eta,B_0,E_1,E_2)=(\eta+1)(\eta+2)B_0E_1^2+2(\eta+1)^2(E_1-E_2). \tag{2.9}$$

Proof If $f(z)\in\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$, then by Definition 1.1 and Lemma 1.3, there exist two analytic functions $u(z)$ and $v(w)\in\mathcal{P}$ so that

$$[\frac{z({}_q\mathcal{I}_sf)'(z)}{{}_q\mathcal{I}_sf(z)}][\frac{({}_q\mathcal{I}_sf)(z)}{z}]^{\eta}-1=\varphi(z)[\phi(u(z))-1] \tag{2.10}$$

and

$$[\frac{z({}_q\mathcal{I}_sg)'(w)}{{}_q\mathcal{I}_sg(w)}][\frac{({}_q\mathcal{I}_sg)(w)}{w}]^{\eta}-1=\varphi(w)[\phi(v(w))-1]. \tag{2.11}$$

Expanding the left half parts of (2.11) and (2.12), we obtain that

$$
\begin{aligned}
&[\frac{z({}_q\mathcal{I}_s f)'(z)}{{}_q\mathcal{I}_s f(z)}][\frac{({}_q\mathcal{I}_s f)(z)}{z}]^{\eta} \\
= \quad & 1+(\eta+1)p_2(q,s)a_2 z+[(\eta+2)p_3(q,s)a_3+\frac{(\eta-1)(\eta+2)}{2}p_2^2(q,s)a_2^2]z^2+\cdots \qquad (2.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&[\frac{z({}_q\mathcal{I}_s g)'(w)}{{}_q\mathcal{I}_s g(w)}][\frac{({}_q\mathcal{I}_s g)(w)}{w}]^{\eta} \\
= \quad & 1-(\eta+1)p_2(q,s)a_2 w+[-(\eta+2)p_3(q,s)a_3+\frac{(\eta+2)(\eta+3)}{2}p_2^2(q,s)a_2^2]w^2+\cdots. \qquad (2.13)
\end{aligned}
$$

In addition, we know that

$$
\varphi(z)[\phi(u(z))-1]=\frac{1}{2}B_0E_1c_1z+[\frac{1}{2}B_1E_1c_1+\frac{1}{2}B_0E_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_0E_2c_1^2]z^2+\cdots \qquad (2.14)
$$

and

$$
\varphi(w)[\phi(v(w))-1]=\frac{1}{2}B_0E_1d_1w+[\frac{1}{2}B_1E_1d_1+\frac{1}{2}B_0E_1(d_2-\frac{d_1^2}{2})+\frac{1}{4}B_0E_2d_1^2]w^2+\cdots. \qquad (2.15)
$$

Therefore, from (2.10)–(2.15) we have that

$$
(\eta+1)p_2(q,s)a_2=\frac{1}{2}B_0E_1c_1, \qquad (2.16)
$$

$$
(\eta+2)p_3(q,s)a_3+\frac{(\eta-1)(\eta+2)}{2}p_2^2(q,s)a_2^2=\frac{1}{2}B_1E_1c_1+\frac{1}{2}B_0E_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_0E_2c_1^2, \qquad (2.17)
$$

$$
-(\eta+1)p_2(q,s)a_2=\frac{1}{2}B_0E_1d_1 \qquad (2.18)
$$

and

$$
-(\eta+2)p_3(q,s)a_3+\frac{(\eta+2)(\eta+3)}{2}p_2^2(q,s)a_2^2=\frac{1}{2}B_1E_1d_1+\frac{1}{2}B_0E_1(d_2-\frac{d_1^2}{2})+\frac{1}{4}B_0E_2d_1^2. \qquad (2.19)
$$

From (2.16) and (2.18), it infers that

$$
a_2=\frac{B_0E_1c_1}{2(\eta+1)p_2(q,s)}=-\frac{B_0E_1d_1}{2(\eta+1)p_2(q,s)}. \qquad (2.20)
$$

Then, we show that

$$
c_1=-d_1 \qquad (2.21)
$$

and

$$
B_0^2E_1^2(c_1^2+d_1^2)=8(\eta+1)^2p_2^2(q,s)a_2^2. \qquad (2.22)
$$

By (2.17) and (2.19), we have that

$$
\frac{1}{4}B_0(E_2-E_1)(c_1^2+d_1^2)+\frac{1}{2}B_0E_1(c_2+d_2)=(\eta+1)(\eta+2)p_2^2(q,s)a_2^2. \qquad (2.23)
$$

Therefore, by (2.22)–(2.23) we obtain that

$$a_2^2 = \frac{B_0^2 E_1^3 (c_2 + d_2)}{2(\eta+1)[(\eta+2)B_0 E_1^2 + 2(\eta+1)(E_1 - E_2)]p_2^2(q,s)}. \tag{2.24}$$

We follow from Lemma 1.3 and (2.22)–(2.24) that

$$| a_2 | \leq \frac{|B_0| E_1}{(\eta+1)|p_2(q,s)|},$$

$$| a_2 | \leq \frac{2|B_0|(|E_2 - E_1| + E_1)}{\sqrt{(\eta+1)(\eta+2)}|p_2(q,s)|}$$

and

$$| a_2 | \leq \frac{|B_0| E_1 \sqrt{2E_1}}{\sqrt{(\eta+1)|(\eta+2)B_0 E_1^2 + 2(\eta+1)(E_1 - E_2)|}|p_2(q,s)|},$$

then (2.7) holds. Similarly, from (2.17), (2.19) and (2.21), it also implies that

$$B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2) = 4(\eta+2)p_3(q,s)a_3 - 4(\eta+2)p_2^2(q,s)a_2^2. \tag{2.25}$$

Hence, from (2.22) and (2.25), we obtain that

$$a_3 = \frac{B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2)}{4(\eta+2)p_3(q,s)} + \frac{B_0^2 E_1^2 (c_1^2 + d_1^2)}{8(\eta+1)^2 p_3(q,s)}.$$

Therefore, from Lemma 1.3 it shows that

$$| a_3 | \leq \frac{(|B_0 + |B_1|)E_1}{(\eta+2)|p_3(q,s)|} + \frac{B_0^2 E_1^2}{(\eta+1)^2 |p_3(q,s)|}.$$

On the other hand, by (2.23) and (2.25), we infer that

$$a_3 = \frac{B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2)}{4(\eta+2)p_3(q,s)} + \frac{B_0(E_2 - E_1)(c_1^2 + d_1^2) + 2B_0 E_1 (c_2 + d_2)}{4(\eta+1)(\eta+2)p_3(q,s)}.$$

Thus, from Lemma 1.3, we see that

$$| a_3 | \leq \frac{(|B_0 + |B_1|)E_1}{(\eta+2)|p_3(q,s)|} + \frac{2|B_0|(|E_2 - E_1| + E_1)}{(\eta+1)(\eta+2)|p_3(q,s)|}.$$

Next, we consider Fekete-Szegö problems for the class $\mathcal{QH}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(\eta;\phi)$.

Theorem 2.2 If $f(z)$ given by (1.1) belongs to the class $\mathcal{QH}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(\eta;\phi)$ and $\delta \in \mathbb{R}$, then

$$| a_3 - \delta a_2^2 | \leq \begin{cases} \frac{|B_0| E_1}{2(\eta+2)|p_3(q,s)|}, & \text{if } 2|B_0|E_1^2(\eta+2)|p_2^2(q,s) - \delta p_3(q,s)| \leq | \Xi | \, p_2^2(q,s), \\ \frac{2|B_0|^2 E_1^3 |p_2^2(q,s) - \delta p_3(q,s)|}{|\Xi p_3(q,s)| p_2^2(q,s)}, & \text{if } 2|B_0|E_1^2(\eta+2)|p_2^2(q,s) - \delta p_3(q,s)| \geq | \Xi | \, p_2^2(q,s), \end{cases}$$

where $\Xi = \Xi(\eta, B_0, E_1, E_2)$ is the same as in Theorem 2.1.

Proof From (2.25), it follows that

$$a_3 - \frac{p_2^2(q,s)a_2^2}{p_3(q,s)} = \frac{B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2)}{4(\eta+2)p_3(q,s)}.$$

By (2.24) we easily obtain that

$$
\begin{aligned}
a_3-\delta a_2^2 &= \frac{B_0E_1[\Xi p_2^2(q,s)+2B_0E_1^2(\eta+2)(p_2^2(q,s)-\delta p_3(q,s))]c_2}{4(\eta+2)\Xi p_3(q,s)p_2^2(q,s)} \\
&\quad+\frac{-B_0E_1[\Xi p_2^2(q,s)+2B_0E_1^2(\eta+2)(p_2^2(q,s)-\delta p_3(q,s))]d_2}{4(\eta+2)\Xi p_3(q,s)p_2^2(q,s)}.
\end{aligned}
$$

Hence, from Lemma 1.3, we imply that

$$
\mid a_3-\delta a_2^2 \mid\leq \begin{cases} \frac{|B_0|E_1}{2(\eta+2)|p_3(q,s)|}, & \text{if } 2|B_0|E_1^2(\eta+2)|p_2^2(q,s)-\delta p_3(q,s)|\leq\mid\Xi\mid p_2^2(q,s), \\ \frac{2|B_0|^2E_1^3|p_2^2(q,s)-\delta p_3(q,s)|}{|\Xi p_3(q,s)|p_2^2(q,s)}, & \text{if } 2|B_0|E_1^2(\eta+2)|p_2^2(q,s)-\delta p_3(q,s)|\geq\mid\Xi\mid p_2^2(q,s). \end{cases}
$$

Corollary 2.3 If $f(z)$ given by (1.1) belongs to the class $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$ and $\delta\in\mathbb{R}$, then

$$
|a_3|\leq \begin{cases} \frac{|B_0|E_1}{2(\eta+2)|p_3(q,s)|}, & \text{if } 2|B_0|E_1^2(\eta+2)\leq|\Xi|, \\ \frac{2|B_0|^2E_1^3}{|\Xi p_3(q,s)|}, & \text{if } 2|B_0|E_1^2(\eta+2)\geq|\Xi|, \end{cases}
$$

where $\Xi=\Xi(\eta,B_0,E_1,E_2)$ is the same as in Theorem 2.1.

Remark 2.4 Without quasi-subordination (i.e., $\varphi(z)\equiv 1$), if we choose some suitable parameters α_k $(k=1,\cdots,q)$, β_j $(j=1,\cdots,s)$ and η, we obtain the following reduced versions for $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$ in Theorem 2.1.

(I) $\mathcal{QH}_{\sum}{}^{a,b}_{c}(\alpha;\phi)=\mathcal{J}_{\sum}^{a,b;c}(\alpha,\phi)$, refer to Patil and Naik [46].

Remark 2.5 Without Dziok-Srivastava operator, we can collect the following reduced versions for $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)$ in Theorem 2.1.

(I) $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\eta;\phi)=\mathcal{J}_{\eta}^{q}(\phi)$ $(\eta\geq 0)$, refer to Goyal et al. [37].

(II) $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(1;\phi)=\mathcal{H}_{\sum}(\phi)$ for $\varphi(z)\equiv 1$, refer to Ali et al. [50] and Tang et al. [51] for Corollary 2.2; $\mathcal{QH}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(0;\phi)=\mathcal{S}^*_{\sum}(\phi)$ for $\varphi(z)\equiv 1$, refer to Brannan and Taha. [19] and Tang et al. [51] for Corollary 2.4.

3 Coefficient Bound Estimates for the Function Class $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$

Now, we study the coefficients for the class $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$ and establish the next theorem.

Theorem 3.1 If $f(z)$ given by (1.1) belongs to the class $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$, then

$$
\mid a_2\mid\leq\min\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\} \tag{3.1}
$$

for

$$
\mathcal{F}_1=\frac{\mid B_0\tau\mid E_1(1+\gamma-\gamma\lambda)^2}{|p_2(q,s)\Psi(\mu,\lambda,\gamma)|},\quad \mathcal{F}_2=\sqrt{\frac{\mid B_0\mid(\mid E_2-E_1\mid+E_1)\mid\tau\mid(1+\gamma-\gamma\lambda)^3}{\mid\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))\mid}}
$$

and

$$
\mathcal{F}_3=\frac{\mid B_0\tau\mid E_1^{3/2}(1+\gamma-\gamma\lambda)^2}{\sqrt{\mid B_0\tau E_1^2(1+\gamma-\gamma\lambda)\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))+(E_1-E_2)p_2^2(q,s)\Psi^2(\mu,\lambda,\gamma)\mid}},
$$

and

$$| a_3 |\leq \min\{\mathcal{G}_1, \mathcal{G}_2\}$$

for

$$\mathcal{G}_1 = \frac{| \tau | (|B_0| + |B_1|)E_1(1+\gamma-\gamma\lambda)^2}{|p_3(q,s)|\Theta(\mu,\lambda,\gamma)} + \frac{| \tau |^2 B_0^2E_1^2(1+\gamma-\gamma\lambda)^4}{p_2^2(q,s)\Psi^2(\mu,\lambda,\gamma)}$$

and

$$\mathcal{G}_2 = \frac{| \tau | (|B_0| + |B_1|)E_1(1+\gamma-\gamma\lambda)^2}{|p_3(q,s)|\Theta(\mu,\lambda,\gamma)} + \frac{| \tau B_0 | (|E_2-E_1|+E_1)(1+\gamma-\gamma\lambda)^3}{|\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))|},$$

where

$$\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s)) = (1+\gamma-\gamma\lambda)\Theta(\mu,\lambda,\gamma)p_3(q,s) - [\lambda(1-\gamma)+2\gamma]\Psi(\mu,\lambda,\gamma)p_2^2(q,s), \tag{3.2}$$

$$\Psi(\mu,\lambda,\gamma) = 2-\gamma\lambda-\lambda+2\mu(1+\gamma-\gamma\lambda) \tag{3.3}$$

and

$$\Theta(\mu,\lambda,\gamma) = 3-2\gamma\lambda-\lambda+6\mu(1+\gamma-\gamma\lambda). \tag{3.4}$$

Proof Here, we follow the method of Theorem 2.1. If $f(z) \in \mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$, then by Definition 1.2 there exist two analytic functions $u(z), v(z) : \Delta \to \Delta$ with $u(0) = 0$ and $v(0) = 0$ such that

$$\frac{1}{\tau}[\frac{z({}_q\mathcal{I}_sf)'(z)+\mu z^2({}_q\mathcal{I}_sf)''(z)}{(1-\lambda)z+\lambda(1-\gamma){}_q\mathcal{I}_sf(z)+\gamma z({}_q\mathcal{I}_sf)'(z)} - \frac{1}{[1+\gamma(1-\lambda)]}] = \varphi(z)[\phi(u(z))-1] \tag{3.5}$$

and

$$\frac{1}{\tau}[\frac{w({}_q\mathcal{I}_sg)'(z)+\mu w^2({}_q\mathcal{I}_sg)''(w)}{(1-\lambda)w+\lambda(1-\gamma)({}_q\mathcal{I}_sg)(w)+\gamma w({}_q\mathcal{I}_sg)'(w)} - \frac{1}{[1+\gamma(1-\lambda)]}] = \varphi(w)[\phi(v(w))-1]. \tag{3.6}$$

Expanding the left half parts of (3.5) and (3.6), we have that

$$\begin{aligned} & \frac{1}{\tau}[\frac{z({}_q\mathcal{I}_sf)'(z)+\mu z^2({}_q\mathcal{I}_sf)''(z)}{(1-\lambda)z+\lambda(1-\gamma){}_q\mathcal{I}_sf(z)+\gamma z({}_q\mathcal{I}_sf)'(z)} - \frac{1}{[1+\gamma(1-\lambda)]}] \\ = & \frac{\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_2(q,s)a_2z \\ & +[\frac{\Theta(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_3(q,s)a_3 - \frac{[\lambda(1-\gamma)+2\gamma]\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^3}p_2^2(q,s)a_2^2]z^2+\cdots \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & \frac{1}{\tau}[\frac{w({}_q\mathcal{I}_sg)'(z)+\mu w^2({}_q\mathcal{I}_sg)''(w)}{(1-\lambda)w+\lambda(1-\gamma)({}_q\mathcal{I}_sg)(w)+\gamma w({}_q\mathcal{I}_sg)'(w)} - \frac{1}{[1+\gamma(1-\lambda)]}] \\ = & -\frac{\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_2(q,s)a_2w \\ & +\frac{1}{\tau}[\frac{\Theta(\mu,\lambda,\gamma)}{(1+\gamma-\gamma\lambda)^2}p_3(q,s)a_3 - \frac{[\lambda(1-\gamma)+2\gamma]\Psi(\mu,\lambda,\gamma)}{(1+\gamma-\gamma\lambda)^3}p_2^2(q,s)a_2^2]w^2+\cdots. \end{aligned} \tag{3.8}$$

Therefore, From (2.14)–(2.15) and (3.5)–(3.8), we get that

$$\frac{\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_2(q,s)a_2=\frac{1}{2}B_0E_1c_1, \tag{3.9}$$

$$\begin{aligned}&\frac{\Theta(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_3(q,s)a_3-\frac{[\lambda(1-\gamma)+2\gamma]\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^3}p_2^2(q,s)a_2^2\\ =\ &\frac{1}{2}B_1E_1c_1+\frac{1}{2}B_0E_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_0E_2c_1^2,\end{aligned} \tag{3.10}$$

$$-\frac{\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_2(q,s)a_2=\frac{1}{2}B_0E_1d_1 \tag{3.11}$$

and

$$\begin{aligned}&\frac{\Theta(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^2}p_3(q,s)(2a_2^2-a_3)-\frac{[\lambda(1-\gamma)+2\gamma]\Psi(\mu,\lambda,\gamma)}{\tau(1+\gamma-\gamma\lambda)^3}p_2^2(q,s)a_2^2\\ =\ &\frac{1}{2}B_1E_1d_1+\frac{1}{2}B_0E_1(d_2-\frac{d_1^2}{2})+\frac{1}{4}B_0E_2d_1^2.\end{aligned} \tag{3.12}$$

From (3.9) and (3.11), we know that

$$a_2=\frac{B_0E_1c_1\tau(1+\gamma-\gamma\lambda)^2}{2p_2(q,s)\Psi(\mu,\lambda,\gamma)}=-\frac{B_0E_1d_1\tau(1+\gamma-\gamma\lambda)^2}{2p_2(q,s)\Psi(\mu,\lambda,\gamma)}. \tag{3.13}$$

Then, it infers that

$$c_1=-d_1 \tag{3.14}$$

and

$$B_0^2E_1^2(c_1^2+d_1^2)=\frac{8p_2^2(q,s)\Psi(\mu,\lambda,\gamma)^2}{\tau^2(1+\gamma-\gamma\lambda)^4}a_2^2. \tag{3.15}$$

By (3.10) and (3.12), we have that

$$\frac{1}{4}B_0(c_1^2+d_1^2)(E_2-E_1)+\frac{1}{2}B_0E_1(c_2+d_2)=\frac{2a_2^2}{\tau(1+\gamma-\gamma\lambda)^3}\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s)). \tag{3.16}$$

Therefore, by (3.15)–(3.16) we know that

$$a_2^2=\frac{\frac{1}{4}B_0^2E_1^3\tau^2(1+\gamma-\gamma\lambda)^4(c_2+d_2)}{B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))+(E_1-E_2)p_2^2(q,s)\Psi^2(\mu,\lambda,\gamma)}. \tag{3.17}$$

Therefore, from (3.15)–(3.17) and Lemma 1.3, we obtain that

$$\begin{aligned}|\ a_2\ | &\leq \frac{|\ B_0\tau\ |\ E_1(1+\gamma-\gamma\lambda)^2}{|p_2(q,s)|\Psi(\mu,\lambda,\gamma)},\\ |\ a_2\ | &\leq \sqrt{\frac{|\ B_0\ |\ (|\ E_2-E_1\ |+E_1)\ |\ \tau\ |\ (1+\gamma-\gamma\lambda)^3}{|\ \Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))\ |}}\end{aligned}$$

and

$$|\ a_2\ |\leq \frac{|\ B_0\tau\ |\ E_1^{3/2}(1+\gamma-\gamma\lambda)^2}{\sqrt{|\ B_0\tau E_1^2(1+\gamma-\gamma\lambda)\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))+(E_1-E_2)p_2^2(q,s)\Psi^2(\mu,\lambda,\gamma)\ |}}.$$

Similarly, from (3.10) and (3.12), it implies that

$$\frac{1}{2}B_1E_1(c_1-d_1)+\frac{1}{2}B_0E_1(c_2-d_2)=\frac{2p_3(q,s)\Theta(\mu,\lambda,\gamma)(a_3-a_2^2)}{\tau(1+\gamma-\gamma\lambda)^2}. \tag{3.18}$$

Hence, by (3.15) and (3.18), it follows that

$$a_3=\frac{\tau[B_1E_1(c_1-d_1)+B_0E_1(c_2-d_2)](1+\gamma-\gamma\lambda)^2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)}+\frac{\tau^2B_0^2E_1^2(1+\gamma-\gamma\lambda)^4(c_1^2+d_1^2)}{8p_2^2(q,s)\Psi^2(\mu,\lambda,\gamma)}.$$

So, we obtain from Lemma 1.3 that

$$\mid a_3\mid\leq\frac{\mid\tau\mid(|B_0|+|B_1|)E_1(1+\gamma-\gamma\lambda)^2}{\mid p_3(q,s)\Theta(\mu,\lambda,\gamma)\mid}+\frac{\mid\tau\mid^2 B_0^2E_1^2(1+\gamma-\gamma\lambda)^4}{p_2^2(q,s)\Psi^2(\mu,\lambda,\gamma)}.$$

On the other hand, by (3.16) and (3.18), we infer that

$$\begin{aligned}a_3 &= \frac{\tau[B_1E_1(c_1-d_1)+B_0E_1(c_2-d_2)](1+\gamma-\gamma\lambda)^2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)}\\ &\quad+\frac{\tau B_0[(E_2-E_1)(c_1^2+d_1^2)+2E_1(c_2+d_2)](1+\gamma-\gamma\lambda)^3}{8\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))}.\end{aligned}$$

Thus, from Lemma 1.3 we see that

$$\mid a_3\mid\leq\frac{\mid\tau\mid(|B_0|+|B_1|)E_1(1+\gamma-\gamma\lambda)^2}{\mid p_3(q,s)\mid\Theta(\mu,\lambda,\gamma)}+\frac{\mid\tau B_0\mid(|E_2-E_1|+E_1)(1+\gamma-\gamma\lambda)^3}{|\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))|}.$$

Next, we consider Fekete-Szegö problems for the class $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$.

Theorem 3.2 Let $f(z)$ given by (1.1) belong to the class $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$ and $\delta\in\mathbb{R}$. Then

$$\mid a_3-\delta a_2^2\mid\leq\frac{|B_1\tau|E_1(1+\gamma-\gamma\lambda)^2}{|p_3(q,s)|\Theta}+\frac{|1-\delta|B_0^2E_1^3|\tau|(1+\gamma-\gamma\lambda)^4}{|B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2|},$$

if

$$|B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2|\leq|(1-\delta)B_0\tau p_3(q,s)|\Theta E_1^2(1+\gamma-\gamma\lambda)^2,$$

or

$$\mid a_3-\delta a_2^2\mid\leq\frac{(|B_0|+|B_1|)|\tau|E_1(1+\gamma-\gamma\lambda)^2}{|p_3(q,s)|\Theta},$$

if

$$|B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2|\geq|(1-\delta)B_0\tau p_3(q,s)|\Theta E_1^2(1+\gamma-\gamma\lambda)^2,$$

where $\Phi=\Phi(\mu,\lambda,\gamma,p_2(q,s),p_3(q,s))$, $\Theta=\Theta(\mu,\lambda,\gamma)$ and $\Psi=\Psi(\mu,\lambda,\gamma)$ are the same as in Theorem 3.1.

Proof From (3.18), it follows that

$$a_3-a_2^2=\frac{B_1E_1\tau(c_1-d_1)(1+\gamma-\gamma\lambda)^2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)}+\frac{B_0E_1\tau(c_2-d_2)(1+\gamma-\gamma\lambda)^2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)}.$$

By (3.17) we easily obtain that

$$a_3-\delta a_2^2=\frac{B_1E_1\tau(c_1-d_1)(1+\gamma-\gamma\lambda)^2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)}$$

$$+\frac{B_0E_1\tau(1+\gamma-\gamma\lambda)^2[B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2+(1-\delta)B_0E_1^2\tau p_3(q,s)\Theta(1+\gamma-\gamma\lambda)^2]c_2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)[B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2]}$$

$$+\frac{B_0E_1\tau(1+\gamma-\gamma\lambda)^2[B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2-(1-\delta)B_0E_1^2\tau p_3(q,s)\Theta(1+\gamma-\gamma\lambda)^2]d_2}{4p_3(q,s)\Theta(\mu,\lambda,\gamma)[B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2]}$$

Hence, from Lemma 1.3, we imply that

$$|\ a_3-\delta a_2^2\ |\le\frac{|B_1\tau|E_1(1+\gamma-\gamma\lambda)^2}{|p_3(q,s)|\Theta}+\frac{|1-\delta|B_0^2E_1^3|\tau|(1+\gamma-\gamma\lambda)^4}{|B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2|},$$

when

$$|B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2|\le|(1-\delta)B_0\tau p_3(q,s)|\Theta E_1^2(1+\gamma-\gamma\lambda)^2,$$

or

$$|\ a_3-\delta a_2^2\ |\le\frac{(|B_0|+|B_1|)|\tau|E_1(1+\gamma-\gamma\lambda)^2}{|p_3(q,s)|\Theta},$$

when

$$|B_0E_1^2\tau(1+\gamma-\gamma\lambda)\Phi+(E_1-E_2)p_2^2(q,s)\Psi^2|\ge|(1-\delta)B_0\tau p_3(q,s)|\Theta E_1^2(1+\gamma-\gamma\lambda)^2.$$

Remark 3.3 In fact, from Theorems 3.1 and 3.2, we may consider the coefficient bound estimates and Fekete-Szegö problem for the classes $\mathcal{QS}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(\tau,\gamma,1,\gamma;\phi)$, $\mathcal{QS}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(\tau,1,0,1;\phi)$, $\mathcal{QS}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(1,\mu,0,0;\phi)$, $\mathcal{QS}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(1,0,1,\gamma;\phi)$ and $\mathcal{QS}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(1,0,\lambda,0;\phi)$, etc. Similarly, if we choose some suitable parameters $\alpha_k(k=1,\cdots,q)$, $\beta_j(j=1,\cdots,s)$, τ,μ,λ and γ without quasi-subordination (i.e., $\varphi(z)\equiv1$), we provide the following reduced versions for $\mathcal{QS}_{\sum}{}_{\beta_1,\cdots,\beta_s}^{\alpha_1,\cdots,\alpha_q}(\tau,\mu,\lambda,\gamma;\phi)$ in Theorem 3.1.

(I) $\mathcal{QS}_{\sum}{}_{a}^{a,1}(1,0,1,0;\phi)=\mathcal{S}_{\sum}(\phi)$, refer to Ma and Minda [52];

(II) $\mathcal{QS}_{\sum}{}_{a}^{a,1}(1,0,\lambda,0;\phi)=\mathcal{G}_{\sum}^{\phi;\phi}(\gamma)$, refer to Magesh and Yamini [53];

(III) $\mathcal{QS}_{\sum}{}_{a}^{a,1}(\tau,\mu,0,0;\phi)=\sum(\tau,\mu,\phi)$, refer to Srivastava and Bansal [35];

(IV) $\mathcal{QS}_{\sum}{}_{a}^{a,1}(\tau,\gamma,1,\gamma;\phi)=\mathcal{S}_{\sum}(\gamma,\tau;\phi)$, refer to Deniz [21];

(V) $\mathcal{QS}_{\sum}{}_{c}^{a,b}(\tau,\gamma,1,\gamma;\phi)\equiv\mathcal{S}_{\sum}^{a,b,c}(\tau,\gamma;\phi)$, $\mathcal{QS}_{\sum}{}_{c}^{a,b}(\tau,\mu,0,0;\phi)\equiv\mathcal{K}_{\sum}^{a,b,c}(\tau,\mu;\phi)$ and $\mathcal{QS}_{\sum}{}_{c}^{a,b}(1,0,1,0;\phi)=\mathcal{J}_{\sum}(0,\phi)$, refer to refer to Patil and Naik [46];

(VI) $\mathcal{QS}_{\sum}{}_{a}^{a,1}(\tau,1,0,1;\phi)=\mathcal{S}_{\sum}(\tau,1,0,1;\phi)$ and $\mathcal{QS}_{\sum}{}_{a}^{a,1}(\tau,\gamma,\lambda,\gamma;\phi)=\mathcal{S}_{\sum}(\tau,\gamma,\lambda,\gamma;A,B)$, refer to srivastava et al. for Corollary 1 and Example 10 in [32], respectively. Here, the function ϕ in the second equality is defined by

$$\phi(z)=\frac{1+Az}{1+Bz}\quad(-1\le A<B\le1).\tag{3.19}$$

Remark 3.4 Let $\varphi(z)\equiv1$ and $\phi(z)=\frac{1+(1-2\beta)z}{1-z}$ for $0\le\beta<1$. If we take some suitable parameters τ,μ,λ and γ, we also have the following reduced versions for

$\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$ in Theorems 3.1 and 3.2. For example, $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,0,1,0;\beta) = {}_q\mathcal{S}_s(\tau,\beta)$, $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,1,1,1;\beta) = {}_q\mathcal{C}_s(\tau,\beta)$, refer to Al-Hawary et al. [47] for the classes of analytic and univalent (but not bi-univalent) functions.

Remark 3.5 In addition, if we only choose some suitable parameters τ,μ,λ and γ, we give some reduced versions for $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\mu,\lambda,\gamma;\phi)$ without Dziok-Srivastava operator in Theorem 3.1. For example, $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(\tau,\gamma,0,0;\phi) = \mathcal{K}^q_{\gamma,\tau}(\phi)(1 \geq \gamma \geq 0)$, $\mathcal{QS}_{\sum}{}^{\alpha_1,\cdots,\alpha_q}_{\beta_1,\cdots,\beta_s}(1,\alpha,1,0;\phi) = \mathcal{H}^q_\alpha(\phi)(\alpha \geq 0)$, refer to Goyal et al. [37].

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与Dziok-Srivastava 算子有关的几类解析双单值函数拟从属子类的Fekete-Szegö 问题

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摘要: 本文介绍了与Dziok-Srivastava 算子有关的解析双单叶类 $\sum$ 的两个拟从属子类, 系数估计和Fekete-Szegö 泛函. 利用微分拟从属和卷积算子理论, 获得了相应函数子类的Fekete-Szegö 泛函不等式和系数a_2 和a_3 的有界估计, 推广和改进了某些早期已知结果.

关键词: Fekete-Szegö 问题; 双单叶函数; Gaussian 超几何函数; Dziok-Srivastava 算子; 拟从属

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