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NONCONFORMING FINITE ELEMENT METHOD FOR THE NONLINEAR KLEIN-GORDON EQUATION WITH MOVING GRIDS

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Abstract: In this paper, the nonlinear Klein-Gordon equation is studied. By using the Crank-Nicolson moving grid nonconforming finite element method, the traditional Riesz projection operator is not needed, interpolation techniques and special properties of the element are used to obtain the corresponding convergence analysis and optimal error estimation.

Keywords: Klein-Gordon equation; anisotropy; moving grids; nonconforming; Crank-Nicolson scheme

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1 Introduction

Moving grids method has important applications in a variety of physical and engineering areas such as solid and fluid dynamics, combustion, heat transfer, material science, etc. This method is more efficient than the fixed grids and does not increase computing cost. We usually apply the finite element methods to the spatial domain, but choose difference methods with respect to the time variable for solving evolution partial differential equations. At the same time, different meshes of domain are used at different time level.

Several moving grids techniques were studied. Such as [1] considered the moving grids finite element method; [2] and [3] constructed and analyzed this method for the oil-water two-phase displacement problem; [4–8] analyzed the parabolic, Stokes problems, parabolic integro-differential equations, generalized nerve conductive equations and fractional diffusion equations with moving grids nonconforming finite element scheme respectively. But the analysis in the above studies relies on the regular condition or quasi-uniform assumption for meshes.

The Klein-Gordon equation is the most basic equation used in relativistic quantum mechanics and quantum field theory to describe a spin-zero particle. The equation is closely related to the physical problem and plays an important role in the study of soliton. In [9],

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authors studied the existence of a unique global solution under the condition that the parameter is small enough. In [10], a display difference scheme was established for one-dimensional Klein-Gordon equation of unbounded region, and the results of stability and convergence of the scheme were obtained by the energy analysis method. In [11], the numerical solution of one-dimensional Klein-Gordon equation was studied. However, the finite element method for the Klein-Gordon equation is rare.

In this paper, we mainly focus on the convergence theory, the finite element method of moving grids is introduced, and the Crank-Nicolson discrete scheme of the nonlinear Klein-Gordon equation is analyzed without requiring the subdivision to satisfy the regular hypothesis, and the corresponding optimal error estimation of the moving grid approach is derived. It is worth mentioning that, in the usual finite element method of moving meshes, it is necessary to use the Riesz projection to approximate the solution of the original problem, and this paper makes use of the particularity of the element structure, that is, $u-\Pi u$ and the elements in the finite element space are orthogonal in the sense of energy mode, and the Riesz projection is used to simplify the proof process of the previous documents.

2 Element Construction

For the sake of convenience, like [12], let $\Omega\subset R^2$ be a rectangular domain with boundary $\partial\Omega$ parallel to the x-axis or y-axis in the plane, Let Γ_h be a family of rectangular subdivisions, i.e., $\overline{\Omega}=\bigcup\limits_{K\in\Gamma_h}K$. For the general element K, we denote the lengths of edges parallel to x-axis and y-axis by $2h_x$ and $2h_y$, respectively, and the barycenter of element K by (x_K,y_K), the four vertices of K are $a_1=(x_K-h_x,y_K-h_y)$, $a_2=(x_K+h_x,y_K-h_y)$, $a_3=(x_K+h_x,y_K+h_y)$ and $a_4=(x_K-h_x,y_K+h_y)$, respectively, and the four edges are $l_i=\overline{a_ia_{i+1}}$ ($i=1,2,3,4$ (mod 4)), $h_K=\max\limits_{K\in\Gamma_h}\{h_x,h_y\}$, $h=\max\limits_{K\in\Gamma_h}h_K$.

Let $\hat{K}=[-1,1]\times[-1,1]$ be reference element on the $\xi-\eta$ plane, the four vertices of $\hat{K}$ are $\hat{a}_1=(-1,-1)$, $\hat{a}_2=(1,-1)$, $\hat{a}_3=(1,1)$ and $\hat{a}_4=(-1,1)$, respectively, $\hat{l}_i=\overline{\hat{a}_i\hat{a}_{i+1}}$ ($i=1,2,3,4$ (mod4)) be the four edges of $\hat{K}$.

We define the finite element $(\hat{K},\hat{P},\hat{\sum})$ (see [13])

$$\hat{\sum}=\{\hat{v}_1,\hat{v}_2,\hat{v}_3,\hat{v}_4,\hat{v}_5\},\quad \hat{P}=\mathrm{span}\{1,\xi,\eta,\varphi(\xi),\varphi(\eta)\},$$

where $\hat{v}_i=\frac{1}{|\hat{l}_i|}\int_{\hat{l}_i}\hat{v}d\hat{s}, i=1,2,3,4, \hat{v}_5=\frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{v}d\xi d\eta, \varphi(t)=\frac{1}{2}(3t^2-1)$.

$\forall\hat{v}\in H^1(\hat{K})$, interpolation functions $\hat{\Pi}\hat{v}$ can be expressed as

$$\hat{\Pi}\hat{v}=\hat{v}_5+\frac{1}{2}(\hat{v}_2-\hat{v}_4)\xi+\frac{1}{2}(\hat{v}_3-\hat{v}_1)\eta+\frac{1}{2}(\hat{v}_2+\hat{v}_4-2\hat{v}_5)\varphi(\xi)+\frac{1}{2}(\hat{v}_3+\hat{v}_1-2\hat{v}_5)\varphi(\eta). \quad (2.1)$$

Lemma 2.1 [14] The interpolation operator $\hat{\Pi}$ defined as (2.1) has the anisotropic property, i.e., for all $\hat{\phi}\in H^2(\hat{K})$ and $\alpha=(a_1,a_2)$, when $|\alpha|=1$, there holds

$$\|\widehat{D}^\alpha(\hat{\phi}-\hat{\Pi}^c\hat{\phi})\|_{0,\widehat{K}}\le C|\widehat{D}^\alpha\hat{\phi}|_{1,\widehat{K}}. \quad (2.2)$$

Here and later, we use C denotes a positive constant real number independent of h_K and $\frac{h_K}{\rho_K}$, possibly different at different occurrences.

Define the affine mapping $F_K : \hat{K} \to K$ by

$$\begin{cases} x = x_K + h_x\xi, \\ y = y_K + h_y\eta. \end{cases} \tag{2.3}$$

Then $\forall K \in \Gamma_h$, the associated finite element spaces V_h can be defined as

$$V_h = \{v_h | \hat{v} = v_h|_K \circ F_K \in \hat{P}, \int_F [v_h] ds = 0, F \subset \partial K\}, \tag{2.4}$$

where $[v_h]$ stands for the jump of v_h across the edge F if F is an internal edge, and it is equal to v_h itself if F belongs to $\partial\Omega$.

We define the following interpolations Π over spaces V_h as

$$\Pi : H^2(\Omega) \to V_h, \Pi|_K = \Pi_K = \hat{\Pi} \circ F_K^{-1}.$$

It is easy to show, $\|\cdot\|_h = (\sum\limits_{K\in\mathcal{T}_h} |\cdot|_{1,K}^2)^{\frac{1}{2}}$ is the norm over V_h.

3 The Moving Grids Approximation of Crank-Nicolson Discretization Scheme

We consider the nonlinear Klein-Gordon equation

$$\begin{cases} u_{tt} + \alpha u_t - \gamma \triangle u + g(u) = f(X,t), & (X,t) \in \Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \\ u_t(X,0) = \varphi_0(X), & X \in \Omega, \\ u(X,t) = u_t(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \end{cases} \tag{3.1}$$

where $X = (x,y)$, $\alpha > 0$, $\gamma > 0$, $g(u)$ satisfies the Lipschitz continuous conditionon on the variable u ,and has the second order bounded partial derivative.

Let $u_t = Q$, (3.1) is equivalent to the following question

$$\begin{cases} \dfrac{\partial Q}{\partial t} + \alpha Q - \gamma \triangle u + g(u) = f(X,t), & (X,t) \in \Omega \times (0,T], \\ \dfrac{\partial u}{\partial t} = Q, & (X,t) \in \Omega \times [0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \\ Q(X,0) = \varphi_0(X), & X \in \Omega, \\ u(X,t) = Q(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{3.2}$$

The variational formulation for problem (3.2) is written as: $\forall v \in H_0^1(\Omega)$,

$$\begin{cases} (\dfrac{\partial Q}{\partial t}, v) + (\alpha Q, v) + a(u,v) + (g(u),v) = (f,v), \\ (\dfrac{\partial u}{\partial t}, v) = (Q,v), \\ (u(0) - u_0, v) = 0, \\ (Q(0) - \varphi_0, v) = 0, \end{cases} \tag{3.3}$$

where $u(0)=u(X,0), Q(0)=Q(X,0), a(u,v)=\gamma\int_\Omega \nabla u\nabla v dX$.

Then the approximation problem corresponding to (3.3) reads as: find $u_h, Q_h\in V_h, \forall v_h\in V_h$, such that

$$\begin{cases} (\frac{\partial Q_h}{\partial t}, v_h)+(\alpha Q_h, v_h)+a_h(u_h, v_h)+(g(u_h), v_h)=(f, v_h),\\ (\frac{\partial u_h}{\partial t}, v_h)=(Q_h, v_h),\\ (u_h(0)-\Pi u_0, v_h)=0,\\ (Q_h(0)-\Pi\varphi_0, v_h)=0, \end{cases} \tag{3.4}$$

where $a_h(u_h, v_h)=\gamma\sum\limits_K\int_K \nabla u_h\cdot\nabla v_h dX$.

In this section we apply the idea of moving grids to problem (3.4) and develop the Crank-Nicolson discretization scheme for anisotropic finite element. Let $0=t_0<t_1<\cdots<t_N=T$ be a partition of the time interval [0,T], $\triangle t=\frac{T}{N}$, $t_n=n\triangle t$ $(n=0,1,2,\cdots,N)$, $V_n^h=\{v(X,t_n); v\in V_h\}$. We choose the approximating space S of $u(X,t)$ in the following way: the approximation solution $u^h(X,t)\in S$ is the piecewise linear function with respect to the time subdivisions $0=t_0<t_1<\cdots<t_N=T$ based on $u^h(x,t_n)\in V_n^h$ $(n=0,1,2,\cdots,N)$.

Now, we introduce the Crank-Nicolson discretization scheme of anisotropic finite element to determine the function values $u_n^h=u^h(X,t_n)$ as follows

$$(\widehat{u}_0^h-u_0, v_h)=0, \quad \forall v_h\in V_1^h \quad \text{for } n=0, \tag{3.5}$$

$$(\widehat{Q}_0^h-Q_0, v_h)=0, \quad \forall v_h\in V_1^h \quad \text{for } n=0, \tag{3.6}$$

$$(\widehat{u}_n^h-u_n^h, v_h)=0, \quad \forall v_h\in V_{n+1}^h \quad \text{for } n>0, \tag{3.7}$$

$$(\widehat{Q}_n^h-Q_n^h, v_h)=0, \quad \forall v_h\in V_{n+1}^h \quad \text{for } n>0, \tag{3.8}$$

$$(Q_{n+1}^h-\widehat{Q}_n^h, v_h)+(\alpha Q_{n+\frac12}^h, v_h)\triangle t+a_h(u_{n+\frac12}^h, v_h)\triangle t+(g(u_{n+\frac12}), v_h)\triangle t=(f_{n+\frac12}, v_h)\triangle t,\ \forall v_h\in V_{n+1}^h, \tag{3.9}$$

$$(\frac{u_{n+1}^h-\widehat{u}_n^h}{\triangle t}, v_h)=(\frac{Q_{n+1}^h+\widehat{Q}_n^h}{2}, v_h), \tag{3.10}$$

where $u_n^h=u^h(X,t_n)$, $u_{n+\frac12}^h=\frac12(\widehat{u}_n^h+u_{n+1}^h)$, $Q_{n+\frac12}^h=\frac12(\widehat{Q}_n^h+Q_{n+1}^h)$, $f_{n+\frac12}=\frac12(f(X,t_n)+f(X,t_{n+1}))$. (3.7), (3.8) implies that $\widehat{u}_n^h=u_n^h, \widehat{Q}_n^h=Q_n^h$ when $V_n^h=V_{n+1}^h$. Furthermore, (3.7), (3.8) is a L^2-projection modification scheme for the former space when the two spaces V_n^h and V_{n+1}^h have different meshes or different interpolation functions. (3.9) is the general trapezoid difference scheme. We get $\widehat{u}_n^h, \widehat{Q}_n^h$ from u_n^h, Q_n^h in (3.7), (3.8) and get u_{n+1}^h, Q_{n+1}^h from $\widehat{u}_n^h, \widehat{Q}_n^h$ in (3.9), (3.10). So by partial differential equation theory u_n^h and Q_n^h can be determined uniquely through (3.5)–(3.10).

4 Error Estimates

The main error between the solution $u(X,t)$ and the approximation solution $u^h(X,t)$ consists of three parts: the interpolation error with respect to the finite element method, the difference error with respect to the time, and the error of moving grids.

Lemma 4.1 [13] For anisotropic meshes, $\forall u \in H^2(\Omega)$, then we have

$$\Big|\sum_K \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds\Big| \le Ch|u|_2\|v_h\|_h, \quad \forall\ v_h \in V_h, \tag{4.1}$$

here and later $n = (n_x, n_y)$ denotes the outward unit normal vector to ∂K.

Lemma 4.2 [13] $\forall u \in H^1(\Omega)$, there hold $a_h(u - \Pi u, v_h) = 0$, $\forall v_h \in V_h$.

Lemma 4.3 [13] $\|v_h\|_0 \le C\|v_h\|_h$, $\forall\ v_h \in V_h$.

Lemma 4.4 Let $u_n = u(X, t_n)$, $\forall v_h \in V^h_{n+1}$,then there holds

$$(Q_{n+1}-Q_n, v_h)+(\alpha Q_{n+\frac{1}{2}}, v_h)\triangle t+a_h(u_{n+\frac{1}{2}}, v_h)\triangle t+(g(u_{n+\frac{1}{2}}), v_h)\triangle t = (f_{n+\frac{1}{2}}, v_h)\triangle t+E_n(v_h), \tag{4.2}$$

where

$$\begin{aligned} |E_n(v_h)| \ \le\ & C[(\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 f}{\partial t^2}\|_0^2 d\tau)^{\frac{1}{2}} + (\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 Q}{\partial t^2}\|_0^2 d\tau)^{\frac{1}{2}} \\ & +(\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 u}{\partial t^2}\|_1^2 d\tau)^{\frac{1}{2}} + (\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 u}{\partial t^2}\|_0^2 d\tau)^{\frac{1}{2}}](\triangle t_n)^{\frac{5}{2}}\|v_h\|_h + Ch\triangle t\|v_h\|_h. \end{aligned} \tag{4.3}$$

Proof From (3.2), for all $v \in V^h_{n+1}$,

$$\begin{aligned} & (Q_{n+1} - Q_n, v_h) + (\int_{t_n}^{t_{n+1}} \alpha Q d\tau, v_h) + a_h(\int_{t_n}^{t_{n+1}} u d\tau, v_h) + (\int_{t_n}^{t_{n+1}} g(u) d\tau, v_h) \\ = \ & (\int_{t_n}^{t_{n+1}} f d\tau, v_h) + \int_{t_n}^{t_{n+1}} \Gamma_h(u, v_h) d\tau. \end{aligned} \tag{4.4}$$

Combining (4.2) and (4.4), we get

$$\begin{aligned} E_n(v_h) = \ & (\int_{t_n}^{t_{n+1}} (f - f_{n+\frac{1}{2}}) d\tau, v_h) - (\int_{t_n}^{t_{n+1}} \alpha(Q - Q_{n+\frac{1}{2}}) d\tau, v_h) - a_h(\int_{t_n}^{t_{n+1}} (u - u_{n+\frac{1}{2}}) d\tau, v_h) \\ & -(\int_{t_n}^{t_{n+1}} (g(u) - g(u_{n+\frac{1}{2}})) d\tau, v_h) + \int_{t_n}^{t_{n+1}} \Gamma_h(u, v_h) d\tau, \end{aligned} \tag{4.5}$$

where $\Gamma_h(u, v_h) = \sum\limits_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds$.

By Cauchy-Schwartz inequality and one-dimensional linear interpolation theory, we deduce that

$$|(\int_{t_n}^{t_{n+1}} (f - f_{n+\frac{1}{2}}) d\tau, v_h)| \le C|(\int_{t_n}^{t_{n+1}} \frac{\partial^2 f}{\partial t^2}(\triangle t)^2 d\tau, v_h)| \le C(\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 f}{\partial t^2}\|_0^2 d\tau)^{\frac{1}{2}}(\triangle t)^{\frac{5}{2}}\|v_h\|_h.$$

Similarly,

$$\begin{aligned} |(\int_{t_n}^{t_{n+1}} \alpha(Q - Q_{n+\frac{1}{2}}) d\tau, v_h)| \ &\le\ C(\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 Q}{\partial t^2}\|_0^2 d\tau)^{\frac{1}{2}}(\triangle t)^{\frac{5}{2}}\|v_h\|_h, \\ |a_h(\int_{t_n}^{t_{n+1}} (u - u_{n+\frac{1}{2}}) d\tau, v_h)| \ &\le\ C(\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 u}{\partial t^2}\|_1^2 d\tau)^{\frac{1}{2}}(\triangle t)^{\frac{5}{2}}\|v_h\|_h, \\ |(\int_{t_n}^{t_{n+1}} (g(u) - g(u_{n+\frac{1}{2}})) d\tau, v_h)| \ &\le\ C(\int_{t_n}^{t_{n+1}} \|\frac{\partial^2 u}{\partial t^2}\|_0^2 d\tau)^{\frac{1}{2}}(\triangle t)^{\frac{5}{2}}\|v_h\|_h. \end{aligned}$$

Applying Lemma 4.1,we obtain

$$|\int_{t_n}^{t_{n+1}}\Gamma_h(u,v_h)d\tau|\leq C(u)h\triangle t\|v_h\|_h.$$

Then (4.3) comes from the combination of above inequalities. Let

$$\begin{array}{llll}\nabla\xi_0=0, & \nabla\eta_0=0, & \lambda_0=0, & \theta_0=0\\ \xi_n=u_n^h-\Pi_n u_n, & \eta_n=u_n-\Pi_n u_n, & n=1,2,\cdots,N, \\ \widehat{\xi}_n=\widehat{u}_n^h-\Pi_{n+1}u_n, & \widehat{\eta}_n=u_n-\Pi_{n+1}u_n, & n=0,1,2,\cdots,N,\\ \lambda_n=Q_n^h-\Pi_n Q_n, & \theta_n=Q_n-\Pi_n Q_n, & n=1,2,\cdots,N,\\ \widehat{\lambda}_n=\widehat{Q}_n^h-\Pi_{n+1}Q_n, & \widehat{\theta}_n=Q_n-\Pi_{n+1}Q_n, & n=0,1,2,\cdots,N.\end{array}$$

Theorem 4.1 For the approximate solution u_n^h,Q_n^h and the solution u_n,Q_n of (3.5) $-$ (3.10), the following optimal error estimate holds

$$\max_{1\leq n\leq N}(\|Q_n^h-Q_n\|_0^2+\|u_n^h-u_n\|_h^2)\leq C(h^2+(\triangle t)^4). \tag{4.6}$$

Proof By definitions of $\xi_n,\eta_n,\lambda_n,\theta_n$, combining (3.9) and (4.2), we have for all $v_h\in V_{n+1}^h$,

$$\begin{aligned}&(\lambda_{n+1}-\widehat{\lambda}_n,v_h)+(\alpha\tfrac{\lambda_{n+1}+\widehat{\lambda}_n}{2},v_h)\triangle t+a_h(\tfrac{\xi_{n+1}+\widehat{\xi}_n}{2},v_h)\triangle t+(g(u_{n+\frac12}^h)-g(u_{n+\frac12}),v_h)\triangle t\\ =\ &(\theta_{n+1}-\widehat{\theta}_n,v_h)+(\alpha\tfrac{\theta_{n+1}+\widehat{\theta}_n}{2},v_h)\triangle t+a_h(\tfrac{\eta_{n+1}+\widehat{\eta}_n}{2},v_h)\triangle t-E_n(v_h).\end{aligned} \tag{4.7}$$

Applying Lemma 4.2, we obtain $a_h(\frac{\eta_{n+1}+\widehat{\eta}_n}{2},v_h)\triangle t=0$. By (3.10), we have

$$\frac{\lambda_{n+1}+\widehat{\lambda}_n}{2}=\frac{\xi_{n+1}-\widehat{\xi}_n}{\triangle t}+\Pi_{n+1}(\frac{u_{n+1}-u_n}{\triangle t}-\frac{Q_{n+1}+Q_n}{2}). \tag{4.8}$$

Substituting (4.8) into (4.7) with $v_h=\lambda_{n+1}+\widehat{\lambda}_n$, we deduce that

$$\begin{aligned}&\|\lambda_{n+1}\|_0^2-\|\widehat{\lambda}_n\|_0^2+\tfrac{\alpha}{2}\|\lambda_{n+1}+\widehat{\lambda}_n\|_0^2\triangle t+\|\xi_{n+1}\|_h^2-\|\widehat{\xi}_n\|_h^2\\ =\ &(\theta_{n+1}-\widehat{\theta}_n,\lambda_{n+1}+\widehat{\lambda}_n)+(\alpha\tfrac{\theta_{n+1}+\widehat{\theta}_n}{2},\lambda_{n+1}+\widehat{\lambda}_n)\triangle t-(g(u_{n+\frac12}^h)-g(u_{n+\frac12}),\lambda_{n+1}+\widehat{\lambda}_n)\triangle t\\ &+a_h(\xi_{n+1}+\widehat{\xi}_n,\Pi_{n+1}(\tfrac{u_{n+1}-u_n}{\triangle t}-\tfrac{Q_{n+1}+Q_n}{2}))\triangle t-E_n(\lambda_{n+1}+\widehat{\lambda}_n).\end{aligned} \tag{4.9}$$

Applying Cauchy-Schwartz inequality and Young inequality to yield

$$\begin{aligned}&|(\theta_{n+1}-\widehat{\theta}_n,\lambda_{n+1}+\widehat{\lambda}_n)|\leq\|\theta_{n+1}-\widehat{\theta}_n\|_0\|\lambda_{n+1}+\widehat{\lambda}_n\|_0\\ \leq\ &C\tfrac{\|\theta_{n+1}-\widehat{\theta}_n\|_0^2}{\triangle t}+C\|\lambda_{n+1}+\widehat{\lambda}_n\|_0^2\triangle t\\ \leq\ &Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2d\tau+C(\|\lambda_{n+1}\|_0^2+\|\widehat{\lambda}_n\|_0^2)\triangle t,\end{aligned} \tag{4.10}$$

$$\begin{aligned}|(\alpha\tfrac{\theta_{n+1}+\widehat{\theta}_n}{2},\lambda_{n+1}+\widehat{\lambda}_n)\triangle t| &\leq C\|\theta_{n+1}+\widehat{\theta}_n\|_0\|\lambda_{n+1}+\widehat{\lambda}_n\|_0\triangle t\\ &\leq Ch^4\triangle t+\tfrac{\alpha}{8}\|\lambda_{n+1}+\widehat{\lambda}_n\|_0^2\triangle t.\end{aligned}\tag{4.11}$$

Using Lemmas 4.3, we deduce that

$$\begin{aligned}&|(g(u^h_{n+\frac12})-g(u_{n+\frac12}),\lambda_{n+1}+\widehat{\lambda}_n)\triangle t|\leq C\|u^h_{n+\frac12}-u_{n+\frac12}\|_0\|\lambda_{n+1}+\widehat{\lambda}_n\|_0\triangle t\\ \leq\ & C(\|\xi_{n+\frac12}\|+\|\eta_{n+\frac12}\|_0\|\lambda_{n+1}+\widehat{\lambda}_n\|_0)\triangle t\\ \leq\ & C(\|\nabla\xi_{n+1}\|_0^2+\|\nabla\xi_n\|_0^2)\triangle t+Ch^4\triangle t+\frac{\alpha}{8}\|\lambda_{n+1}+\widehat{\lambda}_n\|_0^2\triangle t.\end{aligned}\tag{4.12}$$

$$\begin{aligned}&|a_h(\xi_{n+1}+\widehat{\xi}_n,\Pi_{n+1}(\frac{u_{n+1}-u_n}{\triangle t}-\frac{Q_{n+1}+Q_n}{2}))\triangle t|\\ \leq\ & \|\nabla(\xi_{n+1}+\widehat{\xi}_n)\|_0\|\nabla\Pi_{n+1}(\frac{u_{n+1}-u_n}{\triangle t}-\frac{Q_{n+1}+Q_n}{2})\|_0\triangle t\\ \leq\ & C\|\nabla(\xi_{n+1}+\widehat{\xi}_n)\|_0\|\nabla\int_{t_n}^{t_{n+1}}(\frac{\partial u}{\partial t}-Q_{n+\frac12})d\tau\|_0\\ \leq\ & C\,(\|\nabla\xi_{n+1}\|_0^2+\|\nabla\widehat{\xi}_n\|_0^2)\triangle t+C(\int_{t_n}^{t_{n+1}}\|\frac{\partial^2 Q}{\partial t^2}\|_1^2d\tau)(\triangle t)^4.\end{aligned}\tag{4.13}$$

By (4.3) and Young inequality, we have

$$\begin{aligned}|E_n(\lambda_{n+1}+\widehat{\lambda}_n)|\leq\ & C(\int_{t_n}^{t_{n+1}}(\|\frac{\partial^2 u}{\partial t^2}\|_0^2+\|\frac{\partial^2 Q}{\partial t^2}\|_0^2+\|\frac{\partial^2 f}{\partial t^2}\|_0^2+\|\frac{\partial^2 u}{\partial t^2}\|_1^2)d\tau)(\triangle t)^4\\ &+\frac{\alpha}{8}\|\lambda_{n+1}+\widehat{\lambda}_n\|_0^2\triangle t+Ch^2\triangle t+\frac{\alpha}{8}\|\lambda_{n+1}+\widehat{\lambda}_n\|_0^2\triangle t.\end{aligned}\tag{4.14}$$

Substituting (4.10) – (4.14) into (4.9), we get

$$(1-C\triangle t)(\|\lambda_{n+1}\|_0^2+\|\xi_{n+1}\|_h^2)-(1+C\triangle t)(\|\widehat{\lambda}_n\|_0^2+\|\widehat{\xi}_n\|_h^2)\leq Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2d\tau+\phi_n,\tag{4.15}$$

where

$$\phi_n=C(\int_{t_n}^{t_{n+1}}(\|\frac{\partial^2 u}{\partial t^2}\|_0^2+\|\frac{\partial^2 Q}{\partial t^2}\|_0^2+\|\frac{\partial^2 f}{\partial t^2}\|_0^2+\|\frac{\partial^2 u}{\partial t^2}\|_1^2)d\tau)(\triangle t)^4+Ch^2\triangle t+Ch^4\triangle t.$$

On the other hand, by (3.8), we obtain

$$(\widehat{\lambda}_n-\lambda_n,v_h)=(\widehat{\theta}_n-\theta_n,v_h),\ \forall\ v_h\in V^h_{n+1}.\tag{4.16}$$

Choosing $v_h=\widehat{\lambda}_n$ into (4.16), applying ε-Cauchy inequality to yield

$$\|\widehat{\lambda}_n\|_0^2\leq\frac{1}{1-\varepsilon}\|\lambda_n\|_0^2+\frac{1}{(1-\varepsilon)\varepsilon}\|\widehat{\theta}_n-\theta_n\|_0^2.\tag{4.17}$$

Similarly, by (3.7), we obtain

$$\|\widehat{\xi}_n\|_0^2\leq\frac{1}{1-\varepsilon}\|\xi_n\|_0^2+\frac{1}{(1-\varepsilon)\varepsilon}\|\widehat{\eta}_n-\eta_n\|_0^2.\tag{4.18}$$

Substituting (4.17) – (4.18) into (4.15), we get

$$\begin{aligned}&\frac{(1-\varepsilon)(1-C\triangle t)}{1+C\triangle t}(\|\lambda_{n+1}\|_0^2+\|\xi_{n+1}\|_h^2)-(\|\lambda_n\|_0^2+\|\xi_n\|_h^2)\\ \leq\ & Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2 d\tau+\phi_n+\frac{1}{\varepsilon}\|\widehat{\theta}_n-\theta_n\|_0^2+\frac{1}{\varepsilon}\|\widehat{\eta}_n-\eta_n\|_0^2.\end{aligned}\tag{4.19}$$

When there is no change between the two layers of the meshes,

$$\frac{1-C\triangle t}{1+C\triangle t}(\|\lambda_{n+1}\|_0^2+\|\xi_{n+1}\|_h^2)-(\|\lambda_n\|_0^2+\|\xi_n\|_h^2)\leq Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2 d\tau+\phi_n.\tag{4.20}$$

We use M to represent the number of mesh changes and assume that M is a bounded number independent of h and $\triangle t$. Assume that the former M-level mesh changes, and the $N-M$ layer mesh does not change, we have

$$\begin{aligned}&\frac{(1-\varepsilon)(1-C\triangle t)}{1+C\triangle t}(\|\lambda_{n+1}\|_0^2+\|\xi_{n+1}\|_h^2)-(\|\lambda_n\|_0^2+\|\xi_n\|_h^2) \qquad (4.21)\\ \leq\ & Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2 d\tau+\phi_n+\frac{1}{\varepsilon}\|\widehat{\theta}_n-\theta_n\|_0^2+\frac{1}{\varepsilon}\|\widehat{\eta}_n-\eta_n\|_0^2,\qquad n=1,2,\cdots,M-1,\\ &\frac{1-C\triangle t}{1+C\triangle t}(\|\lambda_{n+1}\|_0^2+\|\xi_{n+1}\|_h^2)-(\|\lambda_n\|_0^2+\|\xi_n\|_h^2) \qquad (4.22)\\ \leq\ & Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2 d\tau+\phi_n,\qquad n=M,M+1,M+2,\cdots,N-1.\end{aligned}$$

Take $n=1$, notice the selection of u_1 and Q_1, $\|\lambda_1\|_0=\|\xi_1\|_0=0$, by (4.21), we obtain

$$\|\lambda_2\|_0^2+\|\xi_2\|_h^2\leq\frac{1+C\triangle t}{(1-\varepsilon)(1-C\triangle t)}(Ch^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2 d\tau+\phi_1+\frac{1}{\varepsilon}\|\widehat{\theta}_1-\theta_1\|_0^2+\frac{1}{\varepsilon}\|\widehat{\eta}_1-\eta_1\|_0^2),\tag{4.23}$$

when $\triangle t$ is full small

$$(\frac{1+C\triangle t}{1-C\triangle t})^n\leq(\frac{1+C\triangle t}{1-C\triangle t})^N=(1+\frac{2C\triangle t}{1-C\triangle t})^{\frac{T}{\triangle t}}\leq e^{2CT}.\tag{4.24}$$

Again by the ϕ_n, we obtain

$$\|\lambda_2\|_0^2+\|\xi_2\|_h^2\leq C_1(h^2+(\triangle t)^4).$$

Furthermore, we have

$$\|\lambda_n\|_0^2+\|\xi_n\|_h^2\leq C_{n-1}(h^2+(\triangle t)^4),\qquad 3\leq n\leq M+1,$$

where C_{n-1} $(n=2,\cdots,M+1)$ are bounded number.

On both sides of (4.22) is multiplied by $(\frac{1-C\triangle t}{1+C\triangle t})^n$, we have

$$\begin{aligned}&(\frac{1-C\triangle t}{1+C\triangle t})^N(\|\lambda_N\|_0^2+\|\xi_N\|_h^2)-(\frac{1-C\triangle t}{1+C\triangle t})^{M+1}(\|\lambda_{M+1}\|_0^2+\|\xi_{M+1}\|_h^2)\\ \leq\ & C\sum_{n=M+1}^{N-1}(h^2\int_{t_n}^{t_{n+1}}\|\frac{\partial Q}{\partial t}\|_1^2 d\tau+\phi_n)(\frac{1-C\triangle t}{1+C\triangle t})^n,\end{aligned}\tag{4.25}$$

that is

$$\|\lambda_N\|_0^2 + \|\xi_N\|_h^2 \leq \quad Ce^{2CT}(\|\lambda_{M+1}\|_0^2 + \|\xi_{M+1}\|_h^2 + T \max_{0\leq n\leq N}(\phi_n + h^2 \int_{t_n}^{t_{n+1}} \|\frac{\partial Q}{\partial t}\|_1^2 d\tau)), \tag{4.26}$$

take $\triangle t$ full small, by Gronwall inequalities, we have

$$\|\lambda_N\|_0^2 + \|\xi_N\|_h^2 \leq C(h^2 + (\triangle t)^4).$$

Furthermore

$$\max_{M+1\leq n\leq N}(\|\lambda_n\|_0^2 + \|\xi_n\|_h^2) \leq C(h^2 + (\triangle t)^4).$$

Again

$$\|Q_n^h - Q_n\|_0^2 + \|u_n^h - u_n\|_h^2 \leq \|\lambda_n\|_0^2 + \|\theta_n\|_0^2 + \|\xi_n\|_h^2 + \|\eta_n\|_h^2,$$

by Lemma 2.1 and the finite element interpolation theory, we obtain (4.6), the proof is completed.

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非线性Klein-Gordon方程的变网格有限元方法

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摘要: 本文研究了非线性Klein-Gordon方程问题, 利用Crank-Nicolson变网格非协调有限元方法, 不需要传统的Riesz投影算子, 利用插值技巧和单元的特殊性质, 得到了相应的收敛性分析和最优误差估计.

关键词: Klein-Gordon方程; 各向异性; 变网格; 非协调; Crank-Nicolson格式

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