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INTEGRAL FORMULAS FOR COMPACT SUBMANIFOLDS IN EUCLID SPACE

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Abstract: In this paper, we study the problem of integral formulas for an oriented and compact n-dimension isometric immersion submanifold M^n without boundary in the $(n+p)$-dimension euclid space R^{n+p}. At first, we define the r-th higher order mean curvature H_r $(0 \leq r \leq n)$ along the direction of the unit mean curvature vector field ξ to M^n, and then we attain a new integral formula, by applying the method of moving frame and exterior differential, which generalizes a classical integral formula in the case of codimension $p=1$, that is in the case of hypersurfaces.

Keywords: euclid space; compact submanifold without boundary; mean curvature vector field; higher order mean curvature; integral formula

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1 Introduction

It is well known that study on hypersurfaces and submanifolds in euclid space is one of fundamental tasks of differential geometry. For oriented and compact isometric immersion hypersurfaces in euclid space, references [1–3] ever established a classical integral formula, that is the following Theorem 1.1.

In this paper, we study an oriented and compact n-dimension isometric immersion submanifold M^n in the $(n+p)$-dimension euclid space R^{n+p}. Let ξ be the unit mean curvature vector field of M^n. At first, we define the higher order mean curvature $H_r(r=0,1,2,\cdots,n)$ along the direction ξ. And then, by applying the method of moving frame and exterior differential, we attain a new integral formula, that is the following Theorem 1.2. When codimension $p=1$, Theorem 1.2 becomes Theorem 1.1.

Theorem 1.1 (see [1–3]) Let $\varphi: M \to R^{n+1}$ be an oriented and compact isometric immersion hypersurface without boundary. Then the following integral formulas hold

$$\int_M (H_k + H_{k+1}\langle\varphi, N\rangle)dM = 0, \ \ k=0,1,2,\cdots,n-1,$$

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here N is the unit normal vector field to M, H_k is the k-th higher order mean curvature of M and $\langle , \rangle$ is the euclid inner product in R^{n+1}, dM is the n-dimension Riemann volume form for M.

Theorem 1.2 Let $\varphi : M^n \to R^{n+p}$ be an oriented and compact n-dimension isometric immersion submanifold without boundary. Then the following integral formulas hold

$$\int_M (H_k + H_{k+1}\langle \varphi, \xi \rangle) dM = 0, \quad k = 0, 1, 2, \cdots, n-1,$$

here ξ is the unit mean curvature vector field to M^n, H_k is the k-th higher order mean curvature along the direction ξ and $\langle , \rangle$ is the euclid inner product in R^{n+p}, dM is the n-dimension Riemann volume form for M^n.

2 Preparation

Let R^{n+p} be the $(n+p)$-dimension euclid space and (M^n, g) be a smooth n-dimension Riemann manifold. Denote by $\varphi : M^n \to R^{n+p}$ a smooth immersion mapping between smooth manifolds. If the equation $g = \varphi^*(\langle , \rangle)$ holds everywhere on M^n, then M^n or $\varphi(M^n)$ is called an isometric immersion submanifold in R^{n+p}. Here $\langle , \rangle$ is the euclid inner product of R^{n+p} and φ^* is the pull-back mapping for the immersion mapping φ.

In this paper, we prescribe the index range as

$$1 \le i, j, k, l \le n, \quad n+1 \le \alpha, \quad \beta \le n+p, \quad 1 \le A, \quad B \le n+p.$$

Denote by $\{e_A\}$ a local unit orthogonal frame field for R^{n+p} such that when being confined onto M^n, $\{e_i\}$ is a local unit tangent frame field to M^n and $\{e_\alpha\}$ is a local unit normal frame field to M^n.

Denote by $\{\omega^A\}$ the dual frame field for $\{e_A\}$, then the second fundamental form II for M^n can be expressed in component form as

$$II = \sum_{\alpha,i,j} h_{ii}^{\alpha}\, \omega^i \otimes \omega^j \otimes e_\alpha.$$

Define the mean curvature vector field σ to M^n as

$$\sigma = \frac{1}{n} \sum_{\alpha,i} h_{ii}^{\alpha} e_\alpha.$$

It is well-known that the definition of σ is independent on the choice of the local unit orthogonal frame field $\{e_A\}$.

We consider the unit mean curvature vector field $\xi = \sigma/|\sigma|$. Let $\{\lambda_i\}$ be the principal curvature functions along the direction ξ, then the r-th higher order mean curvature H_r $(r = 1, 2, ..., n)$ is defined as

$$H_r = \tbinom{n}{r}^{-1} \cdot \Big(\sum_{1 \le i_1 < \ldots < i_r \le n} \lambda_{i_1} \cdot \lambda_{i_2} \cdots \lambda_{i_r} \Big),$$

here $\binom{n}{r}$ is the ordinary combination number. At the same time we define $H_0 \equiv 1$.

Reference [3] ever attained a fundamental integral formula that is the integral of the Codazzi tensor field on an isometric immersion hypersurface in R^{n+1}. Similar to reference [3], for n-dimension isometric immersion submanifold M^n of R^{n+p}, we attain the following Lemma 2.2. Here we firstly recall some relevant fundamental concepts and properties. Assume that S is a tensor field of type (k,k) on a Riemann manifold (M^n, g). If S is anti symmetric both to its each pair of covariant indices and to its each pair of contravariant indices, then we write

$$S \in \Gamma(\mathrm{End}\Lambda^k(TM)).$$

For $S \in \Gamma(\mathrm{End}\,\Lambda^k(TM))$, $T \in \Gamma(\mathrm{End}\,\Lambda^l(TM))$, we also consider the tensor field of type $(k+l, k+l)$,

$$S * T \in \Gamma(\mathrm{End}\,\Lambda^{k+l}(TM)),$$

and the definition of $S * T$ is that the exterior product of covariant components of S and the covariant components of T, and respectively the exterior product of contravariant components of S and the contravariant components of T. And by reference [3], this product $*$ is associative and commutative.

Definition 2.1 (see [3], Codazzi tensor field) Let (M^n, g) be a n-dimension Riemann manifold and $S \in \Gamma(\mathrm{End}\,\Lambda^k(TM))$. If for all C^∞ vector field $X_1, X_2, \cdots, X_{k+1} \in \Gamma(TM)$ we have

$$\sum_j (-1)^{j+1} (\nabla_{X_j} S(X_1 \wedge X_2 \wedge ... \wedge X_{j-1} \wedge X_{j+1} \wedge ... \wedge X_{k+1})) = 0,$$

then S is called a Codazzi tensor field on M^n, here ∇ is the Levi-Civita connection of (M^n, g).

According to reference [3], we know that if S and T are Codazzi tensor field respectively of type (k,k) and type (l,l) on (M^n, g) , then $S * T$ must be a Codazzi field tensor field of type $(k+l, k+l)$ on M^n . From reference [3], we also define a Codazzi tensor field A of type $(1,1)$ on M^n . Let (M^n, g) be a n-dimension Riemann manifold and $\psi : M^n \to R^{n+p}$ be an isometric immersion mapping. Let Y be the position vector field of $\psi(M^n)$ in R^{n+p}, then the Codazzi tensor field A of type $(1,1)$ is determined by

$$\langle A(X), Z\rangle = \langle A(Z), X\rangle = \langle II(X,Z), Y\rangle = \langle II(X,Z), \psi\rangle, \ \ \forall X, Z \in \Gamma(TM),$$

here $\langle , \rangle$ is the euclid inner product of R^{n+p} and II is the second fundamental form for (M^n, g).

Now we are ready to prove the following Lemma 2.2.

Lemma 2.2 let (M^n, g) be a n-dimension Riemann manifold and $\psi : M^n \to R^{n+p}$ be an oriented and isometric immersion mapping. Let $\psi(M^n)$ be compact and be without boundary. Assume that S is a Codazzi tensor field of type (k,k) on M^n , then the following integral formulas hold

$$\int_M \{(n-k) \cdot \mathrm{trace}(S) - \mathrm{trace}(S * A)\} dV = 0, \ \ k = 0, 1, 2, \cdots, n-1.$$

Here dV is the n-dimension Riemann volume form of M^n and *trace* is the trace operator.

Proof Denote by dV the n-dimension Riemann volume form of M^n, then the following equation

$$\alpha(X_1, X_2, \cdots, X_{n-1}) = dV(Y^{\tan}, X_1, X_2, \cdots, X_{n-1}), \ \forall X_1, X_2, \cdots, X_{n-1} \in \Gamma(TM)$$

determines a $(n-1)$ form and it is written as $\alpha = Y^{\tan} \rfloor dV$, here $Y^{\tan}$ is the tangent component to M^n of the position vector Y for $\psi(M^n)$ in R^{n+p}.

From reference [3] and direct computation, we have

$$\nabla_X \alpha = \{X - A(X)\} \rfloor dV, \ \forall X \in \Gamma(TM),$$

here ∇ is the Levi-Civita connection of (M^n, g). At first we assume that S is a Codazzi tensor field of type $(n-1, n-1)$.

Let $\{e_i\}$ is an unit orthogonal frame for M^n and write

$$E_j = e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n, \ E = e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

We can see $\omega = \alpha \circ S$ as a $(n-1)$ form which takes value in $\Gamma(\mathrm{End}\,\Lambda^k(TM)$. By the computation in reference [3], we have

$$\begin{aligned} d\omega(e_1, e_2, \cdots, e_n) &= \sum (-1)^{j+1} (\nabla_{e_j} \omega)(E_j) \\ &= \sum (-1)^{j+1} (\nabla_{e_j} \alpha) \circ S(E_j) + \alpha \circ \sum (-1)^{j+1} (\nabla_{e_j} S)(E_j). \end{aligned}$$

Because S is a Codazzi tensor field, the second term of the above equation vanishes and so we have

$$\begin{aligned} d\omega(e_1, e_2, \cdots, e_n) &= \sum (-1)^{j+1} \{e_j - A(e_j)\} \rfloor dV \circ S(E_j) \\ &= \sum (-1)^{j+1} \langle e_j \wedge S(e_j), E\rangle - \sum (-1)^{j+1} \langle A(e_j) \wedge S(E_j), E\rangle \\ &= \sum \langle e_j \wedge S(E_j), e_j \wedge E_j\rangle - \sum \langle A(e_j) \wedge S(E_j), e_j \wedge E_j\rangle \\ &= \mathrm{trace}\, S - \mathrm{trace}\, S * A. \end{aligned}$$

Because M^n is compact and is without boundary, $\int_M \omega dV = 0$. And so Lemma 2.1 holds in the case that S is a Codazzi tensor field of type $(n-1, n-1)$. Now we assume that S is a Codazzi tensor field of type (k, k). Denote by I the identity element of $\Gamma(\mathrm{End}\,\Lambda^{n-k-1}(TM))$.

Because I is parallel, $I * S$ is a Codizza tensor field of type $(n-1, n-1)$. So from the above conclusion we have

$$\int_M \{ \mathrm{trace}\,(S * I) - \mathrm{trace}\,(S * A * I) \} dV = 0, \ \ k = 0, 1, \cdots, n-1.$$

Finally we notice that

$$\mathrm{trace}\,(S * I) = (n-k)\, \mathrm{trace}\,(S)\,, \ \ \mathrm{trace}\,(S * A * I) = \mathrm{trace}\,(S * A),$$

we already finish the proof of Lemma 2.2.

3 Proof of Theorem 1.2

Theorem 1.2 Let $\varphi : M^n \to R^{n+p}$ be an oriented and compact n-dimension isometric immersion submanifold without boundary. Then the following integral formulas hold.

$$\int_M (H_k + H_{k+1}\langle \varphi, \xi \rangle) dM = 0, \quad k = 0, 1, 2, \cdots, n-1,$$

here ξ is the unit mean curvature vector field to M^n, H_k is the k-th higher order mean curvature along the direction ξ and $\langle , \rangle$ is the euclid inner product in R^{n+p}, dM is the n-dimension Riemann volume form for M^n.

Proof Let T_ξ be the shape operator of M^n along the direction of the unit mean curvature vector field ξ, that is to say, T_ξ is a tensor field of type $(1,1)$ on M^n defined by

$$T_\xi(X) = -(\overline{\nabla}_X \xi)^\top, \quad \forall\, X \in \Gamma(TM),$$

here $\overline{\nabla}$ is the Levi-Civita connection of R^{n+p}.

Because the Levi-Civita connection $\overline{\nabla}$ is flat, by the Codazzi equation for submanifold (see [4]), we know that T_ξ is a Codazzi tensor field of type $(1,1)$ on M^n.

Denote by $\lambda_1, \lambda_2, \cdots, \lambda_n$ the characteristic values of T_ξ and by σ_r the r-th fundamental homogeneous symmetry polynomial, that is

$$\sigma_r = \sum_{j_1 < \ldots < j_r} \lambda_{j_1} \cdots \lambda_{j_r}, \quad r = 1, 2, \cdots, n\,.$$

Let $S = T_\xi^k = T_\xi * T_\xi * \ldots * T_\xi$ be the k-times $*$ product. According to reference [3], we know that S is a Codazzi tensor field of type (k,k). And by direct computation, we have

$$\mathrm{trace}\,(S) = k!\,\sigma_k\,. \tag{3.1}$$

Denote by

$$h = h(x) = \langle \varphi(x), \xi(x) \rangle, \quad x \in M$$

the support function of M^n along the direction ξ. Then it is easy to see $A = -hT_\xi$. By direct computation, we have

$$\mathrm{trace}\,(S * A) = -h\,(k+1)!\,\sigma_{k+1}\,. \tag{3.2}$$

Now we recall once again the definition of the higher order mean curvature H_r along the unit mean curvature vector field ξ

$$H_k = \tbinom{n}{k}^{-1} \sigma_k, \quad k = 1, 2, \cdots, n.$$

We notice the above (3.1), (3.2) and then we apply Lemma 2.2, we already finish the proof for Theorem 1.2.

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欧氏空间中紧致子流形的积分公式

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摘要: 本文研究了 $(n+p)$ 维欧氏空间 R^{n+p} 中 n 维定向紧致无边子流形 M^n 的积分公式的问题. 首先定义了 M^n 沿其单位平均曲率向量场 ξ 方向的高阶平均曲率 H_r $(0 \le r \le n)$; 然后, 利用活动标架与外微分法, 获得了关于M^n 的一个新的积分公式. 新公式推广了余维数 $p=1$ 即超曲面情况下的经典积分公式.

关键词: 欧氏空间; 紧致无边子流形; 平均曲率向量场; 高阶平均曲率; 积分公式

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