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REGULARIZATION METHOD FOR AN ILL-POSED CAUCHY PROBLEM OF NONLINEAR ELLIPTIC EQUATION

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Abstract: This paper considers an ill-posed Cauchy problem of nonlinear elliptic equation. By using a regularization method to overcome the ill-posedness, we obtain the existence, uniqueness, stability and convergence result of the regularization solution, and an iterative scheme is constructed to calculate the regularization solution, which is an extension on the related research results of existing literature in the aspect of regularization theory and algorithm for Cauchy problem of elliptic equation.

Keywords: ill-posed problem; Cauchy problem; nonlinear elliptic equation; regularization method

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1 Introduction

The direct problems for elliptic equations (Dirichlet, Neumann or mixed boundary value problems) extensively studied. However, in some practical problems, the whole boundary data often can not be known, we only know the noisy data on a part of the boundary or at some interior points of the concerned domain, which will lead to some inverse problems.

This paper considers the following inverse problem of nonlinear elliptic equation

$$\begin{cases} u_{xx}+u_{yy}=f(x,y,u(x,y)), & 0<x<\pi, 0<y<T, \\ u(x,0)=\varphi(x), & 0\le x\le\pi, \\ u_y(x,0)=0, & 0\le x\le\pi, \\ u(0,y)=u(\pi,y)=0, & 0\le y\le T, \end{cases} \tag{1.1}$$

where $\varphi(x)\in L^2(0,\pi)$ is known function, $f:\mathbb{R}\times\mathbb{R}\times L^2(0,\pi)\to L^2(0,\pi)$ is an uniform Lipschitz continuous function, i.e., existing $L>0$ independent of $x,y\in\mathbb{R}$, $u,v\in L^2(0,\pi)$, such that

$$\|f(x,y,u)-f(x,y,v)\|\le L\|u-v\|. \tag{1.2}$$

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Our assignment is to determine $u(\cdot, y)$ from (1.1)–(1.2).

This problem is called the Cauchy problem. As $f(x, y, u) = -k^2 u$, it is the Helmholtz equation that has important applications in acoustic, hydrodynamic and electromagnetic waves [1, 2], etc. If f is independent of u, it becomes as the Poisson equation. If setting $f(x, y, u) = \sin u$, we deduce that the classical nonlinear elliptic sine-Gordon equation which mainly appears in the theory of Josephson effects, superconductors, spin waves in ferromagnets, see [3, 4].

Problem(1.1) is ill-posed in the sense that the solution (even if it exists) does not depend continuously on the given Cauchy data [5, 6]. It causes great difficulty in doing the numerical calculations, thus some special regularization techniques are required to stabilize numerical computations, see [6, 7]. Note that, as $f = 0$, the problem is the Cauchy problem of Laplace equation, many regularization methods were presented to study it in past years, such as the quasi-reversibility method [8–11], Tikhonov method [12], discretization method [13, 14], the fundamental solution method [15], non-local boundary value problem method [16, 17], etc. In 2014–2018, [18–21] respectively used modified boundary Tikhonov-type, Fourier truncated, filtering function and Generalized Tikhonov-type regularization methods to solve the abstract Cauchy problem of semi-linear elliptic equation in the general bounded domain. On the other similar works for the Cauchy problem of nonlinear elliptic equation, please see [22–28].

In 2012, [29] considered the Cauchy problem of Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < T, \\ u(0, y) = u(\pi, y) = 0, \quad 0 \le y \le T, \\ u(x, 0) = \varphi(x), \quad 0 \le x \le \pi, \\ u_y(x, 0) = 0, \quad 0 \le x \le \pi, \end{cases} \tag{1.3}$$

and for $p \ge 1$, we defined the regularization solution as

$$u_\alpha^\delta(x, y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\cosh(ny)}{1 + \alpha n^p \sinh(nT)} c_n^\delta X_n(x), \quad p \text{ is odd}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\cosh(ny)}{1 + \alpha n^p \cosh(nT)} c_n^\delta X_n(x), \quad p \text{ is even}, \end{cases} \tag{1.4}$$

i.e., the so-called improved non-local boundary value problem method, where $c_n^\delta = \int_0^\pi \varphi^\delta(\xi) X_n(\xi) d\xi$, $X_n(x) := \sqrt{\frac{2}{\pi}} \sin(nx) (n \ge 1)$, and $\{X_n(x)\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(0, \pi)$, φ^δ denotes the measured data and satisfies $\|\varphi^\delta - \varphi\| \le \delta$, $\|\cdot\|$ denotes the L^2-norm and $\delta > 0$ is a noise level, α is the regularization parameter. Inspired by this method, in the present paper we investigate the nonlinear problem (1.1)–(1.2) and adopting a similar technique to overcome its ill-poseness (see Section 2), this is an extension to the work in [29].

The article is constructed as below. Section 2 uses our method to treat problem (1.1) and proves the well-posedness of regularized problem. In Section 3, the convergence estimate for this method is derived. An iterative scheme is proposed to calculate the regularization solution and make some numerical verifications in Section 4. Some conclusions are given in Section 5.

2 Regularization Method and some Well-Posed Results

2.1 Regularization Method

Adopting the similar method as in [19], we can derive that the unique solution to problem (1.1) satisfies the nonlinear integral equation

$$u(x,y)=\sum_{n=1}^{\infty}\left(\cosh(ny)\varphi_n+\int_0^y\frac{\sinh(n(y-\tau))}{n}f_n(u)(\tau)d\tau\right)\sin(nx), \tag{2.1}$$

where

$$\varphi_n=\frac{2}{\pi}\int_0^{\pi}\varphi(x)\sin(nx)dx,\quad f_n(u)(y)=\frac{2}{\pi}\int_0^{\pi}f(x,y,u(x,y))\sin(nx)dx. \tag{2.2}$$

(2.1) means that $\cosh(ny)$, $\sinh(n(y-\tau))/n\to\infty$ $(n\to\infty)$, so in order to restore the stability of problem (1.1), we must eliminate the high frequency parts of two functions to design the regularization solution of original problem.

Based on the analysis above and considering the compatibility of physical dimension, when $p\geq 1$ is even, we approximate the nonlinear problem (1.1) as

$$\begin{cases}(u_\alpha^\delta)_{xx}+(u_\alpha^\delta)_{yy}=\displaystyle\sum_{n=1}^{\infty}\frac{1/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}f_n(u_\alpha^\delta)(y)\sin(nx), & 0<x<\pi,0<y<T,\\ u_\alpha^\delta(x,0)=\displaystyle\sum_{n=1}^{\infty}\frac{1/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}\varphi_n^\delta\sin(nx), & 0\leq x\leq\pi,\\ (u_\alpha^\delta)_y(x,0)=0, & 0\leq x\leq\pi,\\ u_\alpha^\delta(0,y)=u_\alpha^\delta(\pi,y)=0, & 0\leq y\leq T,\end{cases} \tag{2.3}$$

when $p\geq 1$ is odd, it is regularized by

$$\begin{cases}(u_\alpha^\delta)_{xx}+(u_\alpha^\delta)_{yy}=\displaystyle\sum_{n=1}^{\infty}\frac{1/\sinh(nT)}{\alpha n^p+1/\sinh(nT)}f_n(u_\alpha^\delta)(y)\sin(nx), & 0<x<\pi,0<y<T,\\ u_\alpha^\delta(x,0)=\displaystyle\sum_{n=1}^{\infty}\frac{1/\sinh(nT)}{\alpha n^p+1/\sinh(nT)}\varphi_n^\delta\sin(nx), & 0\leq x\leq\pi,\\ (u_\alpha^\delta)_y(x,0)=0, & 0\leq x\leq\pi,\\ u_\alpha^\delta(0,y)=u_\alpha^\delta(\pi,y)=0, & 0\leq y\leq T,\end{cases} \tag{2.4}$$

where

$$\varphi_n^\delta = \frac{2}{\pi}\int_0^\pi \varphi^\delta(x)\sin(nx)dx, \quad f_n(u_\alpha^\delta)(y) = \frac{2}{\pi}\int_0^\pi f(x,y,u_\alpha^\delta(x,y))\sin(nx)dx, \tag{2.5}$$

and $\alpha > 0$ is the regularization parameter, the measured data $\varphi^\delta \in L^2(0,\pi)$ satisfies $\|\varphi^\delta - \varphi\| \le \delta$, $\delta > 0$ is the error level, $\|\cdot\|$ denotes L^2-norm. In fact, it is easily verified that the solution to problem (2.3) satisfies the nonlinear integral equation

$$\begin{aligned} & u_\alpha^\delta(x,y) \\ = \quad & \sum_{n=1}^\infty \frac{1/\cosh(nT)}{\alpha n^p + 1/\cosh(nT)}\left(\cosh(ny)\varphi_n^\delta + \int_0^y \frac{\sinh(n(y-\tau))}{n} f_n(u_\alpha^\delta)(\tau)d\tau\right)\sin(nx), \end{aligned} \tag{2.6}$$

the solution of problem (2.4) satisfies the nonlinear integral equation

$$u_\alpha^\delta(x,y) = \sum_{n=1}^\infty \frac{1/\sinh(nT)}{\alpha n^p + 1/\sinh(nT)}\left(\cosh(ny)\varphi_n^\delta + \int_0^y \frac{\sinh(n(y-\tau))}{n} f_n(u_\alpha^\delta)(\tau)d\tau\right)\sin(nx), \tag{2.7}$$

where φ_n^δ and $f_n(u_\alpha^\delta)(y)$ are given by (2.5).

2.2 Some Well-Posed Results

According to the regularization theory, in order to ensure that the regularization solution is a stable approximation to exact solution, we need proof the existence, uniqueness, and stability for the solution of (2.6) and (2.7).

Now, let $x > 0$, $0 \le \tau \le y \le T$, we define two functions

$$h(x) = \frac{1}{\alpha x + e^{-xT}}, \quad g(x) = \frac{e^{(y-\tau-T)x}}{\alpha x + e^{-Tx}}. \tag{2.8}$$

Noting that when $\alpha < T$, $h(x)$ attain unique maximum at the point x_0 such that

$$h(x) \le h(x_0) = h\left(\frac{\ln(T/\alpha)}{T}\right) = \frac{T}{\alpha(1+\ln(T/\alpha))}, \tag{2.9}$$

and it is also proved that

$$g(x) \le T^{\frac{y-\tau}{T}}(\alpha\ln(T/\alpha))^{\frac{\tau-y}{T}}, \tag{2.10}$$

clearly, when $\tau = 0$, we have

$$g(x) = \frac{e^{(y-T)x}}{\alpha x + e^{-Tx}} \le T^{\frac{y}{T}}(\alpha\ln(T/\alpha))^{-\frac{y}{T}}. \tag{2.11}$$

Theorem 2.1 Let $\varphi^\delta \in L^2(0,\pi)$, f satisfies (1.2), then problem (2.6) exists a unique solution $u_\alpha^\delta \in C([0,T];L^2(0,\pi))$.

Proof For $w \in C([0,T];L^2(0,\pi))$, we define the operator $\Gamma(w)(\cdot,y)$ as

$$\Gamma(w)(\cdot,y) = \sum_{n=1}^\infty\left(\frac{\cosh(ny)\varphi_n^\delta}{1+\alpha n^p\cosh(nT)} + \int_0^y \frac{\sinh(n(y-\tau))f_n(w)(\tau)}{n\,(1+\alpha n^p\cosh(nT))}d\tau\right)\sin(nx), \tag{2.12}$$

then for $w, v \in C([0,T]; L^2(0,\pi))$, $q \geq 1$, we prove the following result

$$|||\Gamma^q(w)(\cdot,y) - \Gamma^q(v)(\cdot,y)||| \leq \frac{(LC_\alpha T)^q}{\sqrt{q!}} \, |||w-v||| \,, \tag{2.13}$$

where $C_\alpha = T/\alpha$, $|||\cdot|||$ denotes the sup norm in $C([0,T]; L^2(0,\pi))$. First, we use the induction method to derive the following estimate

$$\|\Gamma^q(w)(\cdot,y) - \Gamma^q(v)(\cdot,y)\| \leq (LC_\alpha)^q \frac{y^{q/2}T^{q/2}}{\sqrt{q!}} \, |||w-v||| \,, \tag{2.14}$$

For $q = 1$, from (2.12), (2.9) and (1.2), and $e^{(y-\tau-T)} \leq 1$, we can get

$$\begin{aligned}
& \|\Gamma(w)(\cdot,y) - \Gamma(v)(\cdot,y)\|^2 \\
= \; & \|\sum_{n=1}^{\infty} \int_0^y \frac{\sinh(n(y-\tau))}{n(1+\alpha n^p \cosh(nT))} (f_n(w)(\tau) - f_n(v)(\tau)) d\tau \sin(nx)\|^2 \\
\leq \; & \frac{\pi}{2} \sum_{n=1}^{\infty} (\int_0^y \frac{\sinh(n(y-\tau))}{n(1+\alpha n^p \cosh(nT))} (f_n(w)(\tau) - f_n(v)(\tau)) d\tau)^2 \\
\leq \; & \frac{\pi}{2} \sum_{n=1}^{\infty} \int_0^y (\frac{e^{n(y-\tau-T)}}{\alpha n + e^{-nT}})^2 d\tau \int_0^y (f_n(w)(\tau) - f_n(v)(\tau))^2 d\tau \\
\leq \; & \frac{\pi}{2} \sum_{n=1}^{\infty} \int_0^y (\frac{1}{\alpha n + e^{-nT}})^2 d\tau \int_0^y (f_n(w)(\tau) - f_n(v)(\tau))^2 d\tau \\
\leq \; & \frac{\pi}{2} y (\frac{T}{\alpha(1+\ln(T/\alpha))})^2 \int_0^y \sum_{n=1}^{\infty} (f_n(w)(\tau) - f_n(v)(\tau))^2 d\tau \\
\leq \; & TC_\alpha^2 \int_0^y \|f(\cdot,\tau,w(\cdot,\tau)) - f(\cdot,\tau,v(\cdot,\tau))\|^2 d\tau \\
\leq \; & L^2 TC_\alpha^2 \int_0^y \|w(\cdot,\tau) - v(\cdot,\tau)\|^2 d\tau \\
\leq \; & (LC_\alpha)^2 yT |||w-v|||^2.
\end{aligned}$$

When $q = i$, suppose that

$$\|\Gamma^i(w)(\cdot,y) - \Gamma^i(v)(\cdot,y)\|^2 \leq (LC_\alpha)^{2i} \frac{y^i T^i}{i!} |||w-v|||^2, \tag{2.15}$$

then for $q = i+1$, by (2.15), and use the similar process, we obtain that

$$\begin{aligned}
& \|\Gamma^{i+1}(w)(\cdot,y) - \Gamma^{i+1}(v)(\cdot,y)\|^2 \\
= \; & \|\sum_{n=1}^{\infty} \int_0^y \frac{\sinh(n(y-\tau))}{n(1+\alpha n^p \cosh(nT))} (f_n(\Gamma^i(w))(\tau) - f_n(\Gamma^i(v))(\tau)) d\tau \sin(nx)\|^2 \\
\leq \; & \frac{\pi}{2} \sum_{n=1}^{\infty} (\int_0^y \frac{\sinh(n(y-\tau))}{n(1+\alpha n^p \cosh(nT))} (f_n(\Gamma^i(w))(\tau) - f_n(\Gamma^i(v))(\tau)) d\tau)^2 \\
\leq \; & TC_\alpha^2 \int_0^y \|f(\cdot,\tau,\Gamma^i(w)(\cdot,\tau)) - f(\cdot,\tau,\Gamma^i(v)(\cdot,\tau))\|^2 d\tau \\
\leq \; & L^2 TC_\alpha^2 \int_0^y \|\Gamma^i(w)(\cdot,\tau) - \Gamma^i(v)(\cdot,\tau)\|^2 d\tau \leq (LC_\alpha)^{2i+2} \frac{y^{i+1}T^{i+1}}{(i+1)!} |||w-v|||^2.
\end{aligned}$$

By the induction principle, we have

$$\|\Gamma^q(w)(\cdot,y)-\Gamma^q(v)(\cdot,y)\| \le (LC_\alpha)^q \frac{y^{q/2}T^{q/2}}{\sqrt{q!}} |||w-v|||. \tag{2.16}$$

Subsequently, it is clearly derived that

$$|||\Gamma^q(w)(\cdot,y)-\Gamma^q(v)(\cdot,y)||| \le \frac{(LC_\alpha T)^q}{\sqrt{q!}} |||w-v|||. \tag{2.17}$$

Consider the operator $\Gamma : C([0,T];L^2(0,\pi)) \to C([0,T];L^2(0,\pi))$, and we know

$$\lim_{q\to\infty} \frac{(LC_\alpha T)^q}{\sqrt{q!}} = 0. \tag{2.18}$$

It exists a positive integer constant q_0 and satisfies $0 < \frac{(LC_\alpha T)^{q_0}}{\sqrt{q_0!}} < 1$, then Γ^{q_0} is a contraction, this indicates that $\Gamma^{q_0}(w) = w$ has a unique solution $u_\alpha^\delta \in C([0,T];L^2(0,\pi))$. Note that $\Gamma\left(\Gamma^{q_0}\left(u_\alpha^\delta\right)\right) = \Gamma\left(u_\alpha^\delta\right)$, hence $\Gamma^{q_0}\left(G\left(u_\alpha^\delta\right)\right) = \Gamma\left(u_\alpha^\delta\right)$. By the uniqueness of the fixed point, we obtain that $\Gamma\left(u_\alpha^\delta\right) = u_\alpha^\delta$, so $\Gamma(w) = w$ exists a unique solution $u_\alpha^\delta \in C([0,T];L^2(0,\pi))$.

Theorem 2.2 Let $u_{\alpha1}^\delta$ and $u_{\alpha2}^\delta$ be the solutions of problem (2.6) corresponding to the noisy data φ_1^δ and φ_2^δ, respectively, then for $\alpha < T$, we have

$$\|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\| \le C_0\left(\alpha\ln(T/\alpha)\right)^{-\frac{y}{T}} \left\|\varphi_1^\delta-\varphi_2^\delta\right\| \tag{2.19}$$

with $T_0 = \max\{T, T^{\frac{y}{T}}, T^{\frac{y-\tau}{T}}\}$, $C_0 = \sqrt{8T_0^2\left(1+2yL^2T_0^3e^{2yL^2T_0^3}\right)}$.

Proof From equation (2.6), we have

$$u_{\alpha1}^\delta(x,y) = \sum_{n=1}^\infty \frac{1/\cosh(nT)}{\alpha n^p + 1/\cosh(nT)}\left(\cosh(ny)\varphi_{1n}^\delta + \int_0^y \frac{\sinh(n(y-\tau))}{n} f_n(u_{\alpha1}^\delta)(\tau)d\tau\right)\sin(nx), \tag{2.20}$$

$$u_{\alpha2}^\delta(x,y) = \sum_{n=1}^\infty \frac{1/\cosh(nT)}{\alpha n^p + 1/\cosh(nT)}\left(\cosh(ny)\varphi_{2n}^\delta + \int_0^y \frac{\sinh(n(y-\tau))}{n} f_n(u_{\alpha2}^\delta)(\tau)d\tau\right)\sin(nx), \tag{2.21}$$

where $\varphi_{\mu n}^\delta = \frac{2}{\pi}\int_0^\pi \varphi_\mu^\delta(x)\sin(nx)dx$, $\mu = 1,2$.

For $n \ge 1$, from (2.20), (2.21), (2.9), (2.10) and (1.2), we get

$$\begin{aligned}
& \|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\|^2 \\
= \ & \frac{\pi}{2}\sum_{n=1}^\infty \|\frac{\cosh(ny)/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}(\varphi_{1n}^\delta-\varphi_{2n}^\delta) + \int_0^y \frac{\sinh(n(y-\tau))/\cosh(nT)}{n(\alpha n^p+1/\cosh(nT))}(f_n(u_{\alpha1}^\delta)(\tau)-f_n(u_{\alpha2}^\delta)(\tau))d\tau\|^2 \\
\le \ & \pi\sum_{n=1}^\infty (\frac{\cosh(ny)/\cosh(nT)}{\alpha n^p+1/\cosh(nT)})^2(\varphi_{1n}^\delta-\varphi_{2n}^\delta)^2 \\
& +\pi\sum_{n=1}^\infty \frac{1}{n^2}(\int_0^y \frac{\sinh(n(y-\tau))/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}(f_n(u_{\alpha1}^\delta)(\tau)-f_n(u_{\alpha2}^\delta)(\tau))d\tau)^2 \\
\le \ & 4\pi\sum_{n=1}^\infty (\frac{e^{n(y-T)}}{\alpha n+e^{-nT}})^2(\varphi_{1n}^\delta-\varphi_{2n}^\delta)^2 + \pi\sum_{n=1}^\infty \frac{1}{n^2}(\int_0^y \frac{e^{n(y-\tau-T)}}{\alpha n+e^{-nT}}(f_n(u_{\alpha1}^\delta)(\tau)-f_n(u_{\alpha2}^\delta)(\tau))d\tau)^2 \\
\le \ & 8T^{\frac{2y}{T}}(\alpha\ln(T/\alpha))^{-\frac{2y}{T}}\|\varphi_1^\delta-\varphi_2^\delta\|^2 + \pi\sum_{n=1}^\infty \int_0^y (\frac{e^{n(y-\tau-T)}}{\alpha n+e^{-nT}})^2 d\tau \int_0^y \|f_n(u_{\alpha1}^\delta)(\tau)-f_n(u_{\alpha2}^\delta)(\tau)\|^2 d\tau
\end{aligned}$$

$$\begin{aligned}\leq\ & 8T^{\frac{2y}{T}}(\alpha\ln(T/\alpha))^{-\frac{2y}{T}}\|\varphi_1^\delta-\varphi_2^\delta\|^2\\&+2yL^2T^{\frac{2y-2\tau}{T}}(\alpha\ln(T/\alpha))^{-\frac{2y}{T}}\int_0^y(\alpha\ln(T/\alpha))^{\frac{2\tau}{T}}\|u_{\alpha1}^\delta(\cdot,\tau)-u_{\alpha2}^\delta(\cdot,\tau)\|^2d\tau.\end{aligned}$$

Then

$$\begin{aligned}&(\alpha\ln(T/\alpha))^{\frac{2y}{T}}\|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\|^2\\\leq\ & 8T_0^2\|\varphi_1^\delta-\varphi_2^\delta\|^2+2L^2T_0^3\int_0^y(\alpha\ln(T/\alpha))^{\frac{2\tau}{T}}\|u_{\alpha1}^\delta(\cdot,\tau)-u_{\alpha2}^\delta(\cdot,\tau)\|^2d\tau.\end{aligned}$$

Using Gronwall's inequality [30], it can be obtained that

$$(\alpha\ln(T/\alpha))^{\frac{2y}{T}}\|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\|^2\leq 8T_0^2(1+2yL^2T_0^3e^{2yL^2T_0^3})\|\varphi_1^\delta-\varphi_2^\delta\|^2. \quad (2.22)$$

From (2.22), estimate (2.19) can be established.

3 Convergence Estimate

Theorem 3.1 Suppose that u is the solution of problem (1.1) and u_α^δ is the solution of problem (2.6). Let the measured data φ^δ satisfy $\|\varphi^\delta-\varphi\|\leq\delta$, and the exact solution u satisfy

$$\frac{\pi}{2}\sum_{n=1}^{\infty}n^{2p}e^{2n(T-y)}|\langle u(\cdot,y),\sin(nx)\rangle|^2\leq E^2, \quad (3.1)$$

and the regularization parameter α is chosen as

$$\alpha=\delta, \quad (3.2)$$

then for fixed $0<y\leq T$ and $\delta<T$, we have the following convergence estimate

$$\|u_\alpha^\delta(\cdot,y)-u(\cdot,y)\|\leq C_2\delta^{1-\frac{y}{T}}(\ln(T/\delta))^{-\frac{y}{T}}, \quad (3.3)$$

where $C_2=C_0+C_1$, $C_1=\sqrt{32T_0^2E^2(1+2yL^2T_0^3e^{2yL^2T_0^3})}$.

Proof Denoting u_α as the corresponding solution of problem (2.6) with the exact data φ. Using the triangle inequality, we have

$$\|u_\alpha^\delta-u\|\leq\|u_\alpha^\delta-u_\alpha\|+\|u_\alpha-u\|. \quad (3.4)$$

From Theorem 2.2, we get

$$\|u_\alpha^\delta(\cdot,y)-u_\alpha(\cdot,y)\|\leq C_0\left(\alpha\ln(T/\alpha)\right)^{-\frac{y}{T}}\left\|\varphi^\delta-\varphi\right\|. \quad (3.5)$$

By (2.1), (2.6), (2.10), (2.11), (3.1), and use the inequality $e^t/2\leq\cosh(t)\leq e^t$ for $t>0$, it can be derived that

$$\begin{aligned}&\|u_\alpha(\cdot,y)-u(\cdot,y)\|^2\\\leq\ & \frac{\pi}{2}\sum_{n=1}^{\infty}\|\frac{\alpha n^p}{\alpha n^p+1/\cosh(nT)}(\varphi_n\cosh(ny)+\frac{1}{n}\int_0^y\sinh(n(y-\tau))f_n(u)(\tau)d\tau)\\&+\frac{1}{n}\int_0^y\frac{\sinh(n(y-\tau))/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}(f_n(u_\alpha)(\tau)-f_n(u)(\tau))d\tau\|^2\\\leq\ & \pi\sum_{n=1}^{\infty}\left(\frac{\alpha\cosh(ny)/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}\right)^2\left(\frac{\cosh(nT)}{\cosh(ny)}\right)^2n^{2p}\left(\varphi_n\cosh(ny)+\frac{1}{n}\int_0^y\sinh(n(y-\tau))f_n(u)(\tau)d\tau\right)^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n^2}\left(\int_0^y \frac{\sinh(n(y-\tau))/\cosh(nT)}{\alpha n^p+1/\cosh(nT)}(f_n(u_\alpha)(\tau)-f_n(u)(\tau))d\tau\right)^2\\
\leq\ & 16\pi\sum_{n=1}^{\infty}\left(\frac{e^{n(y-T)}}{\alpha n+e^{-nT}}\right)^2\alpha^2 n^{2p}e^{2n(T-y)}\left(\varphi_n\cosh(ny)+\frac{1}{n}\int_0^y\sinh(n(y-\tau))f_n(u)(\tau)d\tau\right)^2\\
&+\pi\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^y\frac{e^{n(y-\tau-T)}}{\alpha n+e^{-nT}}(f_n(u_\alpha)(\tau)-f_n(u)(\tau))d\tau\right)^2\\
\leq\ & 32T^{\frac{2y}{T}}(\alpha\ln(T/\alpha))^{-\frac{2y}{T}}\alpha^2E^2+\pi\sum_{n=1}^{\infty}\int_0^y\left(\frac{e^{n(y-\tau-T)}}{\alpha n+e^{-nT}}\right)^2d\tau\int_0^y\|f_n(u_\alpha)(\tau)-f_n(u)(\tau))\|^2\,d\tau\\
\leq\ & 32T^{\frac{2y}{T}}(\alpha\ln(T/\alpha))^{-\frac{2y}{T}}\alpha^2E^2+2yL^2T^{\frac{2y-2\tau}{T}}(\alpha\ln(T/\alpha))^{-\frac{2y}{T}}\int_0^y(\alpha\ln(T/\alpha))^{\frac{2\tau}{T}}\|u_\alpha(\cdot,\tau)-u(\cdot,\tau)\|^2d\tau.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(\alpha\ln(T/\alpha))^{\frac{2y}{T}}\|u_\alpha(\cdot,y)-u(\cdot,y)\|^2\\
\leq\ & 32T_0^2E^2\alpha^2+2L^2T_0^3\int_0^y(\alpha\ln(T/\alpha))^{\frac{2\tau}{T}}\|u_\alpha(\cdot,\tau)-u(\cdot,\tau)\|^2d\tau.
\end{aligned}\tag{3.6}
$$

Using Gronwall's inequality [30], it can be derived that

$$
(\alpha\ln(T/\alpha))^{\frac{2y}{T}}\|u_\alpha(\cdot,y)-u(\cdot,y)\|^2\leq 32T_0^2E^2\left(1+2yL^2T_0^3e^{2yL^2T_0^3}\right)\alpha^2. \tag{3.7}
$$

i.e.,

$$
\|u_\alpha(\cdot,y)-u(\cdot,y)\|\leq C_1\alpha^{1-\frac{y}{T}}(\ln(T/\alpha))^{-\frac{y}{T}}. \tag{3.8}
$$

From (3.2), (3.4), (3.5) and (3.8), we have

$$
\|u_\alpha^\delta(\cdot,y)-u(\cdot,y)\|\leq(C_0+C_1)\delta^{1-\frac{y}{T}}(\ln(T/\delta))^{-\frac{y}{T}}=C_2\delta^{1-\frac{y}{T}}(\ln(T/\delta))^{-\frac{y}{T}}. \tag{3.9}
$$

For the case that p is odd, since the proof method is similar to the above procedure, we only give the conclusions of the stability and convergence estimate.

Theorem 3.2 Let $u_{\alpha1}^\delta$ and $u_{\alpha2}^\delta$ be the solutions of problem (2.7) corresponding to the data φ_1^δ and φ_2^δ, respectively, then for $\alpha<T$, we have

$$
\|u_{\alpha1}^\delta(\cdot,y)-u_{\alpha2}^\delta(\cdot,y)\|\leq C_3\left(\alpha\ln(T/\alpha)\right)^{-\frac{y}{T}}\left\|\varphi_1^\delta-\varphi_2^\delta\right\|, \tag{3.10}
$$

and suppose that u is the solution of problem (1.1) and u_α^δ is the solution of problem (2.7). Let the measured data φ^δ satisfy $\|\varphi^\delta-\varphi\|\leq\delta$, the exact solution u satisfy

$$
\frac{\pi}{2}\sum_{n=1}^{\infty}n^{2p}e^{2n(T-y)}\left|\langle u(\cdot,y),\sin(nx)\rangle\right|^2\leq E^2, \tag{3.11}
$$

if the regularization parameter α is chosen as

$$
\alpha=\delta, \tag{3.12}
$$

then for fixed $0 < y \le T$ and $\delta < T$, we have the following convergence estimate

$$\|u_\alpha^\delta(\cdot, y) - u(\cdot, y)\| \le C_5 \delta^{1-\frac{y}{T}} (\ln(T/\delta))^{-\frac{y}{T}}, \tag{3.13}$$

where $C_3 = \sqrt{8C_4^2T_0^2\left(1 + 2yL^2C_4^2T_0^3 e^{2yL^2C_4^2T_0^3}\right)}$, $C_4 = \frac{2}{1-e^{-2T}}$, and $C_5 = C_3 + C_6$, $C_6 = \sqrt{32T_0^2E^2\left(1 + 2yL^2C_4^2T_0^3 e^{2yL^2C_4^2T_0^3}\right)}$.

Remark 3.3 In the research of ill-posed problems, we often impose an a-priori assumption on the exact solution to make the convergence estimate of one regularization method. For our problem, the difficulty of imposing the a-priori assumption mainly lie in the nonlinear property and the application of regularization method. In view of the factors above, here we impose a-priori assumption (3.1), (3.11), and combine with (2.10), (2.11) to proof Theorems 3.1–3.2.

Remark 3.4 Our given method can be applied to investigate the cases of nonhomogeneous Neumann data ($u(x, 0) = 0$, $u_y(x, 0) = \phi(x) \ne 0$) or nonhomogeneous Dirichlet and Neumann datum ($u(x, 0) = \varphi(x) \ne 0$, $u_y(x, 0) = \phi(x) \ne 0$). Since the constructed procedure of regularization method is similar with the one in Subsection 2.1, here we skip it.

4 Numerical Experiments

In this section, an iterative scheme is proposed to calculate the regularization solution and a numerical example is performed to verify the efficiency of our method. Consider the nonlinear problem

$$\begin{cases} u_{xx} + u_{yy} = \cos(u) + g(x, y), & 0 < x < \pi,\ 0 < y < 1, \\ u(x, 0) = \varphi(x), & 0 \le x \le \pi, \\ u_y(x, 0) = 0, & 0 \le x \le \pi, \\ u(0, y) = u(\pi, y) = 0, & 0 \le y \le 1. \end{cases} \tag{4.1}$$

It is clear that $u(x, y) = x(x - \pi)(2 + y^2)$ is an exact solution of problem (4.1), thus $g(x, y) = 2x(x - \pi) + 2(2 + y^2) - \cos(x(x - \pi)(2 + y^2))$, $\varphi(x) = u(x, 0) = 2x(x - \pi)$, and the measured data is given by $\varphi^\delta(x) = \varphi(x)\left(1 + \varepsilon(x/2 - 1)\right)$.

Let $0 = y_0 < y_1 < \ldots < y_k < \ldots < y_M = 1$ for $k = 0, 1, 2, \ldots, M$, the regularization solution $u_\alpha^\delta(x, y_k)$ with $y_k = \frac{k}{M}$ can be computed by the following iteration scheme

$$u_\alpha^\delta(x, y_k) = v_k(x) = w_{1,k}\sin(x) + w_{2,k}\sin(2x) + \ldots + w_{m,k}\sin(mx), \tag{4.2}$$

here, for the case that p is even, $w_{j,k} = \frac{1/\cosh(j)}{\alpha j^p + 1/\cosh(j)} a_{j,k}$, when p is odd, $w_{j,k} = \frac{1/\sinh(j)}{\alpha j^p + 1/\sinh(j)} a_{j,k}$, and

$$\begin{aligned} a_{j,k} = {} & \cosh(jy_k)\varphi_j^\delta + \tfrac{2}{\pi j}\int_{y_{k-1}}^{y_k}\int_0^\pi \sinh\left(j(y_k - \tau)\right)\left(\cos(v_{k-1}(x)) + g(x, \tau)\right)\sin(jx)dxd\tau \\ & + \tfrac{2}{\pi j}\int_{y_{k-2}}^{y_{k-1}}\int_0^\pi \sinh\left(j(y_k - \tau)\right)\left(\cos(v_{k-2}(x)) + g(x, \tau)\right)\sin(jx)dxd\tau \\ & + \ldots + \tfrac{2}{\pi j}\int_{y_1}^{y_2}\int_0^\pi \sinh\left(j(y_k - \tau)\right)\left(\cos(v_1(x)) + g(x, \tau)\right)\sin(jx)dxd\tau \\ & + \tfrac{2}{\pi j}\int_0^{y_1}\int_0^\pi \sinh\left(j(y_k - \tau)\right)\left(\cos(v_0(x)) + g(x, \tau)\right)\sin(jx)dxd\tau, \end{aligned} \tag{4.3}$$

$$v_0(x) = \varphi^\delta(x) = \varphi(x)\left(1 + \varepsilon(x/2 - 1)\right), \ \varphi_j^\delta = \frac{2}{\pi}\int_0^\pi \varphi^\delta(x)\sin(jx)dx. \tag{4.4}$$

Adopt the above algorithm, we choose $y_k = \frac{k}{M}$ with $M = 50$, for $k = 1, 2, \cdots, 50, j = 1, \cdots, m = 5$ to compute $u_\alpha^\delta(\cdot, y)$ at $y = 0.4$, 1 ($k = 20, 50$). In the computational procedure, the regularization α is chosen by (3.2) or (3.12). For $p = 1, 2, 3, 4$, the numerical results for $\varepsilon = 0.01$ are shown in Figure 1–2, respectively. For $p = 1, 2, 3$, the relative root mean square errors between the exact and regularization solutions are defined by

$$\epsilon(u) = \frac{\sqrt{\frac{1}{N}\sum_{\ell=1}^N \left(u(x_\ell) - u_\alpha^\delta(x_\ell)\right)^2}}{\sqrt{\frac{1}{N}\sum_{\ell=1}^N \left(u(x_\ell)\right)^2}}, \quad x_\ell = \frac{(\ell - 1)}{N - 1}\pi, \quad \ell = 1, 2, \cdots, N, \tag{4.5}$$

the computed results are shown in Table 1.

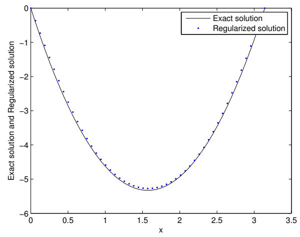
Figure 1–2 and Table 1 show that our proposed method is effective and stable. Meanwhile, we note that the more small ε is, the better the calculation effect is, this is a common phenomenon in the computation of ill-posed problems. Table 1 indicates that the computational effect of our method is more better when p is an odd. Finally, we mention that the iteration scheme (4.2) can be explained as a Fourier series, since the exact solution is a polynomial function, during the procedure of computing the regularization solution, the node number M should be chosen as a larger number relatively, here we take $M = 50$, and according to the numerical results in [18, 19], the truncated term number m has no necessity to be taken too big, the best value should be 5 or 6, here we take it as 5.

5 Conclusions

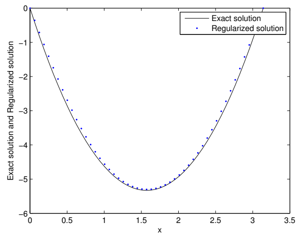
We use a regularization method to solve a Cauchy problem of nonlinear elliptic equation. The existence, uniqueness and stability of the regularization solution are proven, under a-priori bound assumptions for the exact solution, the convergence estimates for $0 < y \le T$ have been obtained. Finally, an iterative scheme is proposed to calculate the regularization solution, and some numerical results indicate that our method is stable and feasible.

Table 1 The relative root mean square errors $\epsilon(u)$ for various noisy levels.

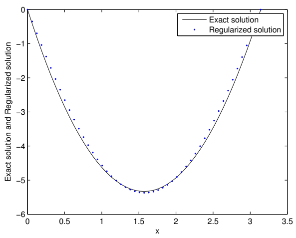
ε	0.0001	0.001	0.01	0.1
$p = 1$	0.0097	0.0099	0.0114	0.0252
$p = 2$	0.0098	0.0102	0.0150	0.0563
$p = 3$	0.0097	0.0100	0.0115	0.0253



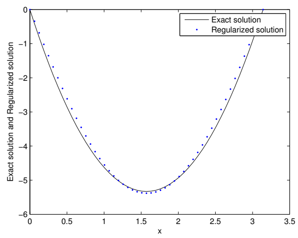
(a) u and u_α^δ with $p=1$.



(b) u and u_α^δ with $p=2$.

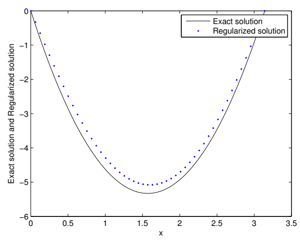


(c) u and u_α^δ with $p=3$.

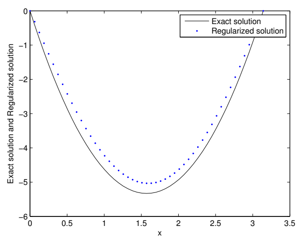


(d) u and u_α^δ with $p=4$.

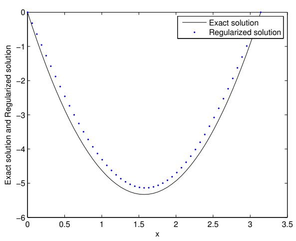
Figure 1 $y=0.4, \varepsilon=0.01$.



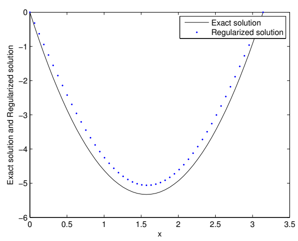
(a) u and u_α^δ with $p=1$.



(b) u and u_α^δ with $p=2$.



(c) u and u_α^δ with $p=3$.



(d) u and u_α^δ with $p=4$.

Figure 2 $y=1, \varepsilon=0.01$.

References

[1] Beskos D E. Boundary element method in dynamic analysis: Part II (1986–1996)[J]. Applied Mechanics Reviews, 1997, 50: 149–197.

[2] Chen J T, Wong F C. Dual formulation of multiple reciprocity method for the acoustic mode of a cavity with a thin partition[J]. Journal of Sound and Vibration, 1998, 217：75–95.

[3] Fokas A S, Pelloni B. The Dirichlet-to-neumann map for the elliptic sine-Gordon equation[J]. Nonlinearity, 2012, 25(4): 1011–1031.

[4] Gutshabash E S, Lipovskii V D. Boundary value problem for the two-dimensional elliptic sine-Gordon equation and its applications to the theory of the stationary josephson effect[J]. Journal of Mathematical Sciences, 1994, 68: 197–201.

[5] Hadamard J. Lectures on Cauchy' s problem in linear partial differential equations[M]. New York: Dover Publications, 1953.

[6] Kirsch A. An Introduction to the mathematical theory of inverse problems[M]. New York: Springer 1996.

[7] Engl H W, Hanke M, Neubauer A. Regularization of inverse problems[M]. Dordrecht: Kluwer Academic Publishers Group, 1996.

[8] Klibanov M V, Santosa F. A computational quasi-reversibility method for Cauchy problems for Laplace' s equation[J]. SIAM Journal on Applied Mathematics, 1991, 51(6): 1653–1675.

[9] Qian Z, Fu C L, Xiong X T. Fourth-order modified method for the Cauchy problem for the Laplace equation[J]. Journal of Computational and Applied Mathematics, 2006, 192(2): 205–218.

[10] Zhao J J, Liu S S, Liu T. Determining surface heat flux for noncharacteristic Cauchy problem for laplace equation[J]. Mathematics and Computers in Simulation, 2016, 129: 69–80.

[11] Zhao J J, Liu S S, Liu T. A new regularization method for Cauchy problem of elliptic equation[J]. Complex Variables and Elliptic Equations, 2014, 59(9): 1302–1314.

[12] Tikhonov A N, Arsenin V Y. Solutions of ill-posed problems[M]. Washington, D. C.: John Wiley and Sons, 1977.

[13] Falk R S, Monk P B. Logarithmic convexity for discrete harmonic functions and the approximation of the Cauchy problem for Poisson's equation[J]. Mathematics of Computation, 1986, 47(175): 135–149.

[14] Reinhardt H J, Han H, Hào D N. Stability and regularization of a discrete approximation to the Cauchy problem for Laplace's equation[J]. SIAM Journal on Numerical Analysis, 1999, 36(3): 890–905.

[15] Wei T, Hon Y C, Ling L. Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators[J]. Engineering Analysis with Boundary Elements, 2007, 31(4): 373–385.

[16] Hào D N, Duc N V, Lesnic D. A non-local boundary value problem method for the Cauchy problem for elliptic equations[J]. Inverse Problems, 2009, 25: 055002.

[17] Vabishchevich P N. Numerical solution of nonlocal elliptic problems[J]. Izv. Vyssh. Uchebn. Zaved. Mat., 1983, 5: 13–19(in Russian).

[18] Zhang H W, Wang R H. Modified boundary Tikhonov-type regularizationmethod for the Cauchy problem of a semi-linear elliptic equation[J]. Inverse Problems in Science and Engineering, 2016, 24(7): 1249–1265.

[19] Zhang H W, Wei T. A Fourier truncated regularization method for a Cauchy problem of a semi-linear elliptic equation[J]. Journal of Inverse and Ill-Posed Problems, 2014, 22(2): 143–168.

[20] Zhang H W, Zhang X J. Filtering function method for the Cauchy problem of a semi-linear elliptic equation[J]. Journal of Applied Mathematics and Physics, 2015, 3(2): 1599–1609.

[21] Zhang H W, Zhang X J. Generalized Tikhonov-type regularization method for the Cauchy problem of a semi-linear elliptic equation[J]. Numerical Algorithms, 2018, DOI:10.1007/s11075-018-0573-4.

[22] Tuan N H, Thang L D, Khoa V A. A modified integral equation method of the nonlinear elliptic equation with globally and locally Lipschitz source[J]. Applied Mathematics and Computation, 2015, 265: 245–265.

[23] Tuan N H, Thang L D, Khoa V A, Tran T. On an inverse boundary value problem of a nonlinear elliptic equation in three dimensions[J]. Journal of Mathematical Analysis and Applications, 2015, 426: 1232–1261.

[24] Tran Q V, Kirane M, Nguyen H T, Nguyen V T. Analysis and numerical simulation of the three-dimensional Cauchy problem for quasi-linear elliptic equations[J]. Journal of Mathematical Analysis and Applications, 2017, 446(1): 470–492.

[25] Tuan N H, Thang L D, Lesnic D. A new general filter regularization method for Cauchy problems for elliptic equations with a locally Lipschitz nonlinear source[J]. Journal of Mathematical Analysis and Applications, 2016, 434: 1376–1393.

[26] Tuan N H, Thang L D, Trong D D, Khoa V A. Approximation of mild solutions of the linear and nonlinear elliptic equations[J]. Inverse Problems in Science and Engineering, 2015, 23(7): 1237–1266.

[27] Khoa V A, Truong M T, Duy N H. A general kernel-based regularization method for coupled elliptic Sine-Gordon equations with Gevrey regularity[J]. Computer Physics Communications, 2015, 183(8): 1813–1821.

[28] Khoa V A, Truong M T, Duy N H, Tuan N H. The Cauchy problem of coupled elliptic Sine-Gordon equations with noise: analysis of a general kernel-based regularization and reliable tools of computing[J]. Computers and Mathematics with Applications, 2017, 73(1): 141–162.

[29] Zhang H W, Wei T. An improved non-local boundary value problem method for a Cauchy problem of the Laplace equation[J]. Numerical Algorithms, 2012, 59: 249–269.

[30] Evans L C. Partial diferential equations[M]. Providence RI: American Mathematical Society, 1998.

一类不适定非线性椭圆方程柯西问题的正则化方法

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摘 要: 本文研究了一类不适定的非线性椭圆方程柯西问题. 利用一种正则化方法克服其不适定性, 获得了正则化解的存在唯一性, 稳定性及收敛性结果, 并构造一种迭代格式计算了正则化解, 推广了已有文献在椭圆方程柯西问题正则化理论与算法方面的相关研究结果.

关键词: 不适定问题; 柯西问题; 非线性椭圆方程; 正则化方法

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