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THE COMMUTATOR TYPE AND THE LEVI FORM TYPE IN $\mathbb{C}^3$

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Abstract: For any fixed $(1,0)$ vector field of a pseudoconvex hypersurface in $\mathbb{C}^3$, we prove that its commutator type and Levi form type are equal to each other. This answers affirmatively a problem of D'Angelo in complex dimension three.

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1 Introduction

The finite type conditions gave their rise from the investigation of the subellipticity of the $\overline{\partial}$-Neumann operator. For any boundary point of a smooth pseudoconvex domain in $\mathbb{C}^2$, Kohn [1] introduced three kinds of integer invariants, which are respectively the regular contact type, the commutator type and the Levi form type. Kohn proved that these invariants are equal to each other. When they are finite at a boundary point, the domain possesses local sub-elliptic estimates near this point. The domain is said to be of finite type if these invariants are finite at each boundary point of the domain.

Ever since then, much attention paid to generalize these finite type conditions to the higher dimensional case. Kohn [2] defined the subelliptic multiplier ideals near each boundary point of a pseudoconvex domain, and if 1 is in any of these ideals, the boundary point is said to be of finite ideal type. In [3], D'Angelo introduced the D'Angelo finite type condition in terms of the order of contact with respect to singular complex analytic varieties. Both of these finite type conditions imply the existence of the sub-elliptic estimates. Bloom [4] generalized Kohn's type conditions in $\mathbb{C}^2$ directly to higher dimensional spaces. More precisely, for a smooth real hypersurface $M \subset \mathbb{C}^n$ and $p \in M$, Bloom defined the regular contact type $a^{(s)}(M,p)$, the commutator type $t^{(s)}(M,p)$ and the Levi-form type $c^{(s)}(M,p)$ of M at p. Bloom conjectured in [4] that these three invariants are the same when the hypersurface is pseudoconvex, which is known as the Bloom conjecture. Bloom-Graham [5] and Bloom [6] proved the conjecture for $s = n-1$. In [4], Bloom showed that $a^{(1)}(M,p) = c^{(1)}(M,p)$ when

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$M \subset \mathbb{C}^3$. Recently, Huang-Yin [7] proved that the Bloom conjecture holds for $s = n-2$. This, in particular, gave a complete solution of the conjecture in complex dimension 3.

For a fixed $(1,0)$ type vector field L at a point p of a smooth real hypersurface, D'Angelo introduced the commutator type $t(L,p)$ and the Levi form type $c(L,p)$.

In fact, let $M \subset \mathbb{C}^n$ be a smooth real hypersurface with $p \in M$, and let ρ be a defining function of M near p. Denote by $\mathcal{M}_1(L)$ the $C^\infty(M)$-module spanned by L and $\overline{L}$. For any $k \geq 1$, we inductively define $\mathcal{M}_k(L)$ to be the $C^\infty(M)$-module spanned by $\mathcal{M}_{k-1}(L)$ and the elements of the of form $[X,Y]$ with $X \in \mathcal{M}_{k-1}(L)$ and $Y \in \mathcal{M}_1(L)$. We say the commutator type $t(L,p) = m$ if $\langle F, \partial\rho\rangle(p) = 0$ for any $F \in \mathcal{M}_{m-1}(L)$ but $\langle G, \partial\rho\rangle(p) \neq 0$ for a certain $G \in \mathcal{M}_m(L)$. We define the Levi form type $c(L,p) = m$ if for any $m-3$ vector fields $F_1, \cdots, F_{m-3}$ of $\mathcal{M}_1(L)$, we have

$$F_{m-3} \cdots F_1 \langle [L, \overline{L}], \partial\rho\rangle(p) = 0;$$

and for a certain choice of $m-2$ vector fields $G_1, \cdots, G_{m-2}$ of $\mathcal{M}_1(L)$, we have

$$G_{m-2} \cdots G_1 \langle [L, \overline{L}], \partial\rho\rangle(p) \neq 0.$$

D'Angelo [8] conjectured that these two types equal to each other when the real hypersurface is pseudoconvex. He confirmed the conjecture when one of the type is exactly 4. The present paper is devoted to proving this conjecture when the real hypersurface is in $\mathbb{C}^3$.

Theorem 1.1 Let M be a smooth pseudoconvex hypersurface in $\mathbb{C}^3$ and $p \in M$. For any fixed $(1,0)$ vector field L near p, we have $t(L,p) = c(L,p)$.

2 Proof of Main Theorems

This section is devoted to the proof of Theorem 1.1.

Let $(z_1, , z_2, w)$ be the coordinates in $\mathbb{C}^3$. Suppose that $p = 0$, and the defining function of M takes the form

$$\rho = -\mathrm{Im} w + E(z, w, \overline{z}, \overline{w}), \ E(z, w, \overline{z}, \overline{w}) = O(|z|^3 + |zw| + |w^2|).$$

For any $j = 1, 2$, write

$$L_j = \frac{\partial}{\partial z_j} - \frac{\partial \rho}{\partial z_j} \cdot (\frac{\partial \rho}{\partial z_j})^{-1} \frac{\partial}{\partial w}.$$

Then L_1 and L_2 form a basis of the complex tangent vector fields of type $(1,0)$ along M near 0. Suppose that $L = A_1(z, w, \overline{z}, \overline{w})L_1 + A_2(z, w, \overline{z}, \overline{w})L_2$. After a linear change of coordinates, we can assume $A_1(0) \neq 0$ and $A_2(0) = 0$. Notice that for any smooth function f on M with $f(0) \neq 0$, we have $t(fL, 0) = t(L, 0)$ and $c(fL, 0) = c(L, 0)$. Thus we can replace L by $A_1^{-1}L$, then L takes the form $L = L_1 + A(z, w, \overline{z}, \overline{w})L_2$.

Denote by $l_0 - 2$ the vanishing order of $\frac{\partial}{\partial \overline{z_1}} A(z_1, 0, \overline{z_1}, 0)$ and denote by $m_0 - 2$ the vanishing order of $\frac{\partial^2}{\partial z_1 \overline{z_1}} E(z_1, 0, \overline{z_1}, 0)$. The proof of main theorem is carried out for three cases, according to the values of l_0 and m_0.

Case I In this case, we assume $l_0 = m_0 = \infty$.

For any fixed integer k, after a holomorphic change of coordinates, we make

$$E(z_1, 0, \overline{z_1}, 0) = O(|z_1|^{k+1}),\ A(z_1, 0, \overline{z_1}, 0) = O(k+1).$$

A direct computation shows that $t(L, 0) \geq k$ and $c(L, 0) \geq k$. By the arbitrariness of k, we obtain $t(L, 0) = c(L, 0) = +\infty$.

Case II In this case, we assume $m_0 < \infty$ and $l_0 > m_0$.

After a holomorphic change of coordinates (see [4] or [7]), we make $E(z_1, 0, \overline{z_1}, 0)$ contains no holomorphic or anti-holomorphic terms, and the terms of degree m_0 in $E(z_1, 0, \overline{z_1}, 0)$ is non-zero. Also, we make the vanishing order of $A(z_1, 0, \overline{z_1}, 0)$ is at least m_0. Now, we introduce the following weighting system

$$\begin{aligned}
&\mathrm{wt}(z_1) = \mathrm{wt}(\overline{z_1}) = 1,\ \mathrm{wt}(z_2) = \mathrm{wt}(\overline{z_2}) = m_0,\ \mathrm{wt}(w) = \mathrm{wt}(\overline{w}) = m_0.\\
&\mathrm{wt}(\frac{\partial}{\partial z_1}) = \mathrm{wt}(\frac{\partial}{\partial \overline{z_1}}) = -1,\ \mathrm{wt}(\frac{\partial}{\partial z_2}) = \mathrm{wt}(\frac{\partial}{\partial \overline{z_2}}) = -m_0,\\
&\mathrm{wt}(\frac{\partial}{\partial w}) = \mathrm{wt}(\frac{\partial}{\partial \overline{w}}) = -m_0.
\end{aligned}$$

Define

$$\rho^{(m_0)} = -\mathrm{Im} w + E^{(m_0)}(z_1, 0, \overline{z_1}, 0),\ L_{-1} = \frac{\partial}{\partial z_1} + 2i\frac{\partial}{\partial z_1}E^{(m_0)}(z_1, 0, \overline{z_1}, 0)\frac{\partial}{\partial w}.$$

Denote by $O_{wt}(k)$ a smooth function or vector field with weighted degree at least k. Then we have

$$\rho = \rho^{(m_0)} + O_{wt}(m_0 + 1),\ L = L_{-1} + O_{wt}(0).$$

Thus for any $1 \leq j \leq m_0 - 2$ and $X_1, \cdots, X_j \in \mathcal{M}_1(L)$, the weighted degree of terms in

$$X_j \cdot X_1 \partial\overline{\partial}\rho(L, \overline{L}) \text{ and } \langle [X_j, [X_{j-1}, \cdots, [X_1, [L, \overline{L}] \cdots], \partial\rho \rangle$$

are at least $m_0 - j + 2$. Hence both of them are 0 when restricted to the origin for any $1 \leq j \leq m_0 - 3$. When $j = m_0 - 2$, by considering the weighted degree, we know

$$X_j \cdots X_1 \partial\overline{\partial}\rho(L, \overline{L})(0) = (X_j)_{-1} \cdots (X_1)_{-1} \partial\overline{\partial}\rho^{(m_0)}(L_{-1}, \overline{L_{-1}})(0) \tag{2.1}$$

and

$$\begin{aligned}
&\langle [X_j, [X_{j-1}, \cdots, [X_1, [L, \overline{L}] \cdots], \partial\rho \rangle(0)\\
=&\langle [(X_j)_{-1}, [(X_{j-1})_{-1}, \cdots, [(X_1)_{-1}, [L_{-1}, \overline{L_{-1}}] \cdots], \partial\rho^{(m_0)} \rangle(0),
\end{aligned} \tag{2.2}$$

here $(X_h)_{-1}$ is the sum of the vector field terms in X_h of weighted degree -1.

Notice that L_{-1} is an $(1, 0)$ tangent field of the real hypersurface defined by $\rho^{(m_0)} = 0$, which must be pseudoconvex. By the finite type theory is dimension 2 (see [9]), we have $t(L_{-1}, 0) = c(L_{-1}, 0) = m_0$. Thus in (2.1) and (2.2), we can choose X_h for $1 \leq h \leq m_0 - 2$ such that the two expressions are non-zero. This means $t(L, 0) = c(L, 0) = m_0$.

Case III In this case, we assume $m_0 < \infty$ and $l_0 \leq m_0 - 1$.

After a holomorphic change of coordinates, we eliminate the holomorphic and anti-holomorphic terms in E up to order m_0, and get rid of the holomorphic terms in A up to order l_0. As in Case II, we define

$$\mathrm{wt}(z_1) = \mathrm{wt}(\overline{z_1}) = 1,\ \mathrm{wt}(z_2) = \mathrm{wt}(\overline{z_2}) = l_0. \tag{2.3}$$

Denote by m_1 the lowest weighted vanishing order of $\rho(z, 0, \overline{z}, 0)$ with the weights given in (2.3). Then $m_1 \leq m_0$. Define

$$\begin{aligned}
&\mathrm{wt}(w) = \mathrm{wt}(\overline{w}) = m_1,\\
&\mathrm{wt}(\frac{\partial}{\partial z_1}) = \mathrm{wt}(\frac{\partial}{\partial \overline{z_1}}) = -1,\ \mathrm{wt}(\frac{\partial}{\partial z_2}) = \mathrm{wt}(\frac{\partial}{\partial \overline{z_2}}) = -l_0,\\
&\mathrm{wt}(\frac{\partial}{\partial w}) = \mathrm{wt}(\frac{\partial}{\partial \overline{w}}) = -m_1.
\end{aligned}$$

Write

$$\begin{aligned}
\rho^{(m_0)} &= -\mathrm{Im} w + E^{(m_0)}(z_1, z_2, 0, \overline{z_1}, \overline{z_2}, 0),\\
L_{-1} &= \frac{\partial}{\partial z_1} + A^{(l_0-1)}\frac{\partial}{\partial z_2} + 2i\left(\frac{\partial \rho^{(m_1)}}{\partial z_1} + A^{(l_0-1)}\frac{\partial \rho^{(m_1)}}{\partial z_2}\right)\frac{\partial}{\partial w}.
\end{aligned}$$

By our construction and definition, we have

$$\begin{aligned}
&\rho = \rho^{(m_1)} + O_{wt}(m_1+1),\ L = L_{-1} + O_{wt}(0),\\
&\rho^{(m_1)} \text{ is not identically 0 and contains no holomorphic terms.}
\end{aligned} \tag{2.4}$$

By a similar weighted degree estimate as in Case II, for any $1 \leq j \leq m_0 - 3$ and $X_1, \cdots, X_j \in \mathcal{M}_1(L)$, we know

$$X_j \cdots X_1 \partial\overline{\partial}\rho(L, \overline{L})(0) = 0 \text{ and } \langle [X_j, [X_{j-1}, \cdots, [X_1, [L, \overline{L}] \cdots], \partial\rho\rangle(0) = 0.$$

Also, for any $X_1, \cdot, X_{m_1-2} \in \mathcal{M}_1(L)$, we have

$$X_{m_1-2} \cdots X_1 \partial\overline{\partial}\rho(L, \overline{L})(0) = (X_j)_{-1} \cdots (X_1)_{-1}\partial\overline{\partial}\rho^{(m_1)}(L_{-1}, \overline{L_{-1}})(0) \tag{2.5}$$

and

$$\begin{aligned}
&\langle [X_{m_1-2}, [X_{j-1}, \cdots, [X_1, [L, \overline{L}] \cdots], \partial\rho\rangle(0)\\
=&\langle [(X_{m_1-2})_{-1}, [(X_{m_1-3})_{-1}, \cdots, [(X_1)_{-1}, [L_{-1}, \overline{L_{-1}}] \cdots], \partial\rho^{(m_1-1)}\rangle(0).
\end{aligned} \tag{2.6}$$

Consider L_{-1} as a complex $(1,0)$ tangent vector field of $M^0 := \{\rho^{(m_1)} = 0\}$. We claim that $t(L_{-1}, 0) = c(L_{-1}, 0) = m_1$.

Since L_{-1} is real analytic, by the Nagano Theorem, $\mathrm{Re}(L_{-1})$, $\mathrm{Im}(L_{-1})$ and their Lie brackets will generate a homogeneous real manifold N^0.

Suppose that $t(L_{-1},0) > m_1$, then for any $X_1, \cdot, X_{m_1-2} \in \mathcal{M}_1(L_{-1})$,

$$X_{m_1-2}\cdots X_1\partial\overline{\partial}\rho(L,\overline{L})(0) = (X_{m_1-2})_{-1}\cdots(X_1)_{-1}\partial\overline{\partial}\rho^{(m_1)}(L_{-1},\overline{L_{-1}})(0) = 0.$$

On the other hand, for any $j \geq 0$, $(X_j)_{-1}\cdots(X_1)_{-1}\partial\overline{\partial}\rho^{(m_1)}(L_{-1},\overline{L_{-1}})(z,w)$ is a weighted homogeneous polynomial of degree $m_1 - j - 2$. Hence it must be 0 when $(z,w) = 0$ and $j \neq m_1 - 2$. Thus $j \geq 0$, we have

$$\langle[(X_{m_1-2})_{-1}, [(X_{m_1-3})_{-1}, \cdots, [(X_1)_{-1}, [L_{-1}, \overline{L_{-1}}]\cdots], \partial\rho^{(m_1-1)}\rangle(0) = 0.$$

This means $\frac{\partial}{\partial \mathrm{Im} v} \notin T_0N^0$. Hence the real dimension of N^0 is at most 4. By our assumption and normalization, A is non-zero and contains no holomorphic terms. By [4, Lemma 4.8], the real dimension of N^0 can only be 3 or 4. From [7], such N^0 exists only if N^0 is Levi-flat, which contradicts to the second line of (2.4). Hence $t(L_{-1},0) = m_1$. Together with (2.5), we obtain $c(L,0) = m_1$.

Next, suppose that $c(L_{-1},0) > m_1$, then for any $j \leq m_1 - 3$, $X_1, \cdots, X_j \in \mathcal{M}_1(L)$, we have

$$X_j\cdots X_1\partial\overline{\partial}\rho(L,\overline{L})(0) = 0. \tag{2.7}$$

A similar weighted degree argument shows that for any $j \geq 0$, $X_1, \cdots, X_j \in \mathcal{M}_1(L_{-1})$, we have

$$X_j\cdots X_1\partial\overline{\partial}\rho(L,\overline{L})(0) = 0.$$

This implies $\partial\overline{\partial}\rho(L,\overline{L}) \equiv 0$ on N^0. Hence for any $q \in N^0$ around 0, we have $\mathrm{Re}(L_{-1})(q)$, $\mathrm{Im}(L_{-1})(q) \in T_q^N N^0$, where

$$T_q^N N^0 := \{S \in T_qN^0 :\ S = \mathrm{Re}(X),\ X \in T_q^{(1,0)}M^0,\ \partial\overline{\partial}\rho(X,\overline{X})(q) \equiv 0\}.$$

By [10, Proposition 2], $T_z^N N^0 = T_zN^0$. In particular, any $S \in T_0N^0$ is in $T_0^h(M^0) := \{Y = \mathrm{Re}(X) :\ X \in T_0^{(1,0)}M^0\}$. Thus $\frac{\partial}{\partial \mathrm{Im} w}|_0 \notin T_0N^0$. By the argument in the proof of $t(L_{-1},0) = m_1$, such N^0 exists only if N^0 is Levi-flat, which again contradicts to (2.4). Hence we obtain $c(L_{-1},0) = m_1$. Together with (2.6), we obtain $c(L,0) = m_1$. Hence in this case, we again obtain $t(L,0) = c(L,0)$.

The proof of Theorem 1.1 is completed.

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$\mathbb{C}^3$中的交换子型和列维型

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摘要: 对于$\mathbb{C}^3$中的超曲面上的任意$(1,0)$切向量场, 证明了其上的交换子型和列维型是相等的. 这在三维情形解决了D'Angelo提出的一个问题.

关键词: 有限型; 拟凸超曲面; Bloom猜测; 次椭圆估计

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