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ON THE STRUCTURE OF SPLIT δ-JORDAN LIE TRIPLE SYSTEMS

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Abstract: The aim of this article is to study the structure of arbitrary split δ-Jordan Lie triple systems by focussing on those with a coherent 0-root space. By developing techniques of connections of roots for this kind of triple system T, we show that such an arbitrary δ-JLTS with a symmetric root system is of the form $T = U + \sum\limits_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ with U a subspace of the 0-root space T_0 and any $I_{[\alpha]}$ a well described ideal of T, satisfying $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Keywords: split δ-Jordan Lie triple system; Lie triple system; δ-Jordan Lie algebra; root system; root space

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1 Introduction

The concept of Lie triple systems (LTSs) was introduced by Nathan Jacobson in 1949 [1], to study subspaces of associative algebras closed under triple commutators $[[u, v], w]$. The role played by LTSs in the theory of symmetric spaces is parallel to that of Lie algebras in the theory of Lie groups. Clearly, every Lie algebra is at the same time a Lie triple system (LTS) by putting $[x, y, z] = [[x, y], z]$. Some applications of LTSs are widely studied recently [2–4]. The notion of δ-Jordan Lie triple systems (δ-JLTSs) was introduced by Susumu Okubo in 1997 [5]. The case of $\delta = 1$ implies δ-JLTSs are LTSs and the other case of $\delta = -1$ gives Jordan Lie triple systems. So a question arises whether some known results on LTSs can be extended to the framework of δ-JLTSs. δ-JLTSs are the natural generalization of LTSs and have important applications. Recently, deformations, Nijenhuis operators, Abelian extensions and T^*-extensions of δ-JLTSs were studied [6].

The class of the split ones is specially related to addition quantum numbers, graded contractions, and deformations. In [7], Calderón introduced techniques of connections of roots in the field of split Lie algebras. In [8], Calderón introduced the concept of split LTSs

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of arbitrary dimension and studied the split LTSs with a coherent 0-root space. Recently, in [9–13], the structures of arbitrary split Leibniz algebras, arbitrary split LTSs, arbitrary split Leibniz triple systems and arbitrary graded Leibniz triple systems were determined by the techniques of connections of roots. In the present paper, we are interested in studying the structure of arbitrary δ-JLTSs with a coherent 0-root space by focussing on the split ones. Our work is essentially motivated by the work on split LTSs [8].

Throughout this paper, δ-JLTSs T with a coherent 0-root space are considered of arbitrary dimension and over an arbitrary field $\mathbb{K}$. This paper proceeds as follows. In Section 2, we establish the preliminaries on split δ-JLTSs theory. In Section 3, we introduce the notion of connections of roots in the framework of split δ-JLTS and study its properties. We also show that such an arbitrary δ-JLTS with a symmetric root system is of the form $T = U + \sum\limits_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ with U a subspace of the 0-root space T_0 and any $I_{[\alpha]}$ a well described ideal of T, satisfying $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

2 Preliminaries

Definition 2.1 [5] A δ-Jordan Lie algebra L is a vector space over a field $\mathbb{K}$ endowed with a bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying

(1) $[x, y] = -\delta[y, x], \quad \delta = \pm 1$,

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in L$.

Remark 2.2 [5] A δ-Jordan Lie algebra L is called a Lie algebra if $\delta = 1$, and a δ-Jordan Lie algebra L is called a Jordan Lie algebra if $\delta = -1$.

Definition 2.3 [5] A δ-JLTS is a vector space T endowed with a trilinear operation $[\cdot,\cdot,\cdot] : T \times T \times T \to T$ satisfying

(1) $[x, y, z] = -\delta[y, x, z], \quad \delta = \pm 1$,

(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (Jacobi identity),

(3) $[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + \delta[a, b, [x, y, c]]$ for $x, y, z, a, b, c \in T$.

When $\delta = 1$, a δ-JLTS is a LTS. So LTSs are special examples of δ-JLTSs.

Example 2.4 If L is a δ-Jordan Lie algebra with product $[\cdot,\cdot]$, then L becomes a δ-JLTS by putting $[x, y, z] = [[x, y], z]$.

Definition 2.5 Let I be a subspace of a δ-JLTS T. Then I is called a subsystem of T, if $[I, I, I] \subseteq I$; I is called an ideal of T, if $[I, T, T] \subseteq I$.

Definition 2.6 [5] The standard embedding of a δ-JLTS T is the $\mathbb{Z}_2$-graded δ-Jordan Lie algebra $L = L^0 \oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{L(x, y).\ x, y \in T\}$, where $L(x, y)$ denotes the left multiplication operator in T, $L(x, y)(z) := [x, y, z]$; $L^1 := T$ and where the product is given by

$$[(L(x, y), z), (L(u, v), w)] := (L([u, v, y], x) - L([u, v, x], y) + L(z, w), [x, y, w] - \delta[u, v, z]).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a δ-Jordan Lie algebra.

Definition 2.7 Let T be a δ-JLTS, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a maximal abelian subalgebra (MASA) of L^0. For a linear functional $\alpha \in (H^0)^*$, we define the root space of T (with respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [h, t_\alpha] = \alpha(h)t_\alpha$ for any h $\in H^0\}$. The elements $\alpha \in (H^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T with respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H^0)^* \setminus \{0\} : T_\alpha \neq 0\}$.

Let us observe that $T_0 = \{t_0 \in T : [h, t_0] = 0$ for any h $\in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L^0_\alpha := \{v^0_\alpha \in L^0 : [h, v^0_\alpha] = \alpha(h)v^0_\alpha$ for any h $\in H^0\} \neq 0$.

Lemma 2.8 Let T be a δ-JLTS, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0. For $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\xi, q \in \Lambda^0 \cup \{0\}$, the following assertions hold.

(1) If $[T_\alpha, T_\beta] \neq 0$, then $\delta(\alpha + \beta) \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L^0_{\delta(\alpha+\beta)}$.

(2) If $[L^0_\xi, T_\alpha] \neq 0$, then $\delta(\xi + \alpha) \in \Lambda^1 \cup \{0\}$ and $[L^0_\xi, T_\alpha] \subseteq T_{\delta(\xi+\alpha)}$.

(3) If $[T_\alpha, L^0_\xi] \neq 0$, then $\delta(\alpha + \xi) \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, L^0_\xi] \subseteq T_{\delta(\alpha+\xi)}$.

(4) If $[L^0_\xi, L^0_q] \neq 0$, then $\delta(\xi + q) \in \Lambda^0 \cup \{0\}$ and $[L^0_\xi, L^0_q] \subseteq L^0_{\delta(\xi+q)}$.

(5) If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, then $\alpha + \beta + \delta\gamma \in \Lambda^1 \cup \{0\}$ and $\{T_\alpha, T_\beta, T_\gamma\} \subseteq T_{\delta^2\alpha+\delta^2\beta+\delta\gamma} = T_{\alpha+\beta+\delta\gamma}$.

Proof (1) For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, \beta(h)y] + \delta[\alpha(h)x, y] = \delta(\alpha + \beta)(h)[x, y]$.

(2) For any $x \in L^0_\xi$, $y \in T_\alpha$ and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, \alpha(h)y] + \delta[\xi(h)x, y] = \delta(\xi + \alpha)(h)[x, y]$.

(3) For any $x \in T_\alpha$, $y \in L^0_\xi$, and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, \xi(h)y] + \delta[\alpha(h)x, y] = \delta(\alpha + \xi)(h)[x, y]$.

(4) For any $x \in L^0_\xi$, $y \in L^0_q$ and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, q(h)y] + \delta[\xi(h)x, y] = \delta(\xi + q)(h)[x, y]$.

(5) It is a consequence of Lemma 2.8 (1) and (2).

Definition 2.9 Let T be a δ-JLTS, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0. We shall call that T is a split δ-JLTS (with respect to H^0) if $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$. We say that Λ^1 is the root system of T.

We also note that the facts $H^0 \subset L^0 = [T, T]$ and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ imply

$$H^0 = \sum_{\alpha\in\Lambda^1} [T_\alpha, T_{-\alpha}]. \tag{2.1}$$

It seems to us that the study of the structure of arbitrary split δ-JLTS is difficult to accomplish at the whole level of generality, so we begin the study of this class of δ-JLTS by considering those with a coherent 0-root space.

Definition 2.10 We say that a split δ-JLTS T has a coherent 0-root space if

(1) $[T_0, T_0, T] = 0$,

(2) $[T_0, T_\alpha, T_0] \neq 0$, for any $\alpha \in \Lambda^1$.

This is a natural relation between T_0 and any T_α, $\alpha \in \Lambda^1$. Observe that, by condition (2), $\alpha \in \Lambda^1$ implies $\alpha \in \Lambda^0$, but not the converse.

Definition 2.11 A root system Λ^1 of a split δ-JLTS T is called symmetric if it satisfies that $\alpha \in \Lambda^1$ implies $-\alpha \in \Lambda^1$.

A similar concept applies to the set Λ^0 of nonzero roots of L^0.

Definition 2.12 A subset Ω^1 of a root system Λ^1, associated to a split δ-JLTS T, is called a root subsystem if it is symmetric, and for $\alpha, \beta, \gamma \in \Omega^1 \cup \{0\}$ such that $\delta(\alpha+\beta) \in \Lambda^0$ and $\alpha + \beta + \delta\gamma \in \Lambda^1$, then $\alpha + \beta + \delta\gamma \in \Omega^1$.

Let Ω^1 be a root subsystem of Λ^1. We define

$$T_{0,\Omega^1} := \mathrm{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma] : \alpha + \beta + \delta\gamma = 0;\ \alpha, \beta, \gamma \in \Omega^1 \cup \{0\}\} \subset T_0$$

and $V_{\Omega^1} := \oplus_{\alpha \in \Omega^1} T_\alpha$. It is straightforward to verify that $T_{\Omega^1} := T_{0,\Omega^1} \oplus V_{\Omega^1}$ is a subsystem of T. We will say that T_{Ω^1} is a subsystem associated to the root subsystem Ω^1.

3 Decompositions

In the following, T denotes a split δ-JLTS having a coherent 0-root space, with a symmetric root system Λ^1, and $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$ the corresponding root decomposition. We begin the study of split δ-JLTS by developing the concept of connections of roots.

Definition 3.1 Let α and β be two nonzero roots, we say that α and β are connected if there exists a family $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\} \subset \Lambda^1 \cup \{0\}$ of roots of T such that

(1) $\{\alpha_1, \delta^2\alpha_1 + \delta^2\alpha_2 + \delta\alpha_3, \delta^4\alpha_1 + \delta^4\alpha_2 + \delta^3\alpha_3 + \delta^2\alpha_4 + \delta\alpha_5, \cdots, \delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1}\} \subset \Lambda^1$;

(2) $\{\delta\alpha_1 + \delta\alpha_2, \delta^3\alpha_1 + \delta^3\alpha_2 + \delta^2\alpha_3 + \delta\alpha_4, \cdots, \delta^{2n-1}\alpha_1 + \cdots + \delta\alpha_{2n}\} \subset \Lambda^0$;

(3) $\alpha_1 = \alpha$ and $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} \in \pm\beta$.

We shall also say that $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β.

We denote by $\Lambda^1_\alpha := \{\beta \in \Lambda^1 : \alpha \,\text{and}\, \beta \text{ are connected}\}$, we can easily get that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore $\pm\alpha \in \Lambda^1_\alpha$.

Proposition 3.2 If Λ^0 is symmetric, then the relation $\sim$ in Λ^1, defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda^1_\alpha$, is of equivalence.

Proof $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$.

If $\alpha \sim \beta$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β, then $\{\delta^{2n}\alpha_1 + \cdots + \delta\alpha_{2n+1}, -\delta\alpha_{2n+1}, -\delta\alpha_{2n}, \cdots, -\delta\alpha_2\}$ is a connection from β to α in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta$, and $\{-\delta^{2n}\alpha_1 - \cdots - \delta\alpha_{2n+1}, \delta\alpha_{2n+1}, \delta\alpha_{2n}, \cdots, \delta\alpha_2\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta$. Therefore $\beta \sim \alpha$.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β and $\{\beta_1, \cdots, \beta_{2m+1}\}$ is a connection from β to γ. If $m \neq 0$, then $\{\alpha_1, \cdots, \alpha_{2n+1}, \beta_2, \cdots, \beta_{2m+1}\}$ is a connection from α to γ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta$, and $\{\alpha_1, \cdots, \alpha_{2n+1}, -\beta_2, \cdots, -\beta_{2m+1}\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta$. If $m = 0$, then $\gamma \in \pm\beta$ and so $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to γ. Therefore $\alpha \sim \gamma$ and $\sim$ is of equivalence.

Proposition 3.3 Let α be a nonzero root and suppose Λ^0 is symmetric. Then Λ^1_α is a root subsystem.

Proof If $\beta \in \Lambda^1_\alpha$ then there exists a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β. It is clear that $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ also connects α to $-\beta$ and therefore $-\beta \in \Lambda^1_\alpha$. Let $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\alpha \cup \{0\}$ be such that $\delta(\beta_1 + \beta_2) \in \Lambda^0$ and $\beta_1 + \beta_2 + \delta\beta_3 \in \Lambda^1$. If $\beta_1 = 0$, as $\delta(\beta_1 + \beta_2) \in \Lambda^0$ then $\beta_2 \neq 0$ and there exists a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β_2. We have $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, 0, \beta_3\}$ is a connection from α to $\beta_2 + \delta\beta_3$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta_2$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, 0, -\beta_3\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta_2$. So $\beta_1 + \beta_2 + \delta\beta_3 = \beta_2 + \delta\beta_3 \in \Lambda^1_\alpha$. Suppose $\beta_1 \neq 0$, then there exists a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β_1. Hence, $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, \beta_2, \beta_3\}$ is a connection from α to $\beta_1 + \beta_2 + \delta\beta_3$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta_1$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, -\beta_2, -\beta_3\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta_1$. Therefore $\beta_1 + \beta_2 + \delta\beta_3 \in \Lambda^1_\alpha$.

Our next goal is to prove that, for a fixed $\alpha_0 \in \Lambda^1$, the subsystem $T_{\Lambda^1_{\alpha_0}}$ associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T. First we need to state a series of lemmas.

Lemma 3.4 The following assertions hold.

(1) If $\alpha, \beta \in \Lambda^1$ with $[T_\alpha, T_\beta] \neq 0$, then α is connected with β.

(2) If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[L^0_\alpha, T_\beta] \neq 0$, then α is connected with β.

(3) If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[T_\alpha, T_{\overline{\beta}}] = 0$ and $[L^0_\alpha, T_{\overline{\beta}}] = 0$ if furthermore $\alpha \in \Lambda^0$.

Proof (1) Suppose $[T_\alpha, T_\beta] \neq 0$, by Lemma 2.8 (1), one gets $\delta(\alpha + \beta) \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, one gets $\{\alpha, \beta, -\delta\alpha\}$ is a connection from α to β.

(2) Suppose $[L^0_\alpha, T_\beta] \neq 0$, by Lemma 2.8 (2), one gets $\delta(\alpha + \beta) \in \Lambda^1 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^1$, we obtain $\{\alpha, 0, -\delta\alpha - \delta\beta\}$ is a connection from α to β.

(3) It is a consequence of Lemma 3.4 (1) and (2).

Lemma 3.5 Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. Then the following assertions hold.

(1) If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1$, $\beta \neq -\alpha$, and $[T_0, T_\alpha, T_\beta] \neq 0$. Then $\alpha + \delta\beta \in \Lambda^1_{\alpha_0}$.

(2) If Λ^0 is symmetric, given $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta, \gamma \in \Lambda^1$ with $\alpha + \beta + \delta\gamma \neq 0$ and $[T_\alpha, T_\beta, T_\gamma] \neq 0$, then $\alpha + \beta + \delta\gamma \in \Lambda^1_{\alpha_0}$.

Proof (1) By Lemma 2.8 (5), we have $\alpha + \delta\beta \in \Lambda^1$. From here, if $\{\alpha_1, \cdots, \alpha_{2n+1}\}$ is a connection from α_0 to α, then $\{\alpha_1, \cdots, \alpha_{2n+1}, 0, \beta\}$ is a connection from α_0 to $\alpha + \delta\beta$ in case $\delta^{2n}\alpha_1 + \cdots + \delta\alpha_{2n+1} = \alpha$ and $\{\alpha_1, \cdots, \alpha_{2n+1}, 0, -\beta\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta\alpha_{2n+1} = -\alpha$. So α_0 is connected with $\alpha + \delta\beta$.

(2) First, let us observe that $\beta, \gamma \in \Lambda^1_{\alpha_0}$. Indeed, as $[T_\alpha, T_\beta] \neq 0$, Lemma 3.4 (1) gives β is connected with α, and by the transitivity of the connection relation $\beta \in \Lambda^1_{\alpha_0}$. By Jacobi identity, either $[T_\alpha, T_\gamma] \neq 0$ or $[T_\beta, T_\gamma] \neq 0$. Then, we have as above $\gamma \in \Lambda^1_{\alpha_0}$. Second, if $\alpha + \beta = 0$, then $\alpha + \beta + \delta\gamma = \delta\gamma \in \Lambda^1_{\alpha_0}$. Suppose $\alpha + \beta \neq 0$, as $[T_\alpha, T_\beta, T_\gamma] \neq 0$ then $\delta(\alpha + \beta) \in \Lambda^0$ and $\alpha + \beta + \delta\gamma \in \Lambda^1$. Hence, as $\Lambda^1_{\alpha_0}$ is a root subsystem, we obtain $\alpha + \beta + \delta\gamma \in \Lambda^1_{\alpha_0}$.

Lemma 3.6 Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha \in \Lambda^1_{\alpha_0}$ and $\overline{\beta} \in \Lambda^1$ with

$\overline{\beta} \notin \Lambda^1_{\alpha_0}$, then for any $\widetilde{\gamma} \in \Lambda^1 \cup \{0\}$ we have $[T_{\widetilde{\gamma}}, T_\alpha, T_{\overline{\beta}}] = 0$. Any permutation of the factors $T_{\widetilde{\gamma}}$, T_α, $T_{\overline{\beta}}$ in the above triple product also makes null the triple product.

Proof Suppose $\widetilde{\gamma} = 0$. If $[T_0, T_\alpha] \neq 0$, as $[T_0, T_\alpha] \subset L^0_{\delta\alpha}$, Lemma 3.4 (3) gives $[T_0, T_\alpha, T_{\overline{\beta}}] = 0$. Suppose $\widetilde{\gamma} \in \Lambda^1_{\alpha_0}$. Since $[T_{\widetilde{\gamma}}, T_\alpha, T_{\overline{\beta}}] \subset [T_\alpha, T_{\overline{\beta}}, T_{\widetilde{\gamma}}] + [T_{\overline{\beta}}, T_{\widetilde{\gamma}}, T_\alpha]$, and by Lemma 3.4 (3), $[T_\alpha, T_{\overline{\beta}}] = [T_{\overline{\beta}}, T_{\widetilde{\gamma}}] = 0$, we have $[T_{\widetilde{\gamma}}, T_\alpha, T_{\overline{\beta}}] = 0$. Finally, if $\widetilde{\gamma} \notin \Lambda^1_{\alpha_0}$, as by Lemma 3.4 (3), $[T_{\widetilde{\gamma}}, T_\alpha] = 0$, we obtain $[T_{\widetilde{\gamma}}, T_\alpha, T_{\overline{\beta}}] = 0$. If we permute the factors in the triple product, we can use either the Jacobi identity or argue in a similar way to get that the triple product is null.

Lemma 3.7 Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$ and $\overline{\xi} \in \Lambda^1$ with $\overline{\xi} \notin \Lambda^1_{\alpha_0}$, then $[[T_\alpha, T_\beta, T_\gamma], T_{\overline{\xi}}] = 0$.

Proof By Jacobi identity, the fact $[T_0, T_0, T] = 0$ and Lemma 3.4 (3), there is no loss of generality in supposing $\alpha, \gamma \neq 0$.

Suppose $[[T_\alpha, T_\beta, T_\gamma], T_{\overline{\xi}}] \neq 0$, that is, $[[[T_\alpha, T_\beta], T_\gamma], T_{\overline{\xi}}] \neq 0$, and so either

$$[[T_\gamma, T_{\overline{\xi}}], [T_\alpha, T_\beta]] \neq 0 \text{ or } [[T_{\overline{\xi}}, [T_\alpha, T_\beta]], T_\gamma] \neq 0.$$

In the first case $[T_\gamma, T_{\overline{\xi}}] \neq 0$, which contradicts Lemma 3.4 (3). In the second case, $[T_{\overline{\xi}}, [T_\alpha, T_\beta]] \neq 0$, which contradicts Lemma 3.6.

Definition 3.8 A δ-JLTS T is said to be simple, if $\{T, T, T\} \neq 0$ and its only ideals are $\{0\}$ and T.

Theorem 3.9 Suppose Λ^0 is symmetric, the following assertions hold.

(1) For any $\alpha_0 \in \Lambda^1$, the subsystem

$$T_{\Lambda^1_{\alpha_0}} = T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}$$

of T associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

(2) If T is simple, then there exists a connection from α to β for any α, $\beta \in \Lambda^1$.

Proof (1) Recall that

$$T_{0,\Lambda^1_{\alpha_0}} := \operatorname{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma] : \alpha + \beta + \delta\gamma = 0;\ \alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}\} \subset T_0$$

and $V_{\Lambda^1_{\alpha_0}} := \oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma$. In order to complete the proof, it is sufficient to show that

$$[T_{\Lambda^1_{\alpha_0}}, T, T] \subset T_{\Lambda^1_{\alpha_0}}.$$

It is easy to see that

$$\begin{aligned}
&[T_{\Lambda^1_{\alpha_0}}, T, T] \\
&= [T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}, T, T] \\
&= \Big[\sum_{\substack{\alpha+\beta+\delta\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma] + \sum_{\alpha \in \Lambda^1_{\alpha_0}} T_\alpha, T_0 + \sum_{\beta \in \Lambda^1_{\alpha_0}} T_\beta + \sum_{\overline{\gamma} \notin \Lambda^1_{\alpha_0}} T_{\overline{\gamma}}, T_0 + \sum_{\xi \in \Lambda^1_{\alpha_0}} T_\xi + \sum_{\overline{\epsilon} \notin \Lambda^1_{\alpha_0}} T_{\overline{\epsilon}} \Big].
\end{aligned}$$

Lemmas 3.5, 3.6 and 3.7 prove the assertion.

(2) The simplicity of T implies $T_{\Lambda^1_{\alpha_0}} = T$. Hence $\Lambda^1_{\alpha_0} = \Lambda^1$.

Lemma 3.10 Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$ with $\alpha + \beta + \delta\gamma = 0$, $\xi \in \Lambda^1_{\alpha_0}$ and $\overline{\epsilon}, \overline{\rho}, \overline{\tau} \in \Lambda^1_{\beta_0} \cup \{0\}$, with $\beta_0 \notin \Lambda^1_{\alpha_0}$ and $\overline{\epsilon} + \overline{\rho} + \delta\overline{\tau} = 0$. Then

(1) $[T_\alpha, T_\beta, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] = 0$.

(2) $[[T_\alpha, T_\beta, T_\gamma], T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] = 0$.

(3) If $\overline{\mu} \notin \Lambda^1_{\alpha_0}$, $[[T_\alpha, T_\beta, T_\gamma], T_{\overline{\mu}}, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] = 0$.

Proof (1) Suppose $[T_\alpha, T_\beta, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] \neq 0$, then either $[T_\beta, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], T_\alpha] \neq 0$ or $[[T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], T_\alpha, T_\beta] \neq 0$, a contradiction with Lemma 3.7 or with $[T_0, T_0, T] = 0$.

(2) Let us suppose $[[T_\alpha, T_\beta, T_\gamma], T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] \neq 0$. By Jacobi identity, we have

$$[[T_\alpha, T_\beta, T_\gamma], T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] \subset [T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], [T_\alpha, T_\beta, T_\gamma]] + [[T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], [T_\alpha, T_\beta, T_\gamma], T_\xi].$$

From $[T_0, T_0, T] = 0$, then

$$[[T_\alpha, T_\beta, T_\gamma], T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] \subset [T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], [T_\alpha, T_\beta, T_\gamma]] \neq 0.$$

From here, the product $[T_\xi, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]]$ is nonzero which contradicts Lemma 3.7.

(3) By Jacobi identity and $[T_0, T_0, T] = 0$, we have

$$[[T_\alpha, T_\beta, T_\gamma], T_{\overline{\mu}}, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}]] \subset [T_{\overline{\mu}}, [T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], [T_\alpha, T_\beta, T_\gamma]] \subset -\delta[[T_{\overline{\epsilon}}, T_{\overline{\rho}}, T_{\overline{\tau}}], T_{\overline{\mu}}, [T_\alpha, T_\beta, T_\gamma]].$$

If $\overline{\mu} \in \Lambda^1_{\beta_0}$, Lemma 3.10 (2) shows the above triple prouct is null. Otherwise, if $\overline{\mu} \notin \Lambda^1_{\beta_0}$, the nullity of the triple product is a consequence of Lemma 3.7.

Theorem 3.11 Suppose Λ^0 is symmetric. Then for a vector space complement U of $\mathrm{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma] : \alpha + \beta + \delta\gamma = 0$, where $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ in T_0, we have

$$T = U + \sum_{[\alpha] \in \Lambda^1/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals $T_{\Lambda^1_\alpha}$ of T described in Theorem 3.9. Moreover $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof Let us denote $\xi_0 :=\mathrm{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma] : \alpha+\beta+\delta\gamma = 0$, where $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ in T_0. By Proposition 3.2, we can consider the quotient set $\Lambda^1/\sim := \{[\alpha] : \alpha \in \Lambda^1\}$. By denoting $I_{[\alpha]} := T_{\Lambda^1_\alpha}$, $T_{0,[\alpha]} := T_{0,\Lambda^1_\alpha}$ and $V_{[\alpha]} := V_{\Lambda^1_\alpha}$, one gets $I_{[\alpha]} := T_{0,[\alpha]} \oplus V_{[\alpha]}$. From

$$T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha) = (U + \xi_0) \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha),$$

it follows $\oplus_{\alpha \in \Lambda^1} T_\alpha = \oplus_{[\alpha] \in \Lambda^1/\sim} V_{[\alpha]}$, $\quad \xi_0 = \sum_{[\alpha] \in \Lambda^1/\sim} T_{0,[\alpha]}$, which implies

$$T = U + \xi_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha) = U + \sum_{[\alpha] \in \Lambda^1/\sim} I_{[\alpha]},$$

where each $I_{[\alpha]}$ is an ideal of T by Theorem 3.9.

Next, it is sufficient to show that $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. Note that,

$$\begin{aligned}[I_{[\alpha]}, T, I_{[\beta]}] &= [\sum_{\substack{\alpha'+\alpha''+\delta\alpha'''=0\\ \alpha',\alpha'',\alpha'''\in\Lambda^1_\alpha\cup\{0\}}} [T_{\alpha'}, T_{\alpha''}, T_{\alpha'''}] + \sum_{\alpha'\in\Lambda^1_\alpha} T_{\alpha'},\\ &\quad T_0 + \sum_{\alpha'\in\Lambda^1_\alpha} T_{\alpha'} + \sum_{\gamma\notin\Lambda^1_\alpha} T_\gamma, \sum_{\substack{\beta'+\beta''+\delta\beta'''=0\\ \beta',\beta'',\beta'''\in\Lambda^1_\beta\cup\{0\}}} [T_{\beta'}, T_{\beta''}, T_{\beta'''}] + \sum_{\beta'\in\Lambda^1_\beta} T_{\beta'}].\end{aligned}$$

Applying Lemmas 3.6, 3.7 and 3.10, we get $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Definition 3.12 The annihilator of a δ-JLTS T is the set $\mathrm{Ann}(T) = \{x \in T : [x, T, T] = 0\}$.

Corollary 3.13 Suppose Λ^0 is symmetric. If $\mathrm{Ann}(T) = 0$ and $[T, T, T] = T$, then T is the direct sum of the ideals given in Theorem 3.11, $T = \oplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$.

Proof From $[T, T, T] = T$ and Theorem 3.11, we have

$$[U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}] = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

Taking into account $U \subset T_0$ and the fact that $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$ (see Theorem 3.11) give us that $U = 0$. That is,

$$T = \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

To finish, it is sufficient to show the direct character of the sum. For $x \in I_{[\alpha]} \cap \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}} I_{[\beta]}$, using again the equation $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ for $[\alpha] \neq [\beta]$, we obtain

$$[x, T, I_{[\alpha]}] = [x, T, \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}} I_{[\beta]}] = 0.$$

So $[x, T, T] = [x, T, I_{[\alpha]} + \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}} I_{[\beta]}] = [x, T, I_{[\alpha]}] + [x, T, \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}} I_{[\beta]}] = 0 + 0 = 0$. That is, $x \in \mathrm{Ann}(T) = 0$. Thus $x = 0$, as desired.

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分裂的δ-JORDAN 李三系的结构

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摘 要: 本文研究了带有相关0根空间的任意分裂的δ-Jordan 李三系的结构.利用这种三系的根连通, 得到了带有对称根系的分裂的δ-Jordan 李三系T可以表示成$T = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$，其中$U$是0根空间$T_0$ 的子空间, 任意$I_{[\alpha]}$为T的理想, 并且满足当$[\alpha] \neq [\beta]$时, $[I_{[\alpha]}, T, I_{[\beta]}] = 0$.

关 键 词: 分裂的δ-Jordan 李三系; 李三系; δ-Jordan 李代数; 根系; 根空间

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