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INEQUALITIES FOR SOLUTIONS OF THE NON-HOMOGENEOUS DIRAC-HARMONIC EQUATIONS IN DIFFERENTIAL FORMS

DAI Zhi-min[1,2], LIU San-yang[1]

(1.School of Mathematics and Statistics, Xidian University, Xi'an 710126, China)

(2.School of Science, Xi'an Technological University, Xi'an 710021, China)

Abstract: In this paper, some inequalities related to the solutions of a class of nonhomogeneous Dirac-harmonic equations in differential forms are studied. By the conditions of the Dirac-harmonic equation and the operation rules of Dirac-harmonic operator D, Poincaré inequality, Caccioppoli inequality and the weak inverse Hölder inequality are obtained. As the applications of related inequalities, the forms of the Poincaré inequality with special weights and in the $L^s(\mu)$-averaging domains are proved. The related inequalities of solutions of homogeneous Dirac-harmonic equation are extended to the case of non-homogeneous Dirac-harmonic equation.

Keywords: Non-homogeneous Dirac-harmonic equations; differential forms; norm inequalities; weights; $L^s(\mu)$-averaging domains

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1 Introduction

As generalizations of the functions, differential forms were widely used in many fields, including potential theory, partial differential equations, quasiconformal mappings and etc. During recent years a series of estimates and inequalities for differential forms, particularly, for the forms satisfying the homogeneous or nonhomogeneous A-harmonic equations, or the conjugate A-harmonic equations in $\mathbb{R}^n (n \geq 2)$, were developed, see [1–8]. These estimates and inequalities are critical tools to investigate the properties of solutions to the nonlinear differential equations and to control oscillatory behavior in domains or on manifolds. However, the nonlinear PDE with the Hodge-Dirac operator for differential forms, that is, the Dirac-harmonic equation has yet to be further developed, where the Dirac operator was initiated by Paul Dirac in order to get a form of quantum theory compatible with special relativity, which was playing a critical role in some fields of mathematics and physics, such as quantum mechanics, Clifford analysis and partial differential equations, see [9–13].

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Biography: Dai Zhimin (1982–), male, born at Xiantao, Hubei, lecturer, major in harmonic analysis and tensor analysis.

The purpose of this paper is to introduce the non-homogeneous Dirac-harmonic equation $d^\star A(x, D\omega) = B(x, D\omega)$ for differential forms and initiate the study of this new type of differential equations, where the Hodge-Dirac operator D is defined by $D = d + d^\star$, d is the exterior differential operator, $d^\star$ is the Hodge codifferential that is formal adjoint operator of d, and A is an operator satisfying certain conditions. Specifically, we establish Poincaré-type inequalities, Caccioppoli-type inequalities and the weak reverse Hölder inequality for differential forms satisfying the non-homogeneous Dirac-harmonic equation. These basic inequalities will form the basis for the study of the L^p-theory of the new introduced Dirac-harmonic equation for differential forms.

Now we introduce some notations and definitions. Let Ω be an open subset of $\mathbb{R}^n (n \geq 2)$ and B be a ball in $\mathbb{R}^n$. Let ρB denote the ball with the same center as B and $\mathrm{diam}(\rho B) = \rho\ \mathrm{diam}(B) (\rho > 0)$. $|\Omega|$ is used to denote the Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n), l = 0, 1, \ldots, n$, be the linear space of all l-forms $\omega(x) = \sum_I \omega_I(x) \mathrm{d}x_I = \sum_I \omega_{i_1 i_2 \cdots i_l}(x) \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_l}$ in $\mathbb{R}^n$, where $I = (i_1, i_2, \ldots, i_l), 1 \leq i_1 < i_2 < \cdots < i_l \leq n$, are the ordered l-tuples. The Grassman algebra $\wedge = \wedge(\mathbb{R}^n) = \oplus_{l=0}^n \wedge^l (\mathbb{R}^n)$ is a graded algebra with respect to the exterior products $\wedge$. Moreover, if each of the coefficient $\omega_I(x)$ of $\omega(x)$ is differential on Ω, then we call $\omega(x)$ a differential l-form on Ω, use $D^{'}(\Omega, \wedge^l)$ to denote the space of all differential l-forms on Ω and $D'(\Omega, \wedge) = \oplus_{l=0}^n D^{'}(\Omega, \wedge^l)$. Analogously $C^\infty(\Omega, \wedge^l)$ denotes the space of smooth l-forms on Ω. The exterior derivative $d : D^{'}(\Omega, \wedge^l) \to D^{'}(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$, is given by

$$d\omega(x) = \sum_I \sum_{j=1}^n \frac{\partial \omega_{i_1 i_2 \cdots i_l}(x)}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_l} \tag{1.1}$$

for all $\omega \in D^{'}(\Omega, \wedge^l)$. The Hodge star operator $\star : \wedge^k \to \wedge^{n-k}$ is defined as follows. If $\omega = \omega_{i_1 i_2 \cdots i_k}(x_1, x_2, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = \omega_I dx_I$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star\omega = \star(\omega_{i_1 i_2 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^{\Sigma(I)} \omega_I dx_J$, where $I = (i_1, i_2, \cdots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = \frac{k(k+1)}{2} + \sum_{j=1}^k i_j$. The Hodge codifferential operator $d^* : D^{'}(\Omega, \wedge^{l+1}) \to D^{'}(\Omega, \wedge^l)$is defined by $d^* = (-1)^{nl+1} \star d \star$ on $D^{'}(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$. For all $\omega \in D^{'}(\Omega, \wedge^l)$, we have $d(d\omega) = d^*(d^*\omega) = 0$. $L^p(\Omega, \wedge^l)(1 \leq p < \infty)$ is a Banach space with the norm $\|\omega\|_{p,\Omega} = (\int_\Omega |\omega(x)|^p \mathrm{d}x)^{1/p} = (\int_\Omega (\sum_I |\omega_I(x)|^2)^{p/2} \mathrm{d}x)^{1/p} < \infty$ and $L^p(\Omega, \wedge) = \oplus_{l=0}^n L^p(\Omega, \wedge^l)$. Similarly, the notations $L^p_{\mathrm{loc}}(\Omega, \wedge^l)$ and $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^l)$ are self-explanatory.

From [14], ω is a differential form in a bounded convex domain Ω, then there is a decomposition

$$\omega = d(T\omega) + T(d\omega), \tag{1.2}$$

where T is called a homotopy operator. For the homotopy operator T, we know that

$$\|T\omega\|_{p,B} \leq C|B|\mathrm{diam}(B)\|\omega\|_{p,B} \tag{1.3}$$

holds for any differential form $\omega \in L^p_{\mathrm{loc}}(\Omega, \wedge^l), l = 1, 2, \ldots, n, 1 < p < \infty$. Furthermore, we can define the l-form $\omega_\Omega \in D'(\Omega, \wedge^l)$ by

$$\omega_\Omega = \begin{cases} |\Omega|^{-1} \int_\Omega \omega(x)\mathrm{d}x, & l = 0, \\ dT(\omega), & l = 1, 2, \ldots, n \end{cases} \tag{1.4}$$

for all $\omega \in L^p(\Omega, \wedge^l), 1 \le p < \infty$.

The theory of differential equations was very well developed during last several decades. Particularly, there was an increasing interest in different types of differential equations for differential forms, see [15–21]. Among these types of equations, the traditional A-harmonic equation for differential forms

$$d^* A(x, d\omega) = 0 \tag{1.5}$$

in $\mathbb{R}^n$, and the corresponding nonhomogeneous A-harmonic equation for differential forms is a nonlinear elliptic equation of the form

$$d^* A(x, d\omega) = B(x, d\omega) \tag{1.6}$$

received much investigation in recent years. In [10], for the purpose of dealing with terms $d\omega$ and $d^*\omega$ simultaneously in many cases , such as in the case of Hodge decomposition of a differential form, Ding introduced the following Dirac-harmonic equation

$$d^* A(x, D\omega) = 0 \tag{1.7}$$

for differential forms. Similarly, we could introduce the corresponding nonhomogeneous Dirac-harmonic equation for differential forms is a nonlinear elliptic equation of the form

$$d^* A(x, D\omega) = B(x, D\omega), \tag{1.8}$$

where the Hodge-Dirac operator D is a Dirac operator defined by $D = d + d^*$, d is the exterior differential operator, d^* is the Hodge codifferential that is formal adjoint operator of d, $A : \Omega \times \wedge(\mathbb{R}^n) \to \wedge(\mathbb{R}^n)$ and $B : \Omega \times \wedge(\mathbb{R}^n) \to \wedge(\mathbb{R}^n)$ satisfy the following conditions

$$|A(x, \xi)| \le a|\xi|^{p-1} \quad , \quad \langle A(x, \xi), \xi\rangle \ge |\xi|^p \quad \text{and} \quad |B(x, \xi)| \le b|\xi|^{p-1} \tag{1.9}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge(\mathbb{R}^n)$, here $a, b > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.8). Let $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1}) = \bigcap W^1_p(\Omega', \wedge^{l-1})$, where the intersection is for all Ω' compactly contained in Ω. A solution to (1.2) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, D\omega), D\varphi\rangle + \langle B(x, D\omega), \varphi\rangle = 0 \tag{1.10}$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

A solution ω of the homogeneous and non-homogeneous A-harmonic equation (1.5) and (1.6) is called a nontrivial solution if $d\omega \ne 0$; otherwise, ω is called a trivial solution of (1.5) and (1.6). Similarly, a solution ω of the homogeneous and non-homogeneous Dirac-harmonic equation (1.7) and (1.8) is called a nontrivial solution if $D\omega \neq 0$; otherwise, ω is called a trivial solution of (1.7) and (1.8). It should be noticed that the Dirac-harmonic equation can be considered as an extension of the traditional A-harmonic equations with operator d being replaced by the Dirac operator $D = d+d^\star$. It is also easy to see that if ω is a function (0-form), both the traditional non-homogeneous A-harmonic equation $d^\star A(x, d\omega) = B(x, d\omega)$and the non-homogeneous Dirac-harmonic equation $d^\star A(x, D\omega) = B(x, D\omega)$ reduce to the usual non-homogeneous A-harmonic equation

$$\mathrm{div}A(x, \nabla\omega) = B(x, \nabla\omega). \tag{1.11}$$

So far although the research on the non-homogeneous A-harmonic equation has gained great attention, we can find most of what we get is the solution of its degenerated equation, that is the special solution of the homogeneous A-harmonic equation, see [1,3, 22–23] for more details. The non-homogeneous Dirac-harmonic equation (1.8), as a kind of more complicated equation compared to the non-homogeneous A-harmonic equation, now we can get some trivial special solutions, that is, the solutions satisfy $D\omega = 0$.

In order to find the trivial solution of equation (1.8), we need the lemmas below.

Lemma 1.1 [9–11] Let ω be any differential form. Then, $D\omega = 0$ if and only if $d\omega = 0$ and $d^\star\omega = 0$. Noticing that $\Delta\omega = (dd^\star+d^\star d)\omega = (d+d^\star)^2\omega = D^2\omega$, we have the proposition.

Proposition 1.1 Let ω be any differential form. Then, $\Delta\omega = 0$ if and only if $D\omega = 0$.

According to Lemma 1.1 and the definitions of the operator d and $d^\star = (-1)^{nl+1} \star d\star$, we can know that $\omega = \sum\limits_I a_I dx_I, a_I \in \mathbb{R}$ is the special solution of equation (1.8). Besides, we can check the following two examples are also the trivial solutions of equation (1.8).

Example 1.1 Let

$$\omega(x_1, x_2) = \frac{-x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2 \tag{1.12}$$

be a 1-form in $\Omega \subset \mathbb{R}^2$ which does not contain the origin $(0, 0)$. Then $\omega(x_1, x_2)$ is a trivial solution of the A-harmonic equation (1.5)–(1.8) in Ω.

Proof By simple calculation, we have

$$\begin{aligned}
d\omega &= \frac{\partial}{\partial x_2}(\frac{-x_2}{x_1^2 + x_2^2}) dx_2 \wedge dx_1 + \frac{\partial}{\partial x_1}(\frac{x_1}{x_1^2 + x_2^2}) dx_1 \wedge dx_2 \\
&= \frac{x_2^2 - x_1^2}{x_1^2 + x_2^2}(dx_2 \wedge dx_1 + dx_1 \wedge dx_2) = 0, \\
\star(\omega) &= \frac{-x_2}{x_1^2 + x_2^2} dx_2 + \frac{-x_1}{x_1^2 + x_2^2} dx_1, \\
d(\star(\omega)) &= \frac{\partial}{\partial x_1}(\frac{-x_2}{x_1^2 + x_2^2}) dx_1 \wedge dx_2 + \frac{\partial}{\partial x_2}(-\frac{x_1}{x_1^2 + x_2^2}) dx_2 \wedge dx_1 \\
&= \frac{2x_1x_2}{x_1^2 + x_2^2}(dx_2 \wedge dx_1 + dx_1 \wedge dx_2) = 0,
\end{aligned} \tag{1.13}$$

$$\begin{aligned} d^\star(\omega) =&(-1)^3 \star d \star(\omega) = 0, \\ D(\omega) =&(d + d^\star)(\omega) = 0. \end{aligned} \tag{1.14}$$

Example 1.2 Let

$$\begin{aligned} \omega(x_1, x_2, x_3) =&\omega_1 dx_1 \wedge dx_2 + \omega_2 dx_2 \wedge dx_3 + \omega_3 dx_3 \wedge dx_1 \\ =&(x_1 + x_2 - 2x_3)dx_1 \wedge dx_2 + (x_1 - x_2 + x_3)dx_2 \wedge dx_3 \\ &+ (x_1 + x_2 - x_3)dx_3 \wedge dx_1 \end{aligned} \tag{1.15}$$

be a 2-form in $\Omega \subset \mathbb{R}^3$. Then $\omega(x_1, x_2)$ is a trivial solution of the A-harmonic equation (1.5)–(1.8) in Ω.

Proof

$$d\omega =(\frac{\partial(w_1)}{\partial x_3} + \frac{\partial(w_2)}{\partial x_1} + \frac{\partial(w_3)}{\partial x_2})dx_1 \wedge dx_2 \wedge dx_3 = 0, \tag{1.16}$$

$$\begin{aligned} \star\omega =&\omega_1 dx_3 + \omega_2 dx_1 - \omega_3 dx_2, \\ d \star \omega =&(\frac{\partial(w_1)}{\partial x_1} - \frac{\partial(w_2)}{\partial x_3})dx_1 \wedge dx_3 - (\frac{\partial(w_3)}{\partial x_1} + \frac{\partial(w_2)}{\partial x_2})dx_1 \wedge dx_2 \\ &+ (\frac{\partial(w_1)}{\partial x_2} - \frac{\partial(w_3)}{\partial x_3})dx_2 \wedge dx_3 = 0, \\ d^\star(\omega) =&(-1)^7 \star d \star(\omega) = 0, \\ D(\omega) =&(d + d^\star)(\omega) = 0. \end{aligned} \tag{1.17}$$

In order to obtain the related inequalities with the solutions of the non-homogeneous Dirac-harmonic equation for differential forms, we need the following generalized Hölder inequality in this paper.

Lemma 1.2 [1] Let $0 < p, q < \infty$ and $s^{-1} = p^{-1} + q^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then

$$\|fg\|_{s,\Omega} \le \|f\|_{p,\Omega}\|g\|_{q,\Omega} \tag{1.18}$$

for any $\Omega \subset \mathbb{R}^n$.

If we select $q = \frac{sp}{p-s}$ and $g \equiv 1$, we can get

$$\|f\|_{s,Q} \le |Q|^{\frac{1}{s}-\frac{1}{p}}\|f\|_{p,Q} \tag{1.19}$$

or

$$\|f\|_{s,Q} \le |Q|^{\frac{p-s}{ps}}\|f\|_{p,Q}. \tag{1.20}$$

2 Poincaré Sobolev and Embedding Inequalities

Different versions of the classical Poincaré inequality were established in the study of the Sobolev space and differential forms, see [1,6]. Susan proved the Poincaré inequality in L^s-averaging domains in [15]. Tadeusz and Lutoborski proved a local Poincaré-type inequality in [14], which plays a crucial role in generalizing the theory of Sobolev functions to differential forms.

In [10], Ding Proved the Poincaré Sobolev and embedding inequalities with the Dirac operator. If appropriate substitution is made, ones can get the new forms of these inequalities.

Lemma 2.1 [10] Let $u \in D'(Q, \wedge^l)$ be a differential form and $Du \in L^p(Q, \wedge)$. Then $u - u_Q$ is in $L^{\frac{np}{(n-p)}}$ and

$$(\int_Q |u - u_Q|^{\frac{np}{n-p}} dx)^{\frac{(n-p)}{np}} \leq C_p(n)(\int_Q |Du|^p dx)^{\frac{1}{p}} \tag{2.1}$$

for Q a cube or a ball in $\mathbb{R}^n$, $l = 0, 1, \ldots, n-1$ and $1 < p < n$.

Considering that the norms $\|u\|_{p,Q}$ and $\|u - u_Q\|_{p,Q}$ are comparable, see [4], namely,

$$\|u - u_Q\|_{p,Q} \leq C_1\|u\|_{p,Q} \leq C_2\|u - u_Q\|_{p,Q}, \tag{2.2}$$

we have

$$(\int_Q |u|^{\frac{np}{n-p}} dx)^{\frac{(n-p)}{np}} \leq C_p(n)(\int_Q |Du|^p dx)^{\frac{1}{p}} \tag{2.3}$$

for Q a cube or a ball in $\mathbb{R}^n$, $l = 0, 1, \ldots, n-1$ and $1 < p < n$.

Replacing u by Du in (2.1) and noting that $\Delta = (d + d^\star)^2 = D^2$, we have the following inequality with the Dirac operator

$$(\int_Q |Du - (Du)_Q|^{\frac{np}{n-p}} dx)^{\frac{(n-p)}{np}} \leq C_p(n)(\int_Q |\Delta u|^p dx)^{\frac{1}{p}}. \tag{2.4}$$

Now, we give the new Poincaré inequality with the Dirac operator.

Theorem 2.1 Let $u \in D'(Q, \wedge^l)$ be a differential form and $Du \in L^p(Q, \wedge), p > 1$. Then, $u - u_Q$ is in $L^p(Q, \wedge)$ and

$$\|u - u_Q\|_{p,Q} \leq C|Q|\text{diam}(Q)\|Du\|_{p,Q} \tag{2.5}$$

for all cubes or balls Q with $Q \subset \mathbb{R}^n$, where C is a constant, independent of u and Du.

Proof

$$\|u - u_Q\|_{p,Q} = \|T(du)\|_{p,Q} \leq C_1|Q|\text{diam}(Q)\|du\|_{p,Q} \leq C_1|Q|\text{diam}(Q)\|Du\|_{p,Q}. \tag{2.6}$$

Noticing that $\text{diam}(Q) \leq C_2|Q|^{\frac{1}{n}}$, we have

$$\|u - u_Q\|_{p,Q} \leq C_3|Q|^{1+\frac{1}{n}}\|Du\|_{p,Q}. \tag{2.7}$$

Similarly, replacing u by Du in (2.5), we have

$$\|Du - (Du)_Q\|_{p,Q} \leq C_3|Q|\text{diam}(Q)\|\Delta u\|_{p,Q}. \tag{2.8}$$

Futhermore, considering that the norms $\|Du\|_{p,Q}$ and $\|Du-(Du)_Q\|_{p,Q}$ are comparable, we have

$$\|Du\|_{p,Q} \leq C_3|Q|\text{diam}(Q)\|\Delta u\|_{p,Q}. \tag{2.9}$$

3 Caccioppoli-Type Inequality

The Caccioppoli-type estimates become powerful tools in analysis and related fields, for the Caccioppoli-type inequalities or estimates provide upper bounds for the norms of ∇u or du in terms of the corresponding norm of u or $u-c$, where u is a differential form or a function satisfying certain conditions. In recent years, different versions of Caccioppoli-type estimates were established; see [1,3,10].

In [10], Ding obtained the Caccioppoli-type inequality with solutions of the Dirac-harmonic equation (1.7) for differential forms. Similarly, we can also get the Caccioppoli-type inequality with solutions of the non-homogeneous Dirac-harmonic equation for differential forms. Now, we first introduce the product rules for the differentiations of exterior product.

Lemma 3.1 [10] (i) Product rule for d holds: let $u = \sum\limits_I u_I dx_I$ be a k-form and $v = \sum\limits_J v_J dx_J$ be l-form in $D'(\Omega, \wedge)$, where u_I and v_J are differentiable functions in Ω. Then

$$d(u \wedge v) = du \wedge v + (-1)^k u \wedge dv. \tag{3.1}$$

(ii) Product rule for $d^\star$ holds: let $u \in D'(\Omega, \wedge)$ be a k-form $u = \sum\limits_I u_I dx_I$, where $I = (i_1, i_2, \cdots, i_k)$ is a set of all k tuples with $i_1 < i_2 < \cdots < i_k$ and $1 \le i_k \le n$. Let η be a differentiable function in Ω, then

$$d^\star(u\eta) = (d^\star u)\eta + (-1)^m \sum_{I-q} u_I \frac{\partial \eta}{\partial x_q} dx_{I-q}, \tag{3.2}$$

$$D(u\eta) = (Du)\eta + (-1)^k u(d\eta) + (-1)^m \sum_{I-q} u_I \frac{\partial \eta}{\partial x_q} dx_{I-q}, \tag{3.3}$$

where m, q are integers, $1 \le q \le n$ and $I-q$ is an abusive notation to represent an $k-1$ tuple with i_q is missing $(i_1, \cdots, \hat{i_q}, \cdots, i_k)$ and $q \notin J$ means $q \ne j_s$ for any j_s in $n-k$ tuples J. Also, $\sum\limits_I$ means the sum of all possible k tuples.

Theorem 3.1 Let $u \in D'(\Omega, \wedge^l), l = 0, 1, \ldots, n$, be a solution of the non-homogeneous Dirac-harmonic equation (1.8) in a bounded domain $\Omega \subset \mathbb{R}^n$, and assume that $1 < p < \infty$ is a fixed exponent associated with equation (1.8). Then there exists a constant C, independent of u and Du, such that

$$\|Du\|_{p,B} \le C|B|^{\frac{-1}{n}} \|u\|_{p,\sigma B} \tag{3.4}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

Proof Let $\eta \in C_0^\infty(\sigma B)$, $0 \le \eta \le 1$ with $\eta \equiv 1$ in B, and $|d\eta| = |\nabla \eta| \le \frac{C_1}{\operatorname{diam}(B)}$. Choosing the test form $\varphi = -u\eta^p$. Then

$$D\varphi = -(Du)\eta^p + (-1)^{l+1} up\eta^{p-1} d\eta + (-1)^{m+1} \sum_{I-q} u_I p\eta^{p-1} \frac{\partial \eta}{\partial x_q} dx_{I-q}. \tag{3.5}$$

From (1.10), we obtain

$$\begin{aligned}\int_{\sigma B}\langle A(x,Du),\eta^p Du\rangle &= -\int_{\sigma B}\langle A(x,Du),(-1)^l u p\eta^{p-1}d\eta\rangle\\ -\int_{\sigma B}\langle A(x,Du),(-1)^m\sum_{I-q}u_I p\eta^{p-1}\frac{\partial\eta}{\partial x_q}dx_{I-q}\rangle &- \int_{\sigma B}\langle B(x,Du),u\eta^p\rangle.\end{aligned} \tag{3.6}$$

Applying (1.9), we have

$$\int_{\sigma B}|\langle A(x,Du),\eta^p Du\rangle| \ge \int_{\sigma B}|\eta|^p|Du|^p dx \tag{3.7}$$

Notice that $|\sum_{I-q}u_I p\eta^{p-1}\frac{\partial\eta}{\partial x_q}dx_{I-q}| \le p|\eta|^{p-1}|u||d\eta|$ and $\operatorname{diam}(B)\le\operatorname{diam}(\Omega)<\infty$.

Using the Hölder inequality (1.18), we obtain

$$\begin{aligned}&\int_{\sigma B}|\eta|^p|Du|^p dx \le \int_{\sigma B}|\langle A(x,Du),\eta^p Du\rangle|\\ &\le \int_{\sigma B}|A(x,Du)|(p|u||\eta|^{p-1}|d\eta|)dx + \int_{\sigma B}|A(x,Du)|(p|u||\eta|^{p-1}|d\eta|)dx + \int_{\sigma B}b|Du|^{p-1}(|u||\eta|^p)dx\\ &\le \frac{C_2}{\operatorname{diam}(B)}\int_{\sigma B}|Du|^{p-1}|\eta|^{p-1}|u|dx + C_3\int_{\sigma B}|Du|^{p-1}|u||\eta|^p dx\\ &\le \frac{C_2}{\operatorname{diam}(B)}\|u\|_{p,\sigma B}(\int_{\sigma B}(|\eta||Du|)^p dx)^{\frac{p-1}{p}} + C_3\|\eta u\|_{p,\sigma B}(\int_{\sigma B}(|\eta||Du|)^p dx)^{\frac{p-1}{p}}\\ &\le \frac{C_2+C_3\operatorname{diam}(B)}{\operatorname{diam}(B)}\|u\|_{p,\sigma B}(\int_{\sigma B}(|\eta||Du|)^p dx)^{\frac{p-1}{p}}\\ &\le \frac{C_4}{\operatorname{diam}(B)}\|u\|_{p,\sigma B}(\int_{\sigma B}(|\eta||Du|)^p dx)^{\frac{p-1}{p}},\end{aligned} \tag{3.8}$$

which is equivalent to

$$\|\eta Du\|_{p,\sigma B} \le \frac{C_4}{\operatorname{diam}(B)}\|u\|_{p,\sigma B}. \tag{3.9}$$

Thus, we have

$$\|Du\|_{p,B} = \|\eta Du\|_{p,B} \le \|\eta Du\|_{p,\sigma B} \le \frac{C_4}{\operatorname{diam}(B)}\|u\|_{p,\sigma B} \le C_5|B|^{\frac{-1}{n}}\|u\|_{p,\sigma B}. \tag{3.10}$$

Let u be a solution of equation (1.8) and c be a harmonic form. Then, $w=u+c$ is also a solution of equation (1.8). Write $u=\omega-c$, then $Du=D\omega$. Thus, we have the following version of (3.10).

Corollary 3.1 Let c be a harmonic form and u is a solution of the non-homogeneous Dirac-harmonic equation (1.8). Then

$$\|Du\|_{p,B} \le C|B|^{-\frac{1}{n}}\|u-c\|_{p,\sigma B} \tag{3.11}$$

for all balls or cubes B with $\sigma B\subset\Omega$, $\sigma>1$ be a constant.

4 Weak Reverse Hölder Inequality

It is known to all that a necessary condition for (1.20) to hold is that $p \geq s$. When p, s are any two positive constants and an inequality weaker than (1.20) holds, which is known as the weak reverse Hölder inequality and plays an important role in establishing the weighted form of the related inequality, see [1] for more details.

In [15], if ω is a solution of the A-harmonic equation (1.6), then the weak reverse Hölder inequality

$$\|\omega\|_{s,Q} \leq C|Q|^{\frac{r-s}{rs}}\|\omega\|_{r,\sigma Q} \tag{4.1}$$

holds for all cubes or balls Q with $\sigma Q \subset \Omega$ and any $0 < r, s < \infty$.

Recently, Ding proved that inequality (4.1) also holds for the solutions of the Dirac-harmonic equation (1.7), see [10]. Now we prove that for any solution ω of the non-homogenious Dirac-harmonic equation (1.8), the weak reverse Hölder inequality still holds. Specifically, we have the following Theorem 4.3.

We define the differential forms ω^+ and ω^- as follows. If $\omega \in \wedge^0(R)$, let $\omega^+ = \max\{\omega, 0\}$ and $\omega^- = \min\{\omega, 0\}$. Otherwise, let $\omega^+ = \sum \omega_I^+ dx_I$ and $\omega^- = \sum \omega_I^- dx_I$.

Theorem 4.1 Let ω be a solution of (1.8) and $\eta \in C_0^\infty(\Omega)$ with $\eta \geq 0$. There exists a constant C, depending only on a, b, and p, such that

$$\int_\Omega |D\omega^+|^p\eta^p dx \leq C(\int_\Omega |\omega^+|^p|d\eta|^p dx + \int_\Omega |\omega^+|^p|\eta|^p dx). \tag{4.2}$$

The same is true for ω^-.

Proof Using the test form $\varphi = -\omega^+\eta^p$ and using inequalities (1.9) and the Hölder inequality (1.18), we get

$$\begin{aligned}\int_\Omega \eta^p|D\omega^+|^p dx \leq& C_1\int_\Omega |D\omega^+|^{p-1}\eta^{p-1}|d\eta||\omega^+|dx + C_2\int_\Omega |D\omega^+|^{p-1}|\omega^+|\eta^p dx\\ \leq& C_1(\int_\Omega |\omega^+|^p|d\eta|^p dx)^{\frac{1}{p}}(\int_\Omega |D\omega^+|^p\eta^p dx)^{\frac{(p-1)}{p}}\\ &+ C_2(\int_\Omega |\omega^+|^p\eta^p dx)^{\frac{1}{p}}(\int_\Omega |D\omega^+|^p\eta^p dx)^{\frac{(p-1)}{p}}.\end{aligned} \tag{4.3}$$

By the following basic inequality

$$\sum_{i=1}^n |a_i|^s \leq n(\sum_{i=1}^n |a_i|)^s \leq n^{s+1}\sum_{i=1}^n |a_i|^s, \tag{4.4}$$

where $s > 0$ is any constant, it follows that

$$\begin{aligned}\int_\Omega \eta^p|D\omega^+|^p dx &\leq (C_1(\int_\Omega |\omega^+|^p|d\eta|^p dx)^{\frac{1}{p}} + C_2(\int_\Omega |\omega^+|^p\eta^p dx)^{\frac{1}{p}})^p\\ &\leq C_3(\int_\Omega |\omega^+|^p|d\eta|^p dx + \int_\Omega |\omega^+|^p\eta^p dx).\end{aligned} \tag{4.5}$$

Theorem 4.2 Let ω be a solution of (1.8) in $\Omega, q>0$. There exists a constant C, depending only on a,p,q,n such that

$$\int_\Omega |\omega^+|^q |D\omega^+|^p \eta^p dx \le C\int_\Omega |\omega^+|^{p+q}(|d\eta|^p+|\eta|^p)dx \tag{4.6}$$

for all nonnegative $\eta\in C_0^\infty(\Omega)$.

The proof of Theorem 4.2 is similar to [15]. For complete purpose and convenience to the readers, we state the proof as follows.

Proof Let $t>0$ be any constant, $T=\sum\limits_I t dx_I$, then $\omega - T$ is also a solution of (1.8) and satisfies (4.2) too. Consider the sets $A=\cup_I\{x|(\omega_I - t)^+>0\}$, $B=\{x|(\omega-T)^+\neq 0\}$, $C=\{x||\omega^+|>t\}$, $D_I=\{x|\omega_I^+>t\}$ and $B=\{x|(\omega-T)^+\neq 0\}$. Then $D_I\subset A=B\subset C$ for all for all I. Let $dv=|D\omega^+|^p\eta^p dx$ and using (4.2) to get

$$\begin{aligned}
&\int_\Omega |\omega^+|^q dv \le C_1\sum_I\int_\Omega |\omega_I^+|^q dv = C_1\sum_I\{q\int_0^\infty t^{q-1}\int_{D_I} dv dt\}\\
\le& C_1\sum_I\{q\int_0^\infty t^{q-1}\int_A |D(\omega_I^+-t)|^p\eta^p dx dt\} \le C_2\int_0^\infty t^{q-1}\int_B |D(\omega-T)^+|^p\eta^p dx dt\\
\le& C_3\int_0^\infty t^{q-1}\int_B |(\omega-T)^+|^p(|d\eta|^p+\eta^p)dx dt \le C_3\int_0^\infty t^{q-1}\int_C |\omega^+|^p(|d\eta|^p+\eta^p)dx dt\\
\le& C_3\int_\Omega |\omega^+|^{q+p}|(|d\eta|^p+\eta^p)dx.
\end{aligned} \tag{4.7}$$

Lemma 4.1 [24] Let $0<s<p$ and $|v|\in L^p_{\text{loc}}(\Omega)$, $\sigma>1$. If there exists a constant C_1 such that

$$(\int_Q |v|^p dx)^{\frac{1}{p}} \le C_1|Q|^{\frac{(s-p)}{sp}}(\int_{2Q}|v|^s dx)^{\frac{1}{s}} \tag{4.8}$$

for all cubes Q with $2Q\subset\Omega$, then for all $r>0$, there exists a constant C_2 depending only on σ,n,p,r and C_1, such that

$$(\int_Q |v|^p dx)^{\frac{1}{p}} \le C_2|Q|^{\frac{(r-p)}{rp}}(\int_{\sigma Q}|v|^r dx)^{\frac{1}{r}} \tag{4.9}$$

for all cubes Q with $\sigma Q\subset\Omega$.

Now, we are ready to present and prove one of our main theorems, the weak reverse Hölder inequality as follows.

Theorem 4.3 Let ω be a solution to the non-homogeneous Dirac-harmonic equation (1.8) associated with $p>1$ in Ω, $\sigma>1$ be some constant, and $0<r,s<\infty$ be any constants. Then, there exists a constant C, independent of ω, such that

$$\|\omega\|_{s,B}\le C|B|^{\frac{(r-s)}{rs}}\|\omega\|_{r,\sigma B} \tag{4.10}$$

for all cubes or balls B with $\sigma B\subset\Omega$.

Proof We prove this theorem in the following two steps. Step (i) is for the case $1<p<n$ and Step (ii) is for the case $p\ge n$.

Step (i) Assume that $1 < p < n$. We first prove that (4.10) holds for any constants r, s with $r > 0, 0 < s < \frac{np}{n-p}$. Let $\max\left\{1, \frac{p}{s}, \frac{n}{s(n-1)}\right\} < \alpha < \frac{np}{s(n-p)}$, and $q = \frac{\alpha ns}{(n+\alpha s)}$. Then

$$q = \frac{\alpha s}{n+\alpha s} \cdot n < n, \quad q - p = \frac{\alpha s(n-p) - np}{n+\alpha s} < 0, \quad q - 1 = \frac{\alpha s(n-1) - n}{n+\alpha s} > 0, \quad (4.11)$$

that is, $1 < q < p < n$, by (2.3), (1.19) and (3.4),

$$\begin{aligned}(\int_B |\omega|^{\frac{nq}{n-q}} dx)^{\frac{n-q}{nq}} &\le C_1(\int_B |D\omega|^q dx)^{\frac{1}{q}} \le C_1|B|^{\frac{1}{q}-\frac{1}{p}}(\int_B |D\omega|^p dx)^{\frac{1}{p}} \\ &\le C_2|B|^{\frac{(n-q)}{nq}-\frac{1}{p}}(\int_{\sigma B} |\omega|^p dx)^{\frac{1}{p}}.\end{aligned} \quad (4.12)$$

Also, $\frac{nq}{n-q} = \alpha s > p$, by Lemma 4.1 and (4.12), for any $r > 0$, we have

$$(\int_B |\omega|^{\frac{nq}{n-q}} dx)^{\frac{n-q}{nq}} \le C_3|B|^{\frac{(n-q)}{nq}-\frac{1}{r}}(\int_{\sigma B} |\omega|^r dx)^{\frac{1}{r}}. \quad (4.13)$$

Since $\frac{nq}{n-q} = \alpha s > s$, by(1.19), it follows that

$$(\int_B |\omega|^s dx)^{\frac{1}{s}} \le |B|^{\frac{1}{s}-\frac{1}{\alpha s}}(\int_B |\omega|^{\frac{nq}{n-q}} dx)^{\frac{n-q}{nq}}. \quad (4.14)$$

Combining (4.13) and (4.14) yields

$$(\int_B |\omega|^s dx)^{\frac{1}{s}} \le C_3|B|^{\frac{r-s}{rs}}(\int_{\sigma B} |\omega|^r dx)^{\frac{1}{r}}, \quad (4.15)$$

that is (4.10) holds for any constants r, s with $r > 0$ and $0 < s < \frac{np}{n-p}$. Next, we show that (4.10) holds for any $r > 0, s \ge \frac{np}{(n-p)}$. We only need to prove that

$$\|\omega^+\|_{s,B} \le C_4|B|^{\frac{(r-s)}{rs}}\|\omega^+\|_{r,\sigma B}. \quad (4.16)$$

The proof for ω^- is similar. The proof follows Theorem 3.34 in [25] with ω^+ to replace u^+ and $D\omega^+$ in replace of du^+. Also by using (4.6) and the Moser iteration technique to get (4.16).

Choose $\alpha > \max\{1, \frac{n}{s(n-1)}\}$ and $q = \frac{\alpha ns}{(n+\alpha s)}$. Then, $\frac{nq}{n-q} = \alpha s > s$ and $1 < q < n$. Let $r_m = \lambda + (1-\lambda)2^{-m}$, $m = 0, 1, \ldots; 0 < \lambda < 1$, then $r_0 = 1$ and $r_m \to \lambda$ as $m \to \infty$. Let $\eta_m \in C_0^\infty(r_m B)$ be a nonnegative function such that $\eta_m = 1$ in $r_m B \setminus r_{m+1}B$ and that $\eta_m \le C_0|d\eta_m| \le C_0 b^m|B|$ over each ball, for some constant C_0 and $1 < b \le \frac{nq}{n-q}$. Let $t_m \ge 0$ be a sequence to be determined later and $\omega_m = \omega^+|\omega^+|^{\frac{t_m}{q}}\eta_m$. Then by product rule (3.3) and exactly the same calculationas in [10], we get

$$\begin{aligned}(\int_{r_m B} |D\omega_m|^q dx)^{\frac{1}{q}} &\le (1 + 2\frac{t_m}{q})(\int_{r_m B} |D\omega^+|^q|\omega^+|^{t_m}\eta_m^q dx)^{\frac{1}{q}} + 2(\int_{r_m B} |\omega^+|^{q+t_m}|d\eta_m|^q dx)^{\frac{1}{q}} \\ &\le [C_5(1 + 2\frac{t_m}{q}) + 2](\int_{r_m B} |\omega^+|^{q+t_m}(|d\eta_m|^q + |\eta_m|^q)dx)^{\frac{1}{q}} \\ &\le C_6(q + t_m)(\int_{r_m B} |\omega^+|^{q+t_m}|d\eta_m|^q dx)^{\frac{1}{q}}. \end{aligned} \quad (4.17)$$

Since the remaining derivation is exactly the same as in [10], we obtain

$$(\int_B |\omega^+|^s dx)^{\frac{1}{s}} \le C_7 |B|^{\frac{r-s}{rs}} (\int_{\sigma B} |\omega^+|^r dx)^{\frac{1}{r}}. \tag{4.18}$$

Similarly, let $\omega_m = -\omega^- |\omega^-|^{\frac{tm}{q}} \eta_m$ with the same method, we can get

$$(\int_B |\omega^-|^s dx)^{\frac{1}{s}} \le C_7 |B|^{\frac{r-s}{rs}} (\int_{\sigma B} |\omega^-|^r dx)^{\frac{1}{r}}. \tag{4.19}$$

Thus, by Minkowski inequality, (4.18) and (4.19), we have

$$\begin{aligned} &\|\omega\|_{s,B} = \|\omega^+ + \omega^-\|_{s,B} \le \|\omega^+\|_{s,B} + \|\omega^-\|_{s,B} \\ \le& C_8 |B|^{\frac{r-s}{rs}} (\|\omega^+\|_{r,\sigma B} + \|\omega^-\|_{r,\sigma B} \le C_8 |B|^{\frac{r-s}{rs}} (\|\omega\|_{r,\sigma B} + \|\omega\|_{r,\sigma B}) \\ \le& C_9 |B|^{\frac{r-s}{rs}} \|\omega\|_{r,\sigma B}, \end{aligned} \tag{4.20}$$

that is, for any $r > 0, s \ge \frac{np}{(n-p)}$.

$$\|\omega\|_{s,B} \le C_9 |B|^{\frac{r-s}{rs}} \|\omega\|_{r,\sigma B}. \tag{4.21}$$

Step (ii) Assume that $p \ge n$. We prove that (4.10) holds for any two positive constants r, s . Let $\max\{1, \frac{p}{s}, \frac{n}{s(n-1)}\} < \alpha$, and $q = \frac{\alpha ns}{(n+\alpha s)}$. we find that (4.11)–(4.14) still hold for such a choice of α. Thus, inequality (4.15) or (4.10) follows immediately for any constants $r, s > 0$. We have completed the proof of Theorem 4.3.

In Theorem 4.3, considering that the norms $\|\omega\|_{s,Q}$ and $\|\omega - \omega_Q\|_{s,Q}$ are comparable, we have the following version of the weak reverse Hölder inequality for differential forms satisfies the non-homogeneous Dirac-harmonic equation (1.8)

$$\|\omega - \omega_Q\|_{s,Q} \le C |Q|^{\frac{r-s}{rs}} \|\omega - \omega_Q\|_{r,\sigma Q}. \tag{4.22}$$

5 Applications

We know the weight functions were widely used in analysis and PDE, so as applications of the Poincaré inequality and weak reverse Hölder inequality established in previous sections, we prove the Poincaré inequality with the $A(\varphi_1(x), \varphi_2(x), \tau, \Omega)$-weights. The $A(\varphi_1(x), \varphi_2(x), \tau, \Omega)$-weight was introduced by Wen in [26].

Definition 5.1 Let $\varphi_1(x)$ and $\varphi_2(x)$ be Young functions, a pair of weights $(w_1(x), w_2(x))$ satisfies the $A(\varphi_1(x), \varphi_2(x), \tau, \Omega)$-condition, write $(w_1(x), w_2(x)) \in A(\varphi_1(x), \varphi_2(x), \tau, \Omega)$, if there exists a constant C, such that

$$\sup_{Q \subset \Omega} \|w_1(x)\|_{\varphi_1(x),Q} \|w_2(x)^{-1}\|^{\tau}_{\varphi_2(x),Q} \le C < \infty, \tag{5.1}$$

where the normalized Luxemburg norm for any function $w(x)$ on a cube or ball Q with Young functions $\varphi : [0, \infty) \to [0, \infty)$ is defined by

$$\|w\|_{\varphi,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q \varphi(\frac{|w(x)|}{\lambda}) dx \le 1\}. \tag{5.2}$$

Remark (i) If $\varphi(t)=t^p$, then $\|w(x)\|_{\varphi,Q}=(\frac{1}{|Q|}\int_Q |w|^p dx)^{\frac{1}{p}}$ and the Luxemburg norm reduce to the L^p-norm. Given a Young function φ, let $\overline{\varphi}$ denote its associate function: the Young function with the property that $t\le \varphi^{-1}(t)\overline{\varphi}^{-1}(t)\le 2t$, $t>0$. If $\varphi(t)=t^p$, $p>1$, and $\frac{1}{p}+\frac{1}{q}=1$, then $\overline{\varphi}(t)=t^q$, and if $\varphi(t)=t^p\log(e+t)^{\alpha}$, then $\overline{\varphi}(t)\approx t^q\log(e+t)^{\frac{-\alpha q}{p}}$.

(ii) From [26], we know that $A(\varphi_1(x),\varphi_2(x),\tau,\Omega)$-weight is the generation of some existing versions of the weight. For example, if $\varphi_1(x)=x$, $\varphi_2(x)=x^{\frac{1}{r-1}}$, $\tau=1$, and $w_1=w_2$ in above definition, then $A(\varphi_1(x),\varphi_2(x),\tau,\Omega)$-weight reduces to the $A_r(\Omega)$-weight, and if $\varphi_1(x)=x^{\lambda}$,$\varphi_2(x)=x^{\frac{\lambda}{r-1}}$,$\tau=1$, then $A(\varphi_1(x),\varphi_2(x),\tau,\Omega)$-weight reduces to the $A_{r,\lambda}(\Omega)$-weight, see more details for [26].

Lemma 5.1 If φ is Young function, then for all functions f and g and any cube Q,

$$\frac{1}{|Q|}\int_Q |fg|dx\le 2\parallel f\parallel_{\varphi,Q}\| g\parallel_{\overline{\varphi},Q}. \tag{5.3}$$

Theorem 5.1 Let $\omega\in L^s_{\rm loc}(\Omega,\wedge^l)$ be a solution to equation (1.8) in a bounded domain $\Omega, l=0,1,\ldots,n-1$, $D\omega\in L^s_{\rm loc}(\Omega,\wedge)$. and let $\sigma>1$ be a constant. If $\varphi_1(x)$ and $\varphi_2(x)$ are Young functions with $1\in L^{\overline{\varphi}_1(x)}(\Omega)\cap L^{\overline{\varphi}_2(x)}(\Omega)$, and the pair of weights $(w_1(x),w_2(x))\in A(\varphi_1(x),\varphi_2(x),\tau,\Omega)$. Then there exists a constant C, independent of ω, such that

$$\|\omega-\omega_B\|_{s,B,w_1^{\alpha}}\le C|B|^{1+\frac{1}{n}}\|D\omega\|_{s,\sigma B,w_2^{\tau\alpha}} \tag{5.4}$$

for all balls B with $\sigma B\subset\Omega$ for some $\sigma>1$ and α is constant with $0<\alpha<1$.

Proof Choose $t=\frac{s}{1-\alpha}$, since $\frac{1}{s}=\frac{1}{t}+\frac{(t-s)}{ts}$ and $t>s$, using the Hölder inequality (1.18), we have

$$\begin{aligned}\|\omega-\omega_B\|_{s,B,w_1^{\alpha}}&=(\int_B(|\omega-\omega_B|w_1^{\frac{\alpha}{s}})^s dx)^{\frac{1}{s}}\le(\int_B|\omega-\omega_B|^t dx)^{\frac{1}{t}}(\int_B w_1^{\frac{\alpha t}{t-s}}dx)^{\frac{t-s}{ts}}\\&=\|\omega-\omega_B\|_{t,B}(\int_B w_1 dx)^{\frac{\alpha}{s}}=\|\omega-\omega_B\|_{t,B}\|w_1\|^{\frac{\alpha}{s}}_{1,B}.\end{aligned} \tag{5.5}$$

Taking $m=\frac{s}{1+\alpha\tau}$, then $m<s<t$. Using (4.22) and (2.7), we have

$$\|\omega-\omega_B\|_{t,B}\le C_2|B|^{\frac{(m-t)}{mt}}\|\omega-\omega_B\|_{m,\sigma B}\le C_3|B|^{1+\frac{1}{n}}|B|^{\frac{(m-t)}{mt}}\|D\omega\|_{m,\sigma B}, \tag{5.6}$$

where $\sigma>1$. Substituting (5.6) in (5.5), we have

$$(\int_B(|\omega-\omega_B|w_1^{\frac{\alpha}{s}})^s dx)^{\frac{1}{s}}\le C_3|B|^{1+\frac{1}{n}}|B|^{\frac{(m-t)}{mt}}\|D\omega\|_{m,\sigma B}\|w_1\|^{\frac{\alpha}{s}}_{1,B}. \tag{5.7}$$

Using (1.18) again yields

$$\begin{aligned}\|D\omega\|_{m,\sigma B}&=(\int_{\sigma B}|D\omega|^m dx)^{\frac{1}{m}}=(\int_{\sigma B}(|D\omega|w_2^{\frac{\tau\alpha}{s}}w_2^{\frac{-\tau\alpha}{s}})^m dx)^{\frac{1}{m}}\\&\le(\int_{\sigma B}(|D\omega|^s w_2^{\tau\alpha}dx)^{\frac{1}{s}}(\int_{\sigma B}(\frac{1}{w_2})^{\frac{\tau\alpha m}{s-m}}dx)^{\frac{s-m}{ms}}\\&=(\int_{\sigma B}(|D\omega|^s w_2^{\tau\alpha}dx)^{\frac{1}{s}}\|\frac{1}{w_2}\|^{\frac{\tau\alpha}{s}}_{1,\sigma B}\end{aligned} \tag{5.8}$$

for all balls B with $\sigma B \subset \Omega$. Substituting (5.8) into (5.7), we find that

$$\|\omega-\omega_B\|_{s,B,w_1^\alpha} \le C_3|B|^{1+\frac{1}{n}}|B|^{\frac{(m-t)}{mt}}\|w_1\|_{1,B}^{\frac{\alpha}{s}}\|\frac{1}{w_2}\|_{1,\sigma B}^{\frac{\tau\alpha}{s}}(\int_{\sigma B}(|D\omega|^s w_2^{\tau\alpha}dx)^{\frac{1}{s}}. \tag{5.9}$$

Since $(w_1(x), w_2(x)) \in A(\varphi_1(x), \varphi_2(x), \tau, \Omega)$,

$$\begin{aligned} &\|w_1\|_{1,B}^{\frac{\alpha}{s}}\|w_2^{-1}\|_{1,\sigma B}^{\frac{\alpha\tau}{s}} = (\|w_1\|_{1,B}\|w_2^{-1}\|_{1,\sigma B}^{\tau})^{\frac{\alpha}{s}} \\ \le&(C_2|B|^{(1+\tau)}(\frac{1}{|\sigma B|}\int_{\sigma B}|w_1(x)|dx)(\frac{1}{|\sigma B|}\int_{\sigma B}|\frac{1}{w_2(x)}|dx)^{\tau})^{\frac{\alpha}{s}} \\ \le&C_3|B|^{\frac{\alpha}{s}(1+\tau)}\|w_1(x)\|_{\varphi_1(x),\sigma B}^{\frac{\alpha}{s}}\|1\|_{\overline{\varphi_1}(x),\sigma B}^{\frac{\alpha}{s}}\|w_2^{-1}(x)\|_{\varphi_2(x),\sigma B}^{\frac{\alpha\tau}{s}}\|1\|_{\overline{\varphi_2}(x),\sigma B}^{\frac{\alpha\tau}{s}} \\ \le&C_4|B|^{\frac{\alpha}{s}(1+\tau)}, \end{aligned} \tag{5.10}$$

noting that

$$\frac{\alpha}{s}(1+\tau)+\frac{1}{t}-\frac{1}{m}=0. \tag{5.11}$$

Combining (5.9)–(5.11), we conclude that $\|\omega-\omega_B\|_{s,B,w_1^\alpha} \le C|B|^{1+\frac{1}{n}}\|D\omega\|_{s,\sigma B,w_2^{\tau\alpha}}$ for all balls B with $\sigma B \subset \Omega$. This ends the proof of Theorem 5.1.

If $\varphi_1(x)=x$,$\varphi_2(x)=x^{\frac{1}{r-1}}$,$\tau=1$, and $w_1=w_2$ in above definition, then $A(\varphi_1(x),\varphi_2(x),\tau,\Omega)$-weight reduces to the $A_r(\Omega)$-weight, then we have the following corollary.

Corollary 5.1 Let $\omega \in L^s_{\text{loc}}(\Omega, \wedge^l)$ be a solution to equation(1.8) in a bounded domain $\Omega, l=0,1,\dots,n-1$, $D\omega \in L^s_{\text{loc}}(\Omega,\wedge)$, and let $\sigma>1$ be a constant. If the weight $w(x) \in A_r(\Omega)$. Then there exists a constant C, independent of ω, such that

$$\|\omega-\omega_B\|_{s,B,w^\alpha} \le C|B|^{1+\frac{1}{n}}\|D\omega\|_{s,\sigma B,w^\alpha} \tag{5.12}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma>1$ and α is constant with $0<\alpha<1$.

As an applications of Corollary 5.1, we can prove the global result in $L^s(\mu)$-averaging domain.

Definition 5.2 [27] A proper subdomain $\Omega \subset \mathbb{R}^n$ is called an $L^s(\mu)$-averaging domain, $s \ge 1$, if $\mu(\Omega)<\infty$ and there exists a constant C such that

$$(\frac{1}{\mu(B_0)}\int_\Omega |u-u_{B_0}|^s d\mu)^{\frac{1}{s}} \le C\sup_{2B\subset\Omega}(\frac{1}{\mu(B)}\int_B|u-u_B|^s d\mu)^{\frac{1}{s}} \tag{5.13}$$

for some ball $B_0 \subset \Omega$ and all $u \in L^s_{\text{loc}}(\Omega;\wedge^l)$. Here the measure μ is defined by $d\mu=w(x)dx$, where $w(x)$ is a weight and $w(x)>0$ a.e., and the supremum is over all balls $B \subset \Omega$.

Theorem 5.2 Let $\omega \in L^s_{\text{loc}}(\Omega,\wedge^l)$ be a solution to equation (1.8) in a bounded domain $\Omega, l=0,1,\dots,n-1$, $D\omega \in L^s_{\text{loc}}(\Omega,\wedge)$. If $1<r<\infty$ and $w \in A_r(\Omega)$ with $w(x) \ge \varepsilon > 0$, then, there exists a constant C, independent of ω and $D\omega$, such that

$$(\frac{1}{\mu(\Omega)}\int_\Omega|\omega-\omega_{B_0}|^s d\mu)^{\frac{1}{s}} \le C(\int_\Omega|D\omega|^s d\mu)^{\frac{1}{s}} \tag{5.14}$$

for any $L^s(\mu)$-averaging domain Ω and some ball B_0 with $2B_0 \subset \Omega$. Here $d\mu=w(x)^\alpha dx$.

Proof For any ball $B \subset \Omega$, $\mu(B) = \int_B w(x)^\alpha dx \geq \int_B \varepsilon^\alpha dx = \varepsilon^\alpha |B|$, so that $\frac{1}{\mu(B)} \leq \frac{C_1}{|B|}$, where $C_1 = \frac{1}{\varepsilon^\alpha}$. By Corollary 5.1 and noticing that $1 + \frac{1}{n} - \frac{1}{s} > 0$, we obtain

$$\begin{aligned} &(\frac{1}{\mu(B)}\int_B |\omega - \omega_B|^s d\mu)^{\frac{1}{s}} = (\mu_B)^{\frac{-1}{s}}(\int_B |\omega - \omega_B|^s d\mu)^{\frac{1}{s}} \qquad (5.15) \\ \leq\ & C_2|B|^{\frac{-1}{s}}(\int_B |\omega - \omega_B|^s d\mu)^{\frac{1}{s}} \leq C_2|B|^{\frac{-1}{s}} C_3 |B|^{1+\frac{1}{n}}(\int_B |D\omega|^s d\mu)^{\frac{1}{s}} = C_4(\int_B |D\omega|^s d\mu)^{\frac{1}{s}}. \end{aligned}$$

Thus, by (5.15) and the definition of $L^s(\mu)$-averaging domain, we deduce that

$$\begin{aligned} &(\frac{1}{\mu(\Omega)}\int_\Omega |\omega - \omega_{B_0}|^s d\mu)^{\frac{1}{s}} \leq (\frac{1}{\mu(B_0)}\int_\Omega |\omega - \omega_{B_0}|^s d\mu)^{\frac{1}{s}} \\ \leq & C_5 \sup_{2B\subset\Omega}(\frac{1}{\mu(B)}\int_B |\omega - \omega_B|^s d\mu)^{\frac{1}{s}} \leq C_5 \sup_{2B\subset\Omega}(C_4(\int_{\sigma B} |D\omega|^s d\mu)^{\frac{1}{s}}) \qquad (5.16) \\ \leq & C_6 \sup_{2B\subset\Omega}(\int_\Omega |D\omega|^s d\mu)^{\frac{1}{s}}) = C_6(\int_\Omega |D\omega|^s d\mu)^{\frac{1}{s}}). \end{aligned}$$

We complete the proof of Theorem 5.2.

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微分形式中的非齐次Dirac-调和方程解的若干不等式

戴志敏 [1,2] , 刘三阳 [1]

(1. 西安电子科技大学 数学与统计学院, 陕西 西安 710126)

(2. 西安工业大学 理学院, 陕西 西安 710021)

摘要: 本文研究了与微分形式中一类非齐次的Dirac-调和方程解相关的不等式问题. 利用非齐次的Dirac-调和方程的条件和Dirac-调和算子D的运算法则, 获得了Poincaré不等式, Caccioppoli不等式和弱逆Hölder不等式. 作为相关不等式的应用, 证明了Poincaré不等式赋特殊权和在$L^s(\mu)$平均域上的形式. 本文的研究将齐次Dirac-调和方程解的相关不等式推广到了对应该方程非齐次的情形.

关键词: 非齐次Dirac-调和方程; 微分形式; 范数不等式; 权; $L^s(\mu)$-平均域

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