Vol. 39 (2019)
No. 6

数 学 杂 志
J. of Math. (PRC)

THE CONCAVITY OF p-RÉNYI ENTROPY POWER FOR THE WEIGHTED DOUBLY NONLINEAR DIFFUSION EQUATIONS ON WEIGHTED RIEMANNIAN MANIFOLDS

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Abstract: In this paper, we study the concavity of the entropy power on Riemannian manifolds. By using the nonlinear Bochner formula and Bakry-Émery method, we prove p-Rényi entropy power is concave for positive solutions to the weighted doubly nonlinear diffusion equations on the weighted closed Riemannian manifolds with $CD(-K,m)$ condition for some $K \geq 0$ and $m \geq n$, which generalizes the cases of porous medium equation and nonnegative Ricci curvature.

Keywords: concavity; p-Rényi entropy power; weighted doubly nonlinear diffusion equations; m-Bakry-Émery Ricci curvature

2010 MR Subject Classification: 58J35; 35K92; 35B40; 35K55.

Document code: A **Article ID:** 0255-7797(2019)06-0791-10

1 Introduction

Let $(M, g, d\mu)$ be a weighted Riemannian manifold equipped with a reference measure $d\mu = e^{-f}\mathrm{vol}$, there is a canonical differential operator associated to the triple given by the weighted Laplacian

$$\Delta_f u \doteq \mathrm{div}_f(\nabla u),$$

where the notation $\doteq$ means definition, $\mathrm{div}_f \doteq e^f \mathrm{div}(e^{-f}\cdot)$ is the weighted divergence operator. The weighted Laplacian Δ_f is a self-adjoint operator on the Hilbert space $L^2(M, d\mu)$. Recall the carré du champ operator of the weighted Laplacian defined by Bakry-Émery [1]

$$\Gamma(u,v) \doteq \frac{1}{2}[\Delta_f(uv) - u\Delta_f v - v\Delta_f u],$$

and the iterated carré du champ operator

$$\Gamma_2(u,v) \doteq \frac{1}{2}[\Delta_f\Gamma(u,v) - \Gamma(u,\Delta_f v) - \Gamma(v,\Delta_f u)].$$

* **Received date:** 2019-08-29 **Accepted date:** 2019-10-9

Foundation item: Supported by National Natural Science Foundation of China (11701347).

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A weighted Riemannian manifold $(M, g, d\mu)$ is said to satisfy the curvature dimensional condition $CD(-K, m)$ if for every function u,

$$\Gamma_2(u,u) \geq \frac{1}{m}(\Delta_f u)^2 - K\Gamma(u,u).$$

In fact, by the enhanced Bochner formula (see P.383 in Villani's book [2])

$$\begin{aligned}&\frac{1}{2}\Delta_f|\nabla u|^2 - \nabla u\cdot\nabla\Delta_f u = |\nabla\nabla u|^2 + \mathrm{Ric}_f(\nabla u,\nabla u)\\ =&\frac{1}{m}(\Delta_f u)^2 + \mathrm{Ric}_f^m(\nabla u,\nabla u) + |\nabla\nabla u - \frac{\Delta u}{n}g|^2 + (\frac{1}{n}-\frac{1}{m})(\Delta u + \frac{n}{m-n}\nabla f\cdot\nabla u)^2,\end{aligned}$$

the curvature dimensional condition $CD(-K, m)$ for $m \in (-\infty, 0) \cup [n, \infty)$ is equivalent to m-dimensional Bakry-Émery Ricci curvature Ric_f^m is bounded below by $-K$, i.e.,

$$\mathrm{Ric}_f^m \doteqdot \mathrm{Ric}_f - \frac{1}{m-n}\nabla f\otimes\nabla f \doteqdot \mathrm{Ric} + \nabla^2 f - \frac{1}{m-n}\nabla f\otimes\nabla f \geq -Kg. \tag{1.1}$$

In this paper, we consider the weighted doubly nonlinear diffusion equation

$$\partial_t u = \Delta_{p,f} u^\gamma, \tag{1.2}$$

where $\Delta_{p,f}\cdot \doteqdot e^f \mathrm{div}(e^{-f}|\nabla\cdot|^{p-2}\nabla\cdot)$ is the weighted p-Laplacian, which appears in non-Newtonian fluids, turbulent flows in porous media, glaciology and other contexts. We are mainly concerned on the concavity of the Rényi type entropy power with respecting to the equation (1.2) on the weighted Riemannian manifolds.

In the classic paper [3], Shannon defined the entropy power $N(X)$ for random vector X on $\mathbb{R}^n$,

$$N(X) \doteqdot e^{\frac{2}{n}H(X)}, \quad H(X) \doteqdot -\int_{\mathbb{R}^n} u\log u dx,$$

where u is the probability density of X and $H(X)$ is the information entropy. Moreover, he proved the entropy power inequality (EPI) for independent random vectors X and Y,

$$N(X+Y) \geq N(X) + N(Y). \tag{1.3}$$

Later, Costa [4] proved EPI when one of a random vector is Gaussian and established an equivalence between EPI and the concavity of Shannon entropy power $N(u)$ when u satisfies heat equation $\partial_t u = \Delta u$, that is,

$$\frac{d^2}{dt^2}N(u(t)) \leq 0.$$

Moreover, Villani [2] gave a short proof by using of the Bakry-Émery identities. In a recent paper, Savaré and Toscani [5] proved the concavity of the Rényi entropy power $N_\gamma(u)$ along the porous medium equation $\partial_t u = \Delta u^\gamma$ on $\mathbb{R}^n$, where $N_\gamma(u)$ is given by

$$N_\gamma(u) \doteqdot (\int_{\mathbb{R}^n} u^\gamma(x)dx)^\alpha, \quad \alpha = \frac{2}{n(\gamma-1)} + 1. \tag{1.4}$$

In [6], the first author and coauthors studied the the concavities of p-Shannon entropy power for positive solutions to p-heat equations on Riemannian manifolds with nonnegative Ricci curvature.

On the other hand, Li and Li [11] proved the concavity of Shannon entropy power for the heat equation $\partial_t u = \Delta_f u$ and the concavity of Rényi entropy power for the porous medium equation $\partial_t u = \Delta_f u^\gamma$ with $\gamma > 1$ on weighted Riemannian manifolds with $CD(0,m)$ or $CD(-K,m)$ condition, and also on $(0,m)$ or $(-K,m)$ super Ricci flows.

Inspired by works mentioned above, we study p-Rényi type entropy power for the weighted doubly nonlinear diffusion equation (1.2) on the weighted Riemannian manifolds and prove its concavity under the curvature dimensional condition $CD(0,m)$ and $CD(-K,m)$.

Let us define the weighted p-Rényi entropy $R_p(u)$ and p-Rényi entropy power $N_p(u)$ on the weighted Riemannian manifolds,

$$R_p(u) \doteqdot -\frac{1}{b}\log\int_M u^{b+1}d\mu \doteqdot -\frac{1}{b}\log(E_p(u)), \tag{1.5}$$

$$N_p(u) \doteqdot \exp(-\sigma b R_p(u)) \doteqdot (E_p(u))^\sigma, \tag{1.6}$$

where b and σ are constants

$$b = \gamma - \frac{1}{p-1}, \quad \sigma = -(p-1) - \frac{p}{mb},$$

and $E_p(u)$ is given by

$$E_p(u) \doteqdot \int_M u^{b+1}d\mu. \tag{1.7}$$

When $p = 2$ and $f=$ const., (1.6) reduces to (1.4).

Theorem 1.1 Let $(M,g,d\mu)$ be a weighted closed Riemannian manifold. If u is a positive solution to (1.2) on $(M,g,d\mu)$, then we have

$$\begin{aligned}\frac{d^2}{dt^2}N_p(u) &= \sigma(1-\sigma)b^2(E_p(u))^{\sigma-1}\int_M(\Delta_{p,f}v + I_p(u))^2u^{b+1}d\mu\\ &+ pb\sigma(E_p(u))^{\sigma-1}\int_M[||\nabla v|^{p-2}\nabla^2 v - \frac{1}{n}(\Delta_p v)a_{ij}|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v,\nabla v)]u^{b+1}d\mu\\ &+ pb\sigma(E_p(u))^{\sigma-1}\int_M[(\frac{1}{n}-\frac{1}{m})(\Delta_p v + \frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2]u^{b+1}d\mu,\end{aligned} \tag{1.8}$$

where a_{ij} is the inverse of $A^{ij} = g^{ij} + (p-2)\frac{\nabla^i v\nabla^j v}{|\nabla v|^2}$ and $I_p(u)$ is the weighted Fisher information in (3.3).

When $m \geq n$, $b \geq -\frac{1}{m}$, $b \neq 0$ and M satisfies the curvature-dimension condition $CD(0,m)$, the weighted p-Rényi entropy power is concave, that is,

$$\frac{d^2}{dt^2}N_p(u) \leq 0.$$

By modified the definition of p-Rényi entropy power, we can also obtain the concavity under the curvature dimensional condition $CD(-K,m)$.

Theorem 1.2 If u is a positive solution to (1.2) on the weighted closed Riemannian manifolds with curvature dimensional condition $CD(-K,m)$ for $K\geq 0$, define p-K-Rényi entropy power $N_{p,K}$ such that

$$\frac{d}{dt}N_{p,K}(u)=\exp\{-\frac{p\gamma}{b+1}\kappa t\}\frac{d}{dt}N_p(u), \tag{1.9}$$

where

$$\kappa \doteqdot \frac{\int_M |\nabla v|^{2p-2}u^{b+1}d\mu}{\int_M |\nabla v|^p u d\mu}K, \tag{1.10}$$

we obtain

$$\begin{aligned}\frac{d^2}{dt^2}N_{p,K}(u)&=\sigma(1-\sigma)b^2(E_p(u))^{\sigma-1}\exp\{-\frac{p\gamma}{b+1}\kappa t\}\int_M |\Delta_{p,f}v+I_p(u)|^2u^{b+1}d\mu\\&+pb\sigma(E_p(u))^{\sigma-1}\exp\{-\frac{p\gamma}{b+1}\kappa t\}\int_M ||\nabla v|^{p-2}\nabla^2 v-\frac{1}{n}(\Delta_p v)a_{ij}|_A^2u^{b+1}d\mu\\&+pb\sigma(E_p(u))^{\sigma-1}\exp\{-\frac{p\gamma}{b+1}\kappa t\}\int_M |\nabla v|^{2p-4}(\mathrm{Ric}_f^m+Kg)(\nabla v,\nabla v)u^{b+1}d\mu\\&+pb\sigma(E_p(u))^{\sigma-1}\exp\{-\frac{p\gamma}{b+1}\kappa t\}\int_M [(\frac{1}{n}-\frac{1}{m})(\Delta_p v+\frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2]u^{b+1}d\mu.\end{aligned} \tag{1.11}$$

When $m\geq n$,$b\geq -\frac{1}{m}$, $b\neq 0$ and $\mathrm{Ric}_f^m\geq -Kg$, we have

$$\frac{d^2}{dt^2}N_{p,K}(u)\leq 0. \tag{1.12}$$

This paper is organized as follows. In section 2, we will prove some identities as lemmas, and they are important tools for proving theorems. In section 3, we will finish the proofs of the main results.

2 Some Useful Lemmas

According to [7], the pressure function is given by

$$v=\frac{\gamma}{b}u^b,\quad b=\gamma-\frac{1}{p-1}, \tag{2.1}$$

then v satisfies the following equation

$$\partial_t v=bv\Delta_{p,f}v+|\nabla v|^p. \tag{2.2}$$

Let us recall the weighted entropy functional

$$E_p(u)=\int_M u^{b+1}d\mu, \tag{2.3}$$

and the weighted p-Rényi entropy

$$R_p(u) = -\frac{1}{b}\log\int_M u^{b+1}d\mu = -\frac{1}{b}\log(E_p(u)), \tag{2.4}$$

where $d\mu = e^{-f}d\mathrm{Vol}$. Hence, the weighted p-Rényi entropy power is given by

$$N_p(u) = \exp(-b\sigma R_p(u)) = (E_p(u))^\sigma, \quad -\frac{1}{\sigma} = \frac{mb}{(p-1)mb+p}. \tag{2.5}$$

In order to obtain our results, we need to prove following lemmas. Motivated by the works of [8], we apply analogous methods in this paper.

Lemma 2.1 For any two functions h and g, let us define the linearized operator of the weighted p-Laplacian at point v,

$$L_{p,f}(\psi) \doteqdot e^f \mathrm{div}(e^{-f}|\nabla v|^{p-2}A(\nabla\psi)),$$

where $A^{ij} = g^{ij} + (p-2)\frac{\nabla^i v\nabla^j v}{|\nabla v|^2}$, then we get

$$L_{p,f}(hg) = g(L_{p,f}h) + (L_{p,f}g)h + 2|\nabla v|^{p-2}\langle\nabla h, \nabla g\rangle_A, \tag{2.6}$$

$$\int_M (L_{p,f}h)gd\mu = \int_M h(L_{p,f}g)d\mu = -\int_M |\nabla v|^{p-2}\langle\nabla h, \nabla g\rangle_A d\mu, \tag{2.7}$$

where $\langle\nabla h, \nabla g\rangle_A = A(\nabla h)\cdot\nabla g = \nabla h\cdot A(\nabla g)$.

Proof The proof is a direct result by the definition of $L_{p,f}$ and integration by parts.

Lemma 2.2 We have the modified weighted p-Bochner formula

$$L_{p,f}|\nabla v|^p = p\Gamma_{2,A}(v) + p|\nabla v|^{p-2}\langle\nabla v, \nabla\Delta_{p,f}v\rangle, \tag{2.8}$$

where

$$\Gamma_{2,A}(v) \doteqdot |\nabla v|^{2p-4}(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v, \nabla v)) \tag{2.9}$$

$$\begin{aligned}&= \frac{1}{m}(\Delta_{p,f}v)^2 + ||\nabla v|^{p-2}\nabla\nabla v - \frac{1}{n}(\Delta_p v)a_{ij}|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v, \nabla v)\\ &\quad + (\frac{1}{n} - \frac{1}{m})(\Delta_p v + \frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2.\end{aligned} \tag{2.10}$$

Proof According to [9], we have

$$L_{p,f}|\nabla v|^p = p|\nabla v|^{2p-4}\left(|\nabla\nabla v|_A^2 + \mathrm{Ric}_f(\nabla v, \nabla v)\right) + p|\nabla v|^{p-2}\langle\nabla v, \nabla\Delta_{p,f}v\rangle,$$

where $|\nabla\nabla v|_A^2 = \frac{(p-2)^2}{4}\frac{\left|\nabla v\cdot\nabla|\nabla v|^2\right|^2}{|\nabla v|^4} + \frac{p-2}{2}\frac{\left|\nabla|\nabla v|^2\right|^2}{|\nabla v|^2} + |\nabla\nabla v|^2$.

Using the definition of m-Bakry- Emery Ricci curvature in (1.1), we find

$$\begin{aligned}\Gamma_{2,A}(v) &= |\nabla v|^{2p-4}|\nabla\nabla v|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v, \nabla v) + \frac{1}{m-n}(|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2\\ &= ||\nabla v|^{p-2}\nabla\nabla v - \frac{1}{n}(\Delta_p v)a_{ij}|_A^2 + \frac{1}{n}(\Delta_p v)^2 + |\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v, \nabla v)\\ &\quad + \frac{1}{m-n}(|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2\\ &= \frac{1}{m}(\Delta_{p,f}v)^2 + ||\nabla v|^{p-2}\nabla\nabla v - \frac{1}{n}\Delta_p v a_{ij}|_A^2 + |\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v, \nabla v)\\ &\quad + (\frac{1}{n} - \frac{1}{m})(\Delta_p v + \frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2,\end{aligned}$$

which completes the proof of (2.8). In particular, when $m > n$ or $m < 0$, we have

$$\Gamma_{2,A}(v) \geq \frac{1}{m}(\Delta_{p,f}v)^2 + |\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v, \nabla v). \tag{2.11}$$

Lemma 2.3 If u,v satisfies equations (1.2) and (2.2), we obtain that

$$\partial_t(\Delta_{p,f}v) = L_{p,f}(\partial_t v), \tag{2.12}$$

$$\partial_t(uv) = \frac{b}{p-1}vL_{p,f}(uv). \tag{2.13}$$

Proof By the definitions of A and $L_{p,f}$, then

$$\begin{aligned}\partial_t(\Delta_{p,f}v) =&\partial_t[e^f\mathrm{div}(e^{-f}|\nabla v|^{p-2}\nabla v)]\\ =&e^f\mathrm{div}(e^{-f}[(p-2)|\nabla v|^{p-4}(\nabla v, \nabla\partial_t v)\nabla v + |\nabla v|^{p-2}\nabla\partial_t v])\\ =&e^f\mathrm{div}(e^{-f}|\nabla v|^{p-2}A(\nabla\partial_t v))\\ =&L_{p,f}(\partial_t v).\end{aligned}$$

By the definition of v in (2.1), we have $u\nabla v = bv\nabla u$. Hence

$$A(\nabla(uv)) = (1+\frac{1}{b})A(u\nabla v) = (p-1)(1+\frac{1}{b})u\nabla v$$

and

$$\begin{aligned}L_{p,f}(uv) =&e^f\mathrm{div}\left(e^{-f}|\nabla v|^{p-2}A(\nabla(uv))\right)\\ =&(p-1)(1+\frac{1}{b})e^f\mathrm{div}\left(e^{-f}u|\nabla v|^{p-2}\nabla v\right)\\ =&(p-1)(1+\frac{1}{b})\partial_t u.\end{aligned} \tag{2.14}$$

A direct calculation shows that

$$\partial_t(uv) = (1+b)v\partial_t u. \tag{2.15}$$

Combining (2.14) with (2.15), we can show (2.13).

Lemma 2.4 Let u be a positive solution to (1.2) and v satisfies (2.2), then

$$\frac{d}{dt}E_p(u) = b\int_M(\Delta_{p,f}v)u^{b+1}d\mu = -\frac{b(b+1)}{\gamma}\int_M|\nabla v|^p ud\mu = -\frac{b(b+1)}{\gamma}\int_M u^{-\frac{1}{p-1}}|\nabla u^\gamma|^p ud\mu \tag{2.16}$$

and

$$\frac{d^2}{dt^2}E_p(u) = pb\int_M[\Gamma_{2,A}(v) + b(\Delta_{p,f}v)^2]u^{b+1}d\mu, \tag{2.17}$$

where $\Gamma_{2,A}$ is defined in (2.10).

Proof By integrating by parts, we can obtain

$$\begin{aligned}\frac{d}{dt}E_p(u)=&(b+1)\int_M u^b\partial_t u d\mu=\frac{b(b+1)}{\gamma}\int_M ve^f\operatorname{div}(e^{-f}u|\nabla v|^{p-2}\nabla v)d\mu\\ =&-\frac{b(b+1)}{\gamma}\int_M|\nabla v|^p ud\mu=b\int_M(\Delta_{p,f}v)u^{b+1}d\mu,\end{aligned}$$

where

$$u\nabla v=bv\nabla u,\quad buv=\gamma u^{b+1}.$$

Using identities (2.2), (2.8), (2.12) and (2.13), we find that

$$\begin{aligned}&\frac{d}{dt}\int_M\Delta_{p,f}v(uv)d\mu=\int_M[\partial_t(\Delta_{p,f}v)(uv)+(\Delta_{p,f}v)\partial_t(uv)]d\mu\\ =&\int_M[L_{p,f}(\partial_t v)(uv)+\frac{b}{p-1}L_{p,f}(uv)(v\Delta_{p,f}v)]d\mu\\ =&\int_M[bL_{p,f}(v\Delta_{p,f})(uv)+L_{p,f}(|\nabla v|^p)(uv)+\frac{b}{p-1}L_{p,f}(v\Delta_{p,f}v)(uv)]d\mu\\ =&p\int_M[\Gamma_{2,A}(v)+|\nabla v|^{p-2}\langle\nabla v,\nabla\Delta_{p,f}v\rangle]uvd\mu+\frac{pb}{p-1}\int_M L_{p,f}(v\Delta_{p,f}v)(uv)d\mu.\end{aligned}$$

According to the properties of the linearized operator $L_{p,f}$ (2.6) and (2.7), we have

$$\begin{aligned}\int_M L_{p,f}(v\Delta_{p,f}v)(uv)d\mu=&\int_M(L_{p,f}(v))(\Delta_{p,f}v)(uv)d\mu\\ &+\int_M L_{p,f}(\Delta_{p,f}v)(uv^2)d\mu+2\int_M|\nabla v|^{p-2}\langle\nabla v,\nabla\Delta_{p,f}v\rangle_A(uv)d\mu\\ =&(p-1)\int_M(\Delta_{p,f}v)^2(uv)d\mu-\int_M|\nabla v|^{p-2}\langle\nabla(uv^2),\nabla\Delta_{p,f}v\rangle_A d\mu\\ &+2\int_M|\nabla v|^{p-2}\langle\nabla v,\nabla\Delta_{p,f}v\rangle_A(uv)d\mu\\ =&(p-1)\int_M(\Delta_{p,f}v)^2(uv)d\mu-\frac{1}{b}\int_M|\nabla v|^{p-2}\langle\nabla v,\nabla\Delta_{p,f}v\rangle_A(uv)d\mu,\end{aligned}$$

where we use the fact

$$\nabla(uv^2)=(\frac{1}{b}+2)uv\nabla v.$$

Hence, we get

$$\begin{aligned}b\frac{d}{dt}\int_M(\Delta_{p,f}v)u^{b+1}d\mu=&\frac{b^2}{\gamma}\frac{d}{dt}\int_M(\Delta_{p,f}v)(uv)d\mu\\ =&\frac{b^2}{\gamma}\int_M[p\Gamma_{2,A}(v)+pb(\Delta_{p,f}v)^2]uvd\mu\\ =&pb\int_M[\Gamma_{2,A}(v)+b(\Delta_{p,f}v)^2]u^{b+1}d\mu,\end{aligned}$$

which is (2.17).

3 Proofs of Theorems

Proof of Theorem 1.1 Let σ be a constant, $N_p(u) = (E_p(u))^\sigma$, a direct computation implies

$$\frac{d}{dt}N_p(u) = \sigma(E_p(u))^{\sigma-1}\frac{d}{dt}E_p(u).$$

By using of identities (2.16), (2.10) and (2.17), we get that

$$\begin{aligned}
&\frac{d^2}{dt^2}N_p(u)\\
=&\sigma(E_p(u))^{\sigma-2}[E_p(u)\frac{d^2}{dt^2}(E_p(u)) + (\sigma-1)(\frac{d}{dt}(E_p(u)))^2]\\
=&\sigma(E_p(u))^{\sigma-1}\int_M pb[\Gamma_{2,A}(v) + b(\Delta_{p,f}v)^2]u^{b+1}d\mu + \sigma(\sigma-1)(E_p(u))^{\sigma-2}(\int_M (b\Delta_{p,f}v)u^{b+1}d\mu)^2\\
=&pb\sigma(E_p(u))^{\sigma-1}\int_M [(\frac{1}{m}+b)(\Delta_{p,f}v)^2 + ||\nabla v|^{p-2}\nabla^2 v - \frac{1}{n}(\Delta_p v)a_{ij}|_A^2\\
&+|\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v,\nabla v)]u^{b+1}d\mu\\
&+pb\sigma(E_p(u))^{\sigma-1}\int_M [(\frac{1}{n}-\frac{1}{m})(\Delta_p v + \frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2]u^{b+1}d\mu\\
&+\sigma(\sigma-1)(E_p(u))^{\sigma-2}(\int_M (b\Delta_{p,f}v)u^{b+1}d\mu)^2.
\end{aligned}\tag{3.1}$$

In particular, when $b > 0$, $\sigma < 0$ or $-\frac{1}{m} \le b < 0$, $\sigma > 1$ and $\mathrm{Ric}_f^m \ge 0$, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\frac{d^2}{dt^2}N_p(u) \le&\sigma(E_b(u))^{\sigma-1}\int_M p(\frac{1}{mb}+1)(b\Delta_{p,f}v)^2u^{b+1}d\mu\\
&+(\sigma-1)(E_p(u))^{\sigma-2}(\int_M (b\Delta_{p,f}v)u^{b+1}d\mu)^2\\
\le&\sigma(E_p(u))^{\sigma-1}[p(\frac{1}{mb}+1)+(\sigma-1)]\int_M (b\Delta_{p,f}v)^2u^{b+1}d\mu.
\end{aligned}$$

We can choose a proper constant σ such that

$$p(\frac{1}{mb}+1)+(\sigma-1) = 0. \tag{3.2}$$

Thus $\frac{d^2}{dt^2}N_p(u) \le 0$, that is the weighted p-Rényi entropy power is concave along the weighted doubly nonlinear diffusion equations (1.2) on $(M, g, d\mu)$ with $CD(0, m)$ condition.

In fact, we can obtain a precise form of the second order derivative of the weighted p-Rényi entropy $N_p(u)$. Let $I_p(u)$ be the weighted Fisher information with respect to $R_p(u)$,

$$I_p(u) \doteqdot \frac{d}{dt}R_p(u) = -\frac{1}{b}\frac{1}{E_p(u)}\frac{d}{dt}E_p(u) = -\frac{1}{\int_M u^{b+1}d\mu}\int_M (\Delta_{p,f}v)u^{b+1}d\mu. \tag{3.3}$$

Applying identities (3.1) and (3.2), we get

$$\frac{d^2}{dt^2}N_p(u)=\sigma(1-\sigma)b^2(E_p(u))^{\sigma-1}[\int_M(\Delta_{p,f}v)^2u^{b+1}d\mu-(E_p(u))^{-1}(\int_M(\Delta_{p,f}v)u^{b+1}d\mu)^2]$$
$$+pb\sigma(E_p(u))^{\sigma-1}\int_M[||\nabla v|^{p-2}\nabla^2v-\frac{1}{n}(\Delta_pv)a_{ij}|_A^2+|\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v,\nabla v)]u^{b+1}d\mu$$
$$+pb\sigma(E_p(u))^{\sigma-1}\int_M[(\frac{1}{n}-\frac{1}{m})(\Delta_pv+\frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2]u^{b+1}d\mu.$$

Combining this with (3.3), one has

$$\frac{d^2}{dt^2}N_p(u)=\sigma(1-\sigma)b^2(E_p(u))^{\sigma-1}\int_M|\Delta_{p,f}v+I_p(u)|^2u^{b+1}d\mu$$
$$+pb\sigma(E_p(u))^{\sigma-1}\int_M[||\nabla v|^{p-2}\nabla^2v-\frac{1}{n}(\Delta_pv)a_{ij}|_A^2+|\nabla v|^{2p-4}\mathrm{Ric}_f^m(\nabla v,\nabla v)]u^{b+1}d\mu$$
$$+pb\sigma(E_p(u))^{\sigma-1}\int_M[(\frac{1}{n}-\frac{1}{m})(\Delta_pv+\frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2]u^{b+1}d\mu.$$

Thus, we obtain an explicit expression about $\frac{d^2}{dt^2}N_p(u)$. Moreover, if $-\frac{1}{m}\le b<0$, $\sigma>1$ and $b>0$, $\sigma<0$, then $\frac{d^2}{dt^2}N_p(u)\le 0$.

Proof of Theorem 1.2 Motivated by [10], formula (1.8) can be rewritten as

$$\frac{d^2}{dt^2}N_p(u)=\sigma(1-\sigma)b^2(E_p(u))^{\sigma-1}\int_M|\Delta_{p,f}v+I_p(u)|^2u^{b+1}d\mu \tag{3.4}$$
$$+pb\sigma(E_p(u))^{\sigma-1}\int_M[||\nabla v|^{p-2}\nabla^2v-\frac{1}{n}(\Delta_pv)a_{ij}|_A^2+|\nabla v|^{2p-4}(\mathrm{Ric}_f^m+Kg)(\nabla v,\nabla v)]u^{b+1}d\mu$$
$$+pb\sigma(E_p(u))^{\sigma-1}\int_M[(\frac{1}{n}-\frac{1}{m})(\Delta_pv+\frac{n}{m-n}|\nabla v|^{p-2}\nabla v\cdot\nabla f)^2]u^{b+1}d\mu+\frac{p\gamma}{b+1}\kappa\frac{d}{dt}N_p(u),$$

where we use the definition of κ in (1.10) and variaitonal formula in (2.16), that is

$$\frac{d}{dt}N_p(u)=\sigma(E_p(u))^{\sigma-1}\frac{d}{dt}E_p(u)=-\frac{\sigma b(b+1)}{\gamma}(E_p(u))^{\sigma-1}\int_M|\nabla v|^pud\mu.$$

Defining a functional $N_{p,K}$ such that [11]

$$\frac{d}{dt}N_{p,K}(u)\doteqdot\exp\{-\frac{p\gamma}{b+1}\kappa t\}\frac{d}{dt}N_p(u).$$

A direct computation yields

$$\frac{d^2}{dt^2}N_{p,K}(u)=\exp\{-\frac{p\gamma}{b+1}\kappa t\}(\frac{d^2}{dt^2}N_p(u)-\frac{p\gamma}{b+1}\kappa\frac{d}{dt}N_p(u)). \tag{3.5}$$

Hence, when $(M,g,d\mu)$ satisfies the curvature dimensional condition $CD(-K,m)$, i.e., $\mathrm{Ric}_f^m\ge -Kg$ with $K>0$, $m\ge n$, by formula (3.5), we obtain the concavity of $N_{p,K}$, that is $\frac{d^2}{dt^2}N_{p,K}(u)\le 0$. Furthermore, we obtain an explicit variational formula (1.11), which finishes the proof of Theorem 1.2.

Acknowledgement The first author would like to thank Professor Xiang-Dong Li for his interest and illuminating discussion.

References

[1] Bakry D, Gentil I, Ledoux M. Analysis and geometry of Markov diffusion operators[M]. Springer, 2014.

[2] Villani C. A short proof of the concavity of entropy power[J]. IEEE Trans. Inform. Theory, 2000, 46: 1695–1696.

[3] Shannon C E. A mathematical theory of communication[J]. Bell System Tech. J., 1948, 27: 379–423; 623–656.

[4] Costa M. A new entropy power inequality[J]. IEEE Trans. Inform. Theory, 1985, 31(6): 751–760.

[5] Savaré G, Toscani G. The concavity of Rényi entropy power[J]. IEEE Trans. Inform. Theory, 2014, 60(5): 2687–2693.

[6] Wang Y Z, Zhang X X. The concavity of p-entropy power and applications in functional inequalities[J]. Nonlinear Anal., 2019, 179: 1–14.

[7] Wang Y Z, Chen W Y. Gradient estimates and entropy formula for doubly nonlinear diffusion equations on Riemannian manifolds[J]. Math. Meth. Appl. Sci., 2014, 37: 2772–2781.

[8] Wang Y Z, Wang Y M. The concavity of Rényi entropy power for the parabolic p-laplace equations and applications[J]. Manuscripta Mathematica, https://doi.org/10.1007/s00229-019-01118-9.

[9] Wang Y Z, Yang J, Chen W Y. Gradient estimates and entropy formulae for weighted p-heat equations on smooth metric measure spaces[J]. Acta Math. Sci., 2013, 33B(4): 963–974.

[10] Kotschwar B, Ni L. Local gradient estimate for p-harmonic functions, $1/H$ flow and an entropy formula[J]. Ann. Sci. Éc. Norm. Supér., 2007, 42(4): 1–36.

[11] Li S, Li X D. Hamilton differential Harnack inequality and W-entropy for Witten Laplacian on Riemannian manifolds[J]. J. Funct. Anal., 2018, 274(11): 3263–3290.

加权黎曼流形上加权双重扩散方程的p-Rényi熵幂的凹性

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摘要: 本文研究了黎曼流形上熵幂的凹性问题. 利用非线性Bochner公式和Bakry-Émery的方法, 证明了当满足曲率维数条件$CD(-K,m)(K\geq 0, m\geq n)$ 时, 对于加权双重扩散方程的正解, 相关的p-Rényi熵幂是凹的, 推广了之前多孔介质方程以及Ricci 曲率非负情形下的结果.

关键词: 凹性; p-Rényi 熵幂; 加权双重扩散方程; m-Bakry-Émery Ricci 曲率

MR(2010)**主题分类号:** 58J35, 35K92, 35B40, 35K55 **中图分类号:** O175.29