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SOME MONOTONIC INEQUALITIES FOR GENERAL L_p-MIXED PROJECTION BODIES AND GENERAL L_p-MIXED CENTROID BODIES

SHI Wei[1], WANG Wei-dong[1,2]

(1.Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, China)

(2.Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China)

Abstract: In this paper, we study the problem of the monotonicity on general L_p-mixed projection bodies and general L_p-mixed centroid bodies. By using analytic inequality theory, some monotonic inequalities of quermassintegrals and dual quermassintegrals for general L_p-mixed projection bodies and general L_p-mixed centroid bodies are obtained, which generalizes the problem of the monotonicity for the form of volume on L_p-projection bodies and L_p-centroid bodies.

Keywords: general L_p-mixed projection body; general L_p-mixed centroid body; monotonic inequality

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1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. The set of convex bodies containing the origin in their interiors, we write $\mathcal{K}^n_o$. $\mathcal{S}^n_o$ denotes the set of star bodies (about the origin) in $\mathbb{R}^n$. The unit ball in $\mathbb{R}^n$ and its surface will be denoted by B and S^{n-1}, respectively. $V(K)$ denotes the n-dimensional volume of a body K and write $V(B) = \omega_n$.

For $K \in \mathcal{K}^n$, its support function, h_K=$h(K, \cdot)$: $\mathbb{R}^n \to \mathbb{R}$, is defined by (see [1, 2])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The conception of L_p-centroid body was introduced by Lutwak and Zhang (see [3]). For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real $p \geq 1$, the L_p-centroid body,

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Biography: Shi Wei (1992–), male, born at Longnan, Gansu, master, major in convex geometry and analysis.

$\Gamma_p K$, of K is an origin-symmetric convex body which support function is defined by

$$\begin{aligned} h^p_{\Gamma_p K}(u) &= \frac{1}{c_{n,p}V(K)}\int_K \mid u\cdot x\mid^p dx \\ &= \frac{1}{c_{n,p}(n+p)V(K)}\int_{S^{n-1}} \mid u\cdot v\mid^p \rho_K^{n+p}(v)dS(v) \end{aligned} \tag{1.1}$$

for any $u\in S^{n-1}$, where the integration is in connection with Lebesgue measure on S^{n-1} and

$$c_{n,p}=\frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}. \tag{1.2}$$

In 2000, Lutwak, Yang and Zhang in [4] put forward the notion of L_p-projection body. For $K\in\mathcal{K}^n_o$ and real $p\geq 1$, the L_p-projection body, $\Pi_p K$, of K is an origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u)=\alpha_{n,p}\int_{S^{n-1}} \mid u\cdot v\mid^p dS_p(K,v) \tag{1.3}$$

for all $u\in S^{n-1}$. Here $S_p(K,\cdot)$ is the L_p-surface area measure of K,

$$\alpha_{n,p}=\frac{1}{n\omega_n c_{n-2,p}}, \tag{1.4}$$

and $c_{n-2,p}$ satisfies (1.2). At the same time, they (see [4]) proved the L_p-Petty projection inequality and L_p-Busemann-Petty centroid inequality. For the L_p-centroid bodies and L_p-projection bodies, some scholars made a series of researches and gained several results (see [5–15]). In particular, Wang, Lu and Leng in [12] established the following monotonic inequalities.

Theorem 1.A Let $K,L\in\mathcal{K}^n_o$ and $p\geq 1$. If for any $Q\in\mathcal{K}^n_o$, $V_p(K,Q)\leq V_p(L,Q)$, then $V(\Pi_p K)\leq V(\Pi_p L)$ with equality for $p=1$ if and only if $\Pi_p K$ and $\Pi_p L$ are translates, for $p>1$ if and only if $\Pi_p K=\Pi_p L$, here $V_p(M,N)$ denotes the L_p-mixed volume of $M,N\in\mathcal{K}^n_o$.

Theorem 1.B Let $K,L\in\mathcal{K}^n_o$ and $p\geq 1$. If for any $Q\in\mathcal{K}^n_o$, $V_p(K,Q)\leq V_p(L,Q)$, then $V(\Pi^*_p K)\geq V(\Pi^*_p L)$ with equality if and only if $\Pi_p K=\Pi_p L$, here $\Pi^*_p M$ denotes the polar of $\Pi_p M$.

Theorem 1.C Let $K,L\in\mathcal{S}^n_o$ and $p\geq 1$. If for any $Q\in\mathcal{S}^n_o$, $\widetilde{V}_{-p}(K,Q)\leq\widetilde{V}_{-p}(L,Q)$, then

$$\frac{V(\Gamma_p K)^{-\frac{p}{n}}}{V(K)}\geq\frac{V(\Gamma_p L)^{-\frac{p}{n}}}{V(L)}$$

with equality for $p=1$ if and only if $\Gamma_p K$ and $\Gamma_p L$ are translates, for $p>1$ if and only if $\Gamma_p K=\Gamma_p L$, here $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M,N\in\mathcal{S}^n_o$.

Theorem 1.D Let $K,L\in\mathcal{S}^n_o$ and $p\geq 1$. If for any $Q\in\mathcal{S}^n_o$, $\widetilde{V}_{-p}(K,Q)\leq\widetilde{V}_{-p}(L,Q)$, then

$$\frac{V(\Gamma^*_p K)^{\frac{p}{n}}}{V(K)}\geq\frac{V(\Gamma^*_p L)^{\frac{p}{n}}}{V(L)}$$

with equality if and only if $\Gamma_p K=\Gamma_p L$, here $\Gamma^*_p M$ denotes the polar of $\Gamma_p M$.

Ludwig (see [16]) introduced a function $\varphi_\tau:\mathbb{R}\to[0,+\infty)$ given by $\varphi_\tau(t)=|t|+\tau t$ for $\tau\in[-1,1]$. Using this function, Ludwig [16] defined general L_p-projection bodies as follows: for $K\in\mathcal{K}_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, general L_p-projection body, $\Pi_p^\tau K\in\mathcal{K}_o^n$, of K with support function by

$$h^p_{\Pi_p^\tau K}(u)=\alpha_{n,p}(\tau)\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p dS_p(K,v), \tag{1.5}$$

where

$$\alpha_{n,p}(\tau)=\frac{2\alpha_{n,p}}{(1+\tau)^p+(1-\tau)^p}, \tag{1.6}$$

and $\alpha_{n,p}$ satisfies (1.4). For every $\tau\in[-1,1]$, the normalization is chosen such that $\Pi_p^\tau B=B$. Clearly, if $\tau=0$, then $\Pi_p^\tau K=\Pi_p^0 K=\Pi_p K$.

Regarding general L_p-projection bodies, Wang and Wan (see [17]) studied the Shephard type problem. Wang and Feng (see [18]) established general L_p-Petty affine projection inequality. Wang and Wang (see [19]) gave the extremums of quermassintegrals and dual quermassintegrals for general L_p-projection bodies and their polar.

Subsequently, according to definition (1.1) of L_p-centroid bodies, Feng, Wang and Lu (see [20]) imported the notion of general L_p-centroid bodies. For $K\in\mathcal{S}_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, the general L_p-centroid body, $\Gamma_p^\tau K\in\mathcal{K}_o^n$, of K which support function is defined by

$$\begin{aligned}h^p_{\Gamma_p^\tau K}(u)&=\frac{1}{c_{n,p}(\tau)V(K)}\int_K\varphi_\tau(u\cdot x)^p dx\\&=\frac{\gamma_{n,p}(\tau)}{V(K)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho_K^{n+p}(v)dS(v),\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}\gamma_{n,p}(\tau)&=\frac{1}{(n+p)c_{n,p}(\tau)},\\ c_{n,p}(\tau)&=\frac{1}{2}c_{n,p}[(1+\tau)^p+(1-\tau)^p],\end{aligned} \tag{1.8}$$

and $c_{n,p}$ satisfies (1.2). The normalization is chosen such that $\Gamma_p^\tau B=B$ for every $\tau\in[-1,1]$, and $\Gamma_p^0 K=\Gamma_p K$.

From the definition of L_p-projection body, Wang and Leng (see [21]) gave the following concept of L_p-mixed projection body. For each $K\in\mathcal{K}_o^n$, real $p\geq 1$ and $i=0,1,\cdots,n-1$, the L_p-mixed projection body, $\Pi_{p,i}K$, of K is an origin-symmetric convex body, which support function is defined by

$$h^p_{\Pi_{p,i}K}(u)=\alpha_{n,p}\int_{S^{n-1}}|\,u\cdot v\,|^p\,dS_{p,i}(K,v) \tag{1.9}$$

for any $u\in S^{n-1}$, the positive Borel measure $S_{p,i}(K,\cdot)$ on S^{n-1} is absolutely continuous with respect to $S_i(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)}=h^{1-p}(K,\cdot). \tag{1.10}$$

By definitions (1.9) and (1.3), we easily know that $\Pi_{p,0}K = \Pi_p K$.

Just as the definition of the L_p-mixed projection body, L_p-mixed centroid body was introduced by Wang, Leng and Lu (see [11]). If $K \subset \mathbb{R}^n$ is compact star-shaped about the origin, $p \geq 1$, $i \in \mathbb{R}$, then the L_p-mixed centroid body, $\Gamma_{p,i}K$, of K is the origin-symmetric convex body whose support function is given by

$$h^p_{\Gamma_{p,i}K}(u) = \frac{1}{(n+p)c_{n,p}V(K)} \int_{S^{n-1}} \mid u \cdot v \mid^p \rho_K^{n+p-i}(v) dS(v)$$

for every $u \in S^{n-1}$. From this and definition (1.1), we have $\Gamma_{p,0}K = \Gamma_p K$.

For the studies of L_p-mixed projection bodies and L_p-mixed centroid bodies, Wang and Leng [21] demonstrated the Petty projection inequality for L_p-mixed projection bodies, and then, Wang, Leng and Lu [11] obtained the forms of quermassintegrals and dual quermass-integrals of Theorem 1.A and Theorem 1.B. Moreover, on one hand, associated with the definition of quermassintegrals, Wang and Leng [10] extended Theorem 1.C to the quermassintegrals; on the other hand, Wang, Lu and Leng [13] gave the dual quermassintegrals form for Theorem 1.D. In regard to the studies of the L_p-mixed projection bodies and the L_p-mixed centroid bodies, see also [22–25].

According to definitions (1.5) and (1.9), general L_p-mixed projection bodies were raised by Wan and Wang [26]. For $K \in \mathcal{K}^n_o$, $p \geq 1$, $\tau \in [-1, 1]$ and $i = 0, 1, \cdots, n-1$, the general L_p-mixed projection bodies, $\Pi^\tau_{p,i}K \in \mathcal{K}^n_o$, whose support function is provided by

$$h^p_{\Pi^\tau_{p,i}K}(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_{p,i}(K, v). \tag{1.11}$$

From (1.11) and (1.5), if $i = 0$, then $\Pi^\tau_{p,0}K = \Pi^\tau_p K$.

Similar to Wan and Wang's idea, we define general L_p-mixed centroid bodies as follows: for $K \in \mathcal{S}^n_o$, $p \geq 1$, $\tau \in [-1, 1]$ and i is any real, the general L_p-mixed centroid body, $\Gamma^\tau_{p,i}K \in \mathcal{K}^n_o$, of K is presented by

$$h^p_{\Gamma^\tau_{p,i}K}(u) = \frac{\gamma_{n,p}(\tau)}{V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho_K^{n+p-i}(v) dS(v), \tag{1.12}$$

where $\gamma_{n,p}(\tau)$ is the same as (1.8). Especially, if $i = 0$, by definitions (1.12) and (1.7), we easily get $\Gamma^\tau_{p,0}K = \Gamma^\tau_p K$.

In this article, we first extend Theorem 1.A and Theorem 1.B to quermassintegrals and dual quermassintegrals, which can be stated as follows.

Theorem 1.1 Let $K, L \in \mathcal{K}^n_o$, $p \geq 1$, $\tau \in [-1, 1]$ and $i, j = 0, 1, \cdots, n-1$. If for any $Q \in \mathcal{K}^n_o$,

$$W_{p,j}(K, Q) \leq W_{p,j}(L, Q),$$

then

$$W_i(\Pi^\tau_{p,j}K) \leq W_i(\Pi^\tau_{p,j}L). \tag{1.13}$$

Equality holds in (1.13) for $p=1$ if and only if $\Pi^{\tau}_{p,j}K$ and $\Pi^{\tau}_{p,j}L$ are translates; for $p>1$ if and only if $\Pi^{\tau}_{p,j}K=\Pi^{\tau}_{p,j}L$. Here $W_{p,j}(M,N)$ $(j=0,1,\cdots,n-1)$ denotes the L_p-mixed quermassintegrals of $M,N\in\mathcal{K}^n_o$.

Theorem 1.2 Let $K,L\in\mathcal{K}^n_o$, $p\geq 1$, $\tau\in[-1,1]$, real $i\neq n$ and $j=0,1,\cdots,n-1$. If for any $Q\in\mathcal{K}^n_o$,

$$W_{p,j}(K,Q)\leq W_{p,j}(L,Q),$$

then for $i<n$,

$$\widetilde{W}_i(\Pi^{\tau,*}_{p,j}K)\geq\widetilde{W}_i(\Pi^{\tau,*}_{p,j}L);\tag{1.14}$$

for $n<i<n+p$ or $i>n+p$,

$$\widetilde{W}_i(\Pi^{\tau,*}_{p,j}K)\leq\widetilde{W}_i(\Pi^{\tau,*}_{p,j}L).\tag{1.15}$$

Equality holds in (1.14) or (1.15) for $i\neq n+p$ if and only if $\Pi^{\tau}_{p,j}K=\Pi^{\tau}_{p,j}L$. For $i=n+p$, inequality (1.15) is identic.

Moreover, we establish the following inequalities of quermassintegrals and dual quermassintegrals for general L_p-mixed centroid bodies, which is regarded as a generalization of Theorem 1.C and Theorem 1.D.

Theorem 1.3 Let $K,L\in\mathcal{S}^n_o$, $p\geq 1$, $\tau\in[-1,1]$, real $i\neq n$ and $j=0,1,\cdots,n-1$. If for any $Q\in\mathcal{S}^n_o$,

$$\widetilde{W}_{-p,i}(K,Q)\leq\widetilde{W}_{-p,i}(L,Q),$$

then

$$\frac{W_j(\Gamma^{\tau}_{p,i}K)^{-\frac{p}{n-j}}}{V(K)}\geq\frac{W_j(\Gamma^{\tau}_{p,i}L)^{-\frac{p}{n-j}}}{V(L)}.\tag{1.16}$$

Equality holds in (1.16) for $p=1$ if and only if $\Gamma^{\tau}_{p,i}K$ and $\Gamma^{\tau}_{p,i}L$ are translates, for $p>1$ if and only if $\Gamma^{\tau}_{p,i}K=\Gamma^{\tau}_{p,i}L$. Here $\widetilde{W}_{-p,j}(M,N)$ $(j\neq n)$ denotes the L_p-dual mixed quermassintegrals of $M,N\in\mathcal{S}^n_o$.

Theorem 1.4 Let $K,L\in\mathcal{S}^n_o$, $p\geq 1$, $\tau\in[-1,1]$, real $i,j\neq n$. If for any $Q\in\mathcal{S}^n_o$,

$$\widetilde{W}_{-p,i}(K,Q)\leq\widetilde{W}_{-p,i}(L,Q),$$

then

$$\frac{\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}K)^{\frac{p}{n-j}}}{V(K)}\geq\frac{\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}L)^{\frac{p}{n-j}}}{V(L)}.\tag{1.17}$$

Equality holds in (1.17) for $j\neq n+p$ if and only if $\Gamma^{\tau}_{p,i}K=\Gamma^{\tau}_{p,i}L$. For $j=n+p$, inequality (1.17) is identic.

Obviously, taking $i=j=\tau=0$ in Theorems 1.1–1.4, then inequalities (1.13)–(1.17) reduce to Theorems 1.A–1.D, respectively.

This paper is organized as follows. In section 2, we provide some basic notions and results. Section 3 gives the proofs of Theorems 1.1–1.4.

2 Basic Notions

2.1 Radial Functions and Polar Bodies

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow [0,+\infty)$, is defined by (see [2])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is positive and continuous, then K is viewed as a star body (about the origin). Two star bodies K and L will be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is a nonempty subset of $\mathbb{R}^n$, then the polar set K^* of K is defined by (see [1, 2])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

If $K \in \mathcal{K}_o^n$, it follows that $(K^*)^* = K$ and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.1}$$

2.2 L_p-Minkowski and L_p-Harmonic Radial Combinations

For $K, L \in \mathcal{K}_o^n$, real $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-Minkowski combination (also called the Firey L_p-combination), $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by (see [27])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p,$$

where the operation $\lambda \cdot K$ denotes Firey scalar multiplication. Obviously, Firey scalar multiplication and usual scalar multiplication are related by $\lambda \cdot K = \lambda^{\frac{1}{p}} K$.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$, of K and L is defined by (see [28])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}.$$

Here $\lambda \star K$ denotes L_p-harmonic radial scalar multiplication, and we can see $\lambda \star K = \lambda^{-\frac{1}{p}} K$. Note that for convex bodies, the L_p-harmonic radial combination was investigated by Firey (see [29]).

2.3 Quermassintegrals and L_p-Mixed Quermassintegrals

If $K \in \mathcal{K}^n$, the quermassintegrals $W_i(K)$ $(i = 0, 1, \cdots, n-1)$ of K are defined by (see [1, 2])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_i(K,u), \tag{2.2}$$

where $S_i(K,\cdot)$ $(i = 0, 1, \cdots, n-1)$ is the mixed surface area measure of $K \in \mathcal{K}^n$, $S_0(K,\cdot)$ is the surface area measure of K. In particular, we easily see that

$$W_0(K) = V(K). \tag{2.3}$$

In [30], Lutwak defined the L_p-mixed quermassintegrals and showed that for $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $i = 0, 1, \cdots, n-1$, the L_p-mixed quermassintegrals $W_{p,i}(K, L)$ has the following integral representation

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_{p,i}(K, u). \tag{2.4}$$

Here $S_{p,i}(K, \cdot)$ $(i = 0, 1, \cdots, n-1)$ satisfies (1.10). The case $i = 0$, $S_{p,0}(K, \cdot)$ is just the L_p-surface area measure $S_p(K, \cdot)$ of $K \in \mathcal{K}_o^n$.

From (2.2), (2.4) and (1.10), it follows immediately that for each $K \in \mathcal{K}_o^n$ and $p \geq 1$,

$$W_{p,i}(K, K) = W_i(K). \tag{2.5}$$

For the L_p-mixed quermassintegrals $W_{p,i}(K, L)$, Lutwak [30] established the following Minkowski inequality

Theorem 2.A If $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $i = 0, 1, \cdots, n-1$, then

$$W_{p,i}(K, L) \geq W_i(K)^{\frac{n-p-i}{n-i}} W_i(L)^{\frac{p}{n-i}} \tag{2.6}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

2.4 Dual Quermassintegrals and L_p-Dual Mixed Quermassintegrals

For $K \in \mathcal{S}_o^n$ and real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by (see [31])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{2.7}$$

Obviously,

$$\widetilde{W}_0(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u) = V(K). \tag{2.8}$$

In 2005, Wang and Leng [32] introduced the L_p-dual mixed quermassintegrals as follows: for $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K, L)$, of K and L are given by

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p-i}(u) \rho_L^{-p}(u) dS(u). \tag{2.9}$$

From formula (2.9) and definition (2.7), we get

$$\widetilde{W}_{-p,i}(K, K) = \widetilde{W}_i(K). \tag{2.10}$$

For the L_p-dual mixed quermassintegrals, Wang and Leng (see [32]) proved the following Minkowski inequality.

Theorem 2.B If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, real $i \neq n$, then for $i < n$ or $n < i < n + p$,

$$\widetilde{W}_{-p,i}(K, L) \geq \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(L)^{-\frac{p}{n-i}}; \tag{2.11}$$

for $i > n+p$,

$$\widetilde{W}_{-p,i}(K,L) \le \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(L)^{-\frac{p}{n-i}}. \tag{2.12}$$

Equality holds in each inequality if and only if K and L are dilates.

3 Proofs of Theorems

In this section, we prove Theorems 1.1–1.4. First, the following lemmas are necessary.

Lemma 3.1 If $K, L \in \mathcal{K}_o^n$, $p \ge 1$, $\tau \in [-1,1]$ and $i, j = 0, 1, \cdots, n-1$, then

$$W_{p,i}(K, \Pi_{p,j}^{\tau} L) = W_{p,j}(L, \Pi_{p,i}^{\tau} K). \tag{3.1}$$

Proof According to definitions (2.4) and (1.11), and using Fubini theorem, we get

$$\begin{aligned} W_{p,i}(K,\Pi_{p,j}^{\tau}L) &= \frac{1}{n}\int_{S^{n-1}} h(\Pi_{p,j}^{\tau}L,u)^p dS_{p,i}(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \alpha_{n,p}(\tau)\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p dS_{p,j}(L,v)dS_{p,i}(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \alpha_{n,p}(\tau)\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p dS_{p,i}(K,u)dS_{p,j}(L,v) \\ &= \frac{1}{n}\int_{S^{n-1}} h(\Pi_{p,i}^{\tau}K,v)^p dS_{p,j}(L,v) \\ &= W_{p,j}(L,\Pi_{p,i}^{\tau}K). \end{aligned}$$

Lemma 3.2 If $K \in \mathcal{K}_o^n$, $p \ge 1$, $\tau \in [-1,1]$, real $i \ne n$ and $j = 0, 1, \cdots, n-1$, then for any $M \in \mathcal{S}_o^n$,

$$W_{p,j}(K, \Gamma_{p,i}^{\tau} M) = \frac{2\omega_n}{V(M)} \widetilde{W}_{-p,i}(M, \Pi_{p,j}^{\tau,*} K). \tag{3.2}$$

Proof From definitions (2.4), (2.9) and (1.12), and using $nc_{n-2,p} = (n+p)c_{n,p}$, we have

$$\begin{aligned} W_{p,j}(K,\Gamma_{p,i}^{\tau}M) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Gamma_{p,i}^{\tau}M}(v)dS_{p,j}(K,v) \\ &= \frac{\gamma_{n,p}(\tau)}{nV(M)}\int_{S^{n-1}}\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p \rho_M^{n+p-i}(u)dS(u)dS_{p,j}(K,v) \\ &= \frac{2\omega_n}{nV(M)}\int_{S^{n-1}} \rho_M^{n+p-i}(u)\rho^{-p}_{\Pi_{p,j}^{\tau,*}K}(u)dS(u) \\ &= \frac{2\omega_n}{V(M)}\widetilde{W}_{-p,i}(M,\Pi_{p,j}^{\tau,*}K). \end{aligned}$$

Lemma 3.3 If $K, L \in \mathcal{S}_o^n$, $p \ge 1$, $\tau \in [-1,1]$ and reals $i, j \ne n$, then

$$\frac{\widetilde{W}_{-p,j}(K, \Gamma_{p,i}^{\tau,*} L)}{V(K)} = \frac{\widetilde{W}_{-p,i}(L, \Gamma_{p,j}^{\tau,*} K)}{V(L)}. \tag{3.3}$$

Proof Due to considerations (2.9), (1.12), (2.1) and Fubini theorem, we obtain

$$\begin{aligned}
&\frac{\widetilde{W}_{-p,j}(K,\Gamma_{p,i}^{\tau,*}L)}{V(K)}\\
=&\frac{1}{nV(K)}\int_{S^{n-1}}\rho_K^{n+p-j}(u)\rho_{\Gamma_{p,i}^{\tau,*}L}^{-p}(u)dS(u)\\
=&\frac{1}{nV(K)}\int_{S^{n-1}}\rho_K^{n+p-j}(u)h_{\Gamma_{p,i}^{\tau}L}^{p}(u)dS(u)\\
=&\frac{\gamma_{n,p}(\tau)}{nV(K)V(L)}\int_{S^{n-1}}\rho_K^{n+p-j}(u)\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho_L^{n+p-i}(v)dS(v)dS(u)\\
=&\frac{\gamma_{n,p}(\tau)}{nV(K)V(L)}\int_{S^{n-1}}\rho_L^{n+p-i}(v)\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho_K^{n+p-j}(u)dS(u)dS(v)\\
=&\frac{1}{nV(L)}\int_{S^{n-1}}\rho_L^{n+p-i}(v)h_{\Gamma_{p,j}^{\tau}K}^{p}(v)dS(v)\\
=&\frac{1}{nV(L)}\int_{S^{n-1}}\rho_L^{n+p-i}(v)\rho_{\Gamma_{p,j}^{\tau,*}K}^{-p}(v)dS(v)\\
=&\frac{\widetilde{W}_{-p,i}(L,\Gamma_{p,j}^{\tau,*}K)}{V(L)}.
\end{aligned}$$

Proof of Theorem 1.1 Since $K,L\in\mathcal{K}_o^n$, $p\ge 1$, $j=0,1,\cdots,n-1$, and for any $Q\in\mathcal{K}_o^n$,

$$W_{p,j}(K,Q)\le W_{p,j}(L,Q), \tag{3.4}$$

thus for any $M\in\mathcal{K}_o^n$, let $Q=\Pi_{p,i}^{\tau}M$, where $\tau\in[-1,1]$ and $i=0,1,\cdots,n-1$, then (3.4) gives

$$W_{p,j}(K,\Pi_{p,i}^{\tau}M)\le W_{p,j}(L,\Pi_{p,i}^{\tau}M). \tag{3.5}$$

By (3.1), we see that (3.5) can be written as the following inequality

$$W_{p,i}(M,\Pi_{p,j}^{\tau}K)\le W_{p,i}(M,\Pi_{p,j}^{\tau}L). \tag{3.6}$$

Taking $M=\Pi_{p,j}^{\tau}L$ in (3.6), and using (2.5) and inequality (2.6), we get

$$W_i(\Pi_{p,j}^{\tau}L)\ge W_{p,i}(\Pi_{p,j}^{\tau}L,\Pi_{p,j}^{\tau}K)\ge W_i(\Pi_{p,j}^{\tau}L)^{\frac{n-p-i}{n-i}}W_i(\Pi_{p,j}^{\tau}K)^{\frac{p}{n-i}},$$

namely,

$$W_i(\Pi_{p,j}^{\tau}L)^{\frac{p}{n-i}}\ge W_i(\Pi_{p,j}^{\tau}K)^{\frac{p}{n-i}}. \tag{3.7}$$

Notice that $0\le i<n$ and $p\ge 1$, then inequality (3.7) can be expressed by

$$W_i(\Pi_{p,j}^{\tau}K)\le W_i(\Pi_{p,j}^{\tau}L),$$

this is just inequality (1.13).

According to the equality conditions of inequality (2.6), we see that equality holds in inequality (1.13) for $p=1$ if and only if $\Pi_{p,j}^{\tau}K$ and $\Pi_{p,j}^{\tau}L$ are translates, for $p>1$ if and only if $\Pi_{p,j}^{\tau}K=\Pi_{p,j}^{\tau}L$.

Proof of Theorem 1.2 For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $j = 0, 1, \cdots, n-1$, and for any $Q \in \mathcal{K}_o^n$,

$$W_{p,j}(K, Q) \leq W_{p,j}(L, Q),$$

so, let $Q = \Gamma_{p,i}^{\tau} M$ for any $M \in \mathcal{S}_o^n$, where $\tau \in [-1, 1]$ and real $i \neq n, n+p$. We get

$$W_{p,j}(K, \Gamma_{p,i}^{\tau} M) \leq W_{p,j}(L, \Gamma_{p,i}^{\tau} M).$$

From (3.2), we know that

$$\widetilde{W}_{-p,i}(M, \Pi_{p,j}^{\tau,*} K) \leq \widetilde{W}_{-p,i}(M, \Pi_{p,j}^{\tau,*} L). \tag{3.8}$$

For $i < n$ or $n < i < n+p$, taking $M = \Pi_{p,j}^{\tau,*} L$ in inequality (3.8), and using (2.10) and inequality (2.11), we obtain that

$$\begin{aligned}\widetilde{W}_i(\Pi_{p,j}^{\tau,*} L) &\geq \widetilde{W}_{-p,i}(\Pi_{p,j}^{\tau,*} L, \Pi_{p,j}^{\tau,*} K) \\ &\geq \widetilde{W}_i(\Pi_{p,j}^{\tau,*} L)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(\Pi_{p,j}^{\tau,*} K)^{-\frac{p}{n-i}},\end{aligned}$$

that is

$$\widetilde{W}_i(\Pi_{p,j}^{\tau,*} K)^{-\frac{p}{n-i}} \leq \widetilde{W}_i(\Pi_{p,j}^{\tau,*} L)^{-\frac{p}{n-i}}. \tag{3.9}$$

Therefore, for $i < n$, inequality (3.9) has the following simple form

$$\widetilde{W}_i(\Pi_{p,j}^{\tau,*} K) \geq \widetilde{W}_i(\Pi_{p,j}^{\tau,*} L),$$

this yields inequality (1.14); for $n < i < n+p$, inequality (3.9) shows

$$\widetilde{W}_i(\Pi_{p,j}^{\tau,*} K) \leq \widetilde{W}_i(\Pi_{p,j}^{\tau,*} L),$$

i.e., inequality (1.15) is obtained.

Similarly, for $i > n+p$, taking $M = \Pi_{p,j}^{\tau,*} K$ in (3.8), and utilizing (2.10) and inequality (2.12), we easily obtain that

$$\begin{aligned}\widetilde{W}_i(\Pi_{p,j}^{\tau,*} K) &\leq \widetilde{W}_{-p,i}(\Pi_{p,j}^{\tau,*} K, \Pi_{p,j}^{\tau,*} L) \\ &\leq \widetilde{W}_i(\Pi_{p,j}^{\tau,*} K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(\Pi_{p,j}^{\tau,*} L)^{-\frac{p}{n-i}},\end{aligned}$$

namely,

$$\widetilde{W}_i(\Pi_{p,j}^{\tau,*} K)^{-\frac{p}{n-i}} \leq \widetilde{W}_i(\Pi_{p,j}^{\tau,*} L)^{-\frac{p}{n-i}},$$

notice that $i > n+p$, we get inequality (1.15).

According to equality conditions of inequalities (2.11) and (2.12), we know that for $i \neq n+p$, equality holds in (1.14) or (1.15) if and only if $\Pi_{p,j}^{\tau,*} K = \Pi_{p,j}^{\tau,*} L$, i.e., $\Pi_{p,j}^{\tau} K = \Pi_{p,j}^{\tau} L$. For $i = n+p$, by (3.8) and (2.9) we know that inequality (1.15) still holds.

Proof of Theorem 1.3 For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, real $i \neq n$ and any $Q \in \mathcal{S}_o^n$, since $\widetilde{W}_{-p,i}(K, Q) \leq \widetilde{W}_{-p,i}(L, Q)$, therefore, for any $M \in \mathcal{K}_o^n$, $\tau \in [-1, 1]$ and $j = 0, 1, \cdots, n-1$, let $Q = \Pi_{p,j}^{\tau,*} M$, we get

$$\widetilde{W}_{-p,i}(K, \Pi_{p,j}^{\tau,*} M) \leq \widetilde{W}_{-p,i}(L, \Pi_{p,j}^{\tau,*} M).$$

Together with (3.2), we obtain

$$V(K)W_{p,j}(M,\Gamma^{\tau}_{p,i}K)\le V(L)W_{p,j}(M,\Gamma^{\tau}_{p,i}L). \tag{3.10}$$

Taking $M=\Gamma^{\tau}_{p,i}L$ in inequality (3.10), and using (2.4) and inequality (2.6), we have

$$\begin{aligned}V(L)W_j(\Gamma^{\tau}_{p,i}L)&\ge V(K)W_{p,j}(\Gamma^{\tau}_{p,i}L,\Gamma^{\tau}_{p,i}K)\\&\ge V(K)W_j(\Gamma^{\tau}_{p,i}L)^{\frac{n-p-j}{n-j}}W_j(\Gamma^{\tau}_{p,i}K)^{\frac{p}{n-j}},\end{aligned}$$

namely,

$$\frac{W_j(\Gamma^{\tau}_{p,i}K)^{-\frac{p}{n-j}}}{V(K)}\ge\frac{W_j(\Gamma^{\tau}_{p,i}L)^{-\frac{p}{n-j}}}{V(L)},$$

this is just inequality (1.16).

According to the condition of equality in (2.6), we know that equality holds in inequality (1.16) for $p=1$ if and only if $\Gamma^{\tau}_{p,i}K$ and $\Gamma^{\tau}_{p,i}L$ are translates, for $p>1$ if and only if $\Gamma^{\tau}_{p,i}K=\Gamma^{\tau}_{p,i}L$.

Proof of Theorem 1.4 For $K,L\in\mathcal{S}^n_o$, $p\ge1$, real $i\ne n$ and any $Q\in\mathcal{S}^n_o$, because $\widetilde{W}_{-p,i}(K,Q)\le\widetilde{W}_{-p,i}(L,Q)$, thus let $Q=\Gamma^{\tau,*}_{p,j}M$ for any $M\in\mathcal{S}^n_o$, where $\tau\in[-1,1]$ and real $j\ne n$, then

$$\widetilde{W}_{-p,i}(K,\Gamma^{\tau,*}_{p,j}M)\le\widetilde{W}_{-p,i}(L,\Gamma^{\tau,*}_{p,j}M).$$

From (3.3), we get

$$V(K)\widetilde{W}_{-p,j}(M,\Gamma^{\tau,*}_{p,i}K)\le V(L)\widetilde{W}_{-p,j}(M,\Gamma^{\tau,*}_{p,i}L). \tag{3.11}$$

For $j<n$ or $n<j<n+p$, taking $M=\Gamma^{\tau,*}_{p,i}L$ in (3.11), and together with inequality (2.11), we have

$$\begin{aligned}V(L)\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}L)&\ge V(K)\widetilde{W}_{-p,j}(\Gamma^{\tau,*}_{p,i}L,\Gamma^{\tau,*}_{p,i}K)\\&\ge V(K)\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}L)^{\frac{n+p-j}{n-j}}\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}K)^{-\frac{p}{n-j}},\end{aligned}$$

i.e.,

$$\frac{\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}K)^{\frac{p}{n-j}}}{V(K)}\ge\frac{\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}L)^{\frac{p}{n-j}}}{V(L)}.$$

This is inequality (1.17).

For $j>n+p$, let $M=\Gamma^{\tau,*}_{p,i}K$ in (3.11), and together with inequality (2.12), we have

$$\begin{aligned}V(K)\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}K)&\le V(L)\widetilde{W}_{-p,j}(\Gamma^{\tau,*}_{p,i}K,\Gamma^{\tau,*}_{p,i}L)\\&\le V(L)\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}K)^{\frac{n+p-j}{n-j}}\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}L)^{-\frac{p}{n-j}},\end{aligned}$$

namely,

$$\frac{\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}K)^{\frac{p}{n-j}}}{V(K)}\ge\frac{\widetilde{W}_j(\Gamma^{\tau,*}_{p,i}L)^{\frac{p}{n-j}}}{V(L)},$$

this yields inequality (1.17).

According to equality conditions of inequalities (2.11) and (2.12), we see that for $j \neq n+p$, equality holds in (1.17) if and only if $\Gamma^{\tau,*}_{p,i}K = \Gamma^{\tau,*}_{p,i}L$, i.e., $\Gamma^{\tau}_{p,i}K = \Gamma^{\tau}_{p,i}L$. For $j = n+p$, by (3.11) and (2.9), we see that inequality (1.17) is still true.

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关于广义L_p-混合投影体与广义L_p-混合质心体的单调不等式

石 伟[1], 王卫东[1,2]

(1.三峡大学理学院数学系, 湖北 宜昌 443002)

(2.三峡大学数学研究中心, 湖北 宜昌 443002)

摘要: 本文研究了广义L_p-混合投影体及广义L_p-混合质心体的单调性问题. 利用解析不等式, 获得了广义L_p-混合投影体与广义L_p-混合质心体的均质积分与对偶均质积分形式的单调不等式, 推广了L_p-投影体及L_p-质心体的体积形式的单调性.

关键词: 广义L_p-混合投影体; 广义L_p-混合质心体; 单调不等式

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