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CHARACTERIZATION OF A CLASS OF PROPER HOLOMORPHIC MAPS FROM $\mathbb{B}^n$ TO $\mathbb{B}^N$

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Abstract: The paper is concerned with the study of rational proper holomorphic maps from the unit ball $\mathbb{B}^n$ to the unit ball $\mathbb{B}^N$. When the geometric rank of the maps are κ_0 and $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$, we give a characterization of their normalized maps.

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1 Introduction

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^N$, and denote by $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ the collection of all proper holomorphic rational maps from $\mathbb{B}^n$ to $\mathbb{B}^N$. Let $\mathbb{H}_n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid \mathrm{Im}(w) > |z|^2\}$ be the Siegel upper half space and denote by $\mathrm{Rat}(\mathbb{H}_n, \mathbb{H}_N)$ the collection of all proper holomorphic rational maps from $\mathbb{H}_n$ to $\mathbb{H}_N$. By the Cayley transform, we can identify $\mathbb{B}^n$ with $\mathbb{H}_n$, and identify $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $\mathrm{Rat}(\mathbb{H}_n, \mathbb{H}_N)$. We say that $f, g \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ are spherically equivalent (or equivalent, for short) if there are $\sigma \in \mathrm{Aut}(\mathbb{B}^n)$ and $\tau \in \mathrm{Aut}(\mathbb{B}^N)$ such that $f = \tau \circ g \circ \sigma$.

The study of proper holomorphic maps can date back to the work of Poincaré [17]. Alexander [1] proved that for equal dimensional case, the map must be an automorphism. For the different dimensional case, by the combining efforts of [5, 8, 19], we know that for $N \in (n, 2n-1)$ with $n \geq 2$, any map $F \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ is spherically equivalent to the map $(z, 0)$. This is now called the first gap theorem.

For the second gap theorem, Huang-Ji-Xu [11] proved that any map $F \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $N \in (2n, 3n-3)$ and $n \geq 4$, must equivalent to a map of the form $(G, 0)$ with $G \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{2n})$. Together with the classification theorems Huang-Ji [10] and Hamada [7], $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{2n})$ must equivalent to $(z, 0)$ or $(G_1, 0)$ or $(G_2, 0)$, where $G_1 \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{2n-1})$ is the Whitney map and $G_2 \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{2n})$ is in the D'Angelo family.

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In [12], Huang-Ji-Yin proved the third gap theorem. Namely, when $N \in (3n, 4n-7)$ and $n \geq 8$, any map $F \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ must be equivalent to a map of the form $(G, 0)$ with $G \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{3n})$. The classification problem for $N = 3n-3$ is achieved by [2]. When $N \geq 3n-2$, the map is no longer monomials and can be very complicated. In fact, [6] constructed a family of maps in $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{3n-2})$, which can not be equivalent to any polynomial maps. Recently, Gul-Ji-Yin [14] gave a characterization of maps in $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^{3n-2})$. The interesting reader can refer to [3, 4, 9, 12, 15, 16, 18] for other related mapping problems between balls.

It seems to be quite less in known for $N \geq 3n-2$. The present paper is devoted to a characterization of proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ with geometric rank κ_0 and $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. We now state our main theorem, with some terminology to be defined in the next section.

Theorem 1.1 (1) Let $F \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with the geometric rank of F being κ_0 and $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Suppose that $\frac{\kappa_0(\kappa_0+1)}{2} < n$ and F is normalized, then it takes the following form

$$\begin{aligned}
f_j &= \frac{1}{q}\big(z_j q + \frac{i}{2}\mu_j z_j w\big) \quad \text{for } 1 \leq j \leq \kappa_0,\ f_k = z_k \quad \text{for } \kappa_0 + 1 \leq k \leq n-1,\\
\phi_{jj} &= \frac{1}{q}\big(\mu_{jj} z_j^2 + e_{j,jj} z_j w\big) \quad \text{for } 1 \leq j \leq \kappa_0,\\
\phi_{jk} &= \frac{1}{q}\big(\mu_{jk} z_j z_k + e_{j,jk} z_k w + e_{k,jk} z_j w\big) \quad \text{for } 1 \leq j < k \leq \kappa_0,\\
\phi_{j\alpha} &= \frac{1}{q}\mu_{j\alpha} z_j z_\alpha \quad \text{for } 1 \leq j \leq \kappa_0 < \alpha \leq n-1,\\
g &= w,
\end{aligned} \tag{1.1}$$

where

$$\begin{aligned}
&q = q(z,w) = 1 + q^{(1)}(z) + Ew,\ q^{(1)}(z) = -2i \sum_{j=1}^{\kappa_0} \frac{\mu_{1j}\overline{e_{1,1j}}}{\mu_1} z_j,\ e_{j,ji} = \frac{\mu_j \mu_{1i}}{\mu_1 \mu_{ij}} e_{1,1i},\\
&0 < \mu_1 \leq \mu_2 = \cdots = \mu_{\kappa_0},\ |e_{1,11}|^2 = \frac{1}{4}(\mu_2^2 - \mu_1^2),\ \ \mathrm{Im}(E) = -\frac{\mu_2^2}{4\mu_1}.
\end{aligned} \tag{1.2}$$

(2) Conversely, if F is defined by (1.1) and (1.2), then the map F is in $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$.

2 Preliminaries

Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant rational CR map from an open subset M of $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0) = 0$. For each $p \in M$ close to 0, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and write $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_N)$ for the map

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle).$$

Then F is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following theorem is important for understanding the geometric properties of $\mathrm{Prop}(\mathbb{H}_n, \mathbb{H}_N)$.

Lemma 2.1 Let $F \in \mathrm{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$ with $2 \le n \le N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in \mathrm{Aut}_0(\mathbb{H}_N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4)$$

with

$$\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Here we use the notation $h^{(k)}(z)$ to denote a polynomial h which has degree k in z, and a function $h(z, \overline{z}, u)$ is said to be quantity $o_{wt}(m)$ if $h(tz, t\overline{z}, t^2 u)/|t|^m \to 0$ uniformly for (z, u) on any small compact subset of 0 as $t (\in \mathbb{R}) \to 0$.

Now, we are in a position to the definition of the geometric rank. Write $\mathcal{A}(p) := -2i\left(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0\right)_{1 \le j,l \le (n-1)}$. Then the geometric rank of F at p is defined to be the rank of the $(n-1) \times (n-1)$ matrix $\mathcal{A}(p)$, which is denoted by $Rk_F(p)$. Notice that $Rk_F(p)$ is a lower semi-continuous function on p, it is independent of the choice of $\tau_p^{**}(p)$, and depends only on p and F. Define the geometric rank of F to be $\kappa_0(F) = \max\limits_{p \in \partial\mathbb{H}_n} Rk_F(p)$. Define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$.

When $1 \le \kappa_0 \le n-2$, a nice normalization was achieved by [9] and [11].

Theorem 2.2 Suppose that $F \in \mathrm{Prop}_3(\mathbb{H}_n, \mathbb{H}_N)$ has geometric rank $1 \le \kappa_0 \le n-2$ with $F(0) = 0$. Then there are $\sigma \in \mathrm{Aut}(\mathbb{H}_n)$ and $\tau \in \mathrm{Aut}(\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$ takes the following form, which is still denoted by $F = (f, \phi, g)$ for convenience of notation

$$\begin{cases} f_l = \sum\limits_{j=1}^{\kappa_0} z_j f_{lj}^*(z, w),\ l \le \kappa_0, \\ f_j = z_j,\ \text{for } \kappa_0 + 1 \le j \le n-1, \\ \phi_{lk} = \mu_{lk} z_l z_k + \sum\limits_{j=1}^{\kappa_0} z_j \phi_{lkj}^* \text{ for } \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = O_{wt}(3), \quad (l,k) \in \mathcal{S}_1, \\ g = w, \\ f_{lj}^*(z,w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + b_{lj}^{(1)}(z) w + O_{wt}(4), \\ \phi_{lkj}^*(z,w) = O_{wt}(2), \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = \sum\limits_{j=1}^{\kappa_0} z_j \phi_{lkj}^* = O_{wt}(3) \ \text{ for } (l,k) \in \mathcal{S}_1. \end{cases} \tag{2.1}$$

Here, for $1 \le \kappa_0 \le n-2$, we write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, the index set for all components of ϕ, where

$$\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le n-1, j \le l\},$$
$$\mathcal{S}_1 = \{(j,l) : j = \kappa_0 + 1, \kappa_0 + 1 \le l \le \kappa_0 + N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\},$$

and

$$\mu_{jl} = \begin{cases} \sqrt{\mu_j + \mu_l} & \text{for } j < l \le \kappa_0; \\ \sqrt{\mu_j} & \text{if } j \le \kappa_0 < l \text{ or if } j = l \le \kappa_0. \end{cases} \tag{2.2}$$

Here we can assume $\mu_1 \le \mu_2 \le \cdots \le \mu_{\kappa_0}$. By [2], we can further assume that

$$e_{1,1\alpha} = 0 \quad \text{for } \kappa_0 + 1 \le \alpha \le n-1. \tag{2.3}$$

A map $F \in \mathrm{Rat}(\mathbb{H}_n, \mathbb{H}_N)$ is called a normalized map if F takes form (2.1) with (2.3). Notice that any $F \in \mathrm{Rat}(\mathbb{H}_n, \mathbb{H}_N)$ can not be further normalized except some rotations in ϕ_{lk} with $(l,k) \in \mathcal{S}_1$.

Next we introduce some notations that will be use during the proof of our main theorem. Let $F = (f, \phi, g) \in \mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ be as in Theorem 1.1. Denote by $\sharp(A)$ the number of elements in the set A. Then we have $N = \sharp(f) + \sharp(\phi) + \sharp(g)$ and $\sharp(\phi) = \sharp(\mathcal{S}_0) + \sharp(\mathcal{S}_1)$. Notice that $\sharp(f) = n-1$, $\sharp(g) = 1$, $\sharp(\mathcal{S}_0) = n\kappa_0 - \frac{(\kappa_0+1)\kappa_0}{2}$. Hence $\sharp(\mathcal{S}_1) = N - n - \sharp(\mathcal{S}_0) = 0$, which means we do not have the $\mathcal{S}_1$ part in the map.

For any rational holomorphic map $H = \frac{(P_1, \cdots, P_m)}{Q}$ on $\mathbb{C}^n$, where $\{P_j, Q\}$ are relatively prime holomorphic polynomials, the degree of H is defined to be

$$\deg(H) := \max\{\deg(P_j), \deg(Q), 1 \le j \le m\}.$$

For any holomorphic map $P(z,w)$, we will use the following notations

$$P(z,w) = \sum_{|\alpha|=j, k\ge 0} P^{(\alpha,k)} z^\alpha w^k = \sum_{j,k} P^{(j,k)}(z) \cdot w^k,$$

where $P^{(j,k)}(z)$ is a homogeneous polynomial with respect to z of degree j.

2 Proof of the Main Theorem

This section is devoted to the proof of Theorem 1.1.

By [15, Theorem 1.1], we know $\deg F = 2$. From (4.2)–(4.4) of [15], we obtain

$$e_{i,jk} = 0, \quad \forall j, k \ne i \text{ or } \kappa_0 + 1 \le j \le n-1 \text{ or } \kappa_0 + 1 \le k \le n-1, \tag{3.1}$$

$$\mu_j \mu_{il} e_{l,il} = \mu_l \mu_{ij} e_{j,ij} \text{ for } 1 \le i, j, l \le \kappa_0. \tag{3.2}$$

From (5.3) of [15], we get

$$\begin{aligned}
&f_\mu(z,0) = z_\mu \text{ for } 1 \le \mu \le n-1, \\
&\phi_{jj}(z,0) = \frac{\mu_{jj} z_j^2}{1 - 2i \sum\limits_{j=1}^{\kappa_0} \overline{A_j} z_j} \text{ for } 1 \le j \le \kappa_0, \\
&\phi_{jk}(z,0) = \frac{\mu_{jk} z_j z_k}{1 - 2i \sum\limits_{j=1}^{\kappa_0} \overline{A_j} z_j} \text{ for } 1 \le j < k \le \kappa_0, \\
&\phi_{j\alpha}(z,0) = \frac{\mu_{j\alpha} z_j z_\alpha}{1 - 2i \sum\limits_{j=1}^{\kappa_0} \overline{A_j} z_j} \text{ for } 1 \le j \le \kappa_0 < \alpha \le n-1.
\end{aligned} \tag{3.3}$$

Here we have set $A_j = \frac{\mu_{1j} e_{1,1j}}{\mu_1}$.

Notice that the degree of F must be 2. Together with the normalization properties in (2.1), the expression of F must have the following form

$$\begin{aligned}
f_j &= \frac{1}{q}\big(z_j q + \frac{\mu_j}{2} i z_j w\big) \quad \text{for } 1 \le j \le \kappa_0,\ f_k = z_k \quad \text{for } \kappa_0 + 1 \le k \le n-1,\\
\phi_{jj} &= \frac{1}{q}\big(\mu_{jj} z_j^2 + e_{j,jj} z_j w\big) \quad \text{for } 1 \le j \le \kappa_0,\\
\phi_{jk} &= \frac{1}{q}\big(\mu_{jk} z_j z_k + e_{j,jk} z_k w + e_{k,jk} z_j w\big) \quad \text{for } 1 \le j < k \le \kappa_0,\\
\phi_{j\alpha} &= \frac{1}{q}\mu_{j\alpha} z_j z_\alpha \quad \text{for } 1 \le j \le \kappa_0 < \alpha \le n-1,\\
g &= w.
\end{aligned} \tag{3.4}$$

Here we have set $q = 1 - 2i \sum_{j=1}^{\kappa_0} \overline{A_j} z_j + Ew$. In what follows, we write $q^{(1)}(z) = -2i \sum_{j=1}^{\kappa_0} \overline{A_j} z_j$.

Since F maps $\partial\mathbb{B}^n$ to $\partial\mathbb{B}^N$, we have the basic equation

$$\mathrm{Im} w = |f|^2 + |\phi|^2 \text{ for } \mathrm{Im} w = |z|^2. \tag{3.5}$$

Substituting (3.4) into this equation, we obtain, for $\mathrm{Im} w = |z|^2$, the following

$$\begin{aligned}
|z|^2|q|^2 &= \sum_{j=1}^{\kappa_0} |z_j q + \frac{i}{2}\mu_j z_j w|^2 + \sum_{j=\kappa_0+1}^{n-1} |z_j q|^2 + \sum_{j=1}^{\kappa_0} |\mu_{jj} z_j^2 + e_{j,jj} z_j w|^2\\
&\quad + \sum_{1 \le j < k \le \kappa_0} |\mu_{jk} z_j z_k + e_{j,jk} z_k w + e_{k,kj} z_j w|^2 + \sum_{1 \le j \le \kappa_0 < \alpha \le n-1} |\mu_{j\alpha} z_j z_\alpha|^2.
\end{aligned} \tag{3.6}$$

After a quick simplification, we obtain

$$\begin{aligned}
&\sum_{j=1}^{\kappa_0} |z_j|^2 \Big\{ 2\mathrm{Re}\Big(\frac{i}{2}\mu_j \overline{w}(1 + q^{(1)}(z) + Ew)\Big) - \frac{\mu_j^2}{4}(u^2 + |z|^4)\Big\} = \sum_{j=1}^{\kappa_0} |\mu_{jj} z_j^2 + e_{j,jj} z_j w|^2\\
&+ \sum_{1 \le j < k \le \kappa_0} |\mu_{jk} z_j z_k + e_{j,jk} z_k w + e_{k,kj} z_j w|^2 + \sum_{j=1}^{\kappa_0} |\mu_j |z_j|^2 \cdot \Big(|z|^2 - \sum_{j=1}^{\kappa_0} |z_j|^2\Big).
\end{aligned} \tag{3.7}$$

Consideration of the Degree 4 Terms By considering the degree 4 terms, we get

$$\begin{aligned}
&\sum_{j=1}^{\kappa_0} |z_j^2| \Big\{ 2\mathrm{Re}\Big(\frac{i}{2}\mu_j(-i|z|^2) + \frac{i}{2}\mu_j u (q^{(1)}(z) + Eu)\Big) - \frac{\mu_j^2}{4} u^2 \Big\}\\
&= \sum_{j=1}^{\kappa_0} |\mu_{jj} z_j^2 + e_{j,jj} z_j u|^2 + \sum_{1 \le j < k \le \kappa_0} |\mu_{jk} z_j z_k + e_{j,jk} z_k u + e_{k,kj} z_j u|^2\\
&\quad + \sum_{j=1}^{\kappa_0} \mu_j z_j|^2 \cdot \Big(|z|^2 - \sum_{j=1}^{\kappa_0} \mu_j |z_j|^2\Big).
\end{aligned} \tag{3.8}$$

Collecting $z^\alpha \overline{z}^\beta u^2$ term with $|\alpha| = |\beta| = 1$, we obtain

$$\sum_{j=1}^{\kappa_0} |z_j^2| \Big\{ 2\mathrm{Re}(\frac{i}{2}\mu_j u \cdot Eu) - \frac{\mu_j^2}{4} u^2 \Big\} = \sum_{j=1}^{\kappa_0} |e_{j,jj} z_j u|^2 + \sum_{1 \le j < k \le \kappa_0} |e_{j,jk} z_k u + e_{k,kj} z_j u|^2. \tag{3.9}$$

By considering the coefficients of $|z_j|^2u$ terms with $1 \le j \le \kappa_0$ and $z_j\overline{z_k}u$ terms with $1 \le j < k \le \kappa_0$, respectively, we get

$$\mathrm{Re}(i\mu_j E) - \frac{1}{4}\mu_j^2 = \sum_{k=1}^{\kappa_0} |e_{j,jk}|^2,\ e_{j,jk}\cdot e_{k,jk} = 0 \text{ for } 1 \le j < k \le \kappa_0. \tag{3.10}$$

Combining this with (3.2), we know only one of $e_{1,1j}$ $1 \le j \le \kappa_0$ is non zero. From (3.2) and (3.10), we obtain

$$\frac{1}{4}\mu_j = \mathrm{Re}(i\mu_j E) - \sum_{k=1}^{\kappa_0} \frac{1}{\mu_j}\Big|\frac{\mu_j\mu_{1k}}{\mu_1\mu_{kj}}e_{1,1k}\Big|^2.$$

Together with $\mu_1 \le \cdots \le \mu_{\kappa_0}$, we get $e_{1,12} = \cdots = e_{1,1\kappa_0} = 0$ and $\mu_2 = \cdots = \mu_{\kappa_0}$. Furthermore, we obtain

$$\begin{aligned} &\frac{1}{4}(\mu_1-\mu_2) = \frac{1}{\mu_2}|e_{2,12}|^2 - \frac{1}{\mu_1}|e_{1,11}|^2 = \frac{\mu_2-(\mu_1+\mu_2)}{\mu_1(\mu_1+\mu_2)}|e_{1,11}|^2 = -\frac{1}{\mu_1+\mu_2}|e_{1,11}|^2. \\ &\mathrm{Im}(E) = -\frac{1}{4}\mu_1 - \frac{1}{\mu_1}\cdot\frac{\mu_2^2-\mu_1^2}{4} = -\frac{\mu_2^2}{4\mu_1}. \end{aligned} \tag{3.11}$$

Thus

$$|e_{1,11}|^2 = \frac{1}{4}(\mu_2^2-\mu_1^2),\quad \mathrm{Im}(E) = -\frac{\mu_2^2}{4\mu_1}. \tag{3.12}$$

Collecting $z^\alpha\overline{z}^\beta$ term with $|\alpha| = |\beta| = 2$, we obtain

$$\begin{aligned} \sum_{j=1}^{\kappa_0} |z_j^2|\mathrm{Re}\Big(\mu_j|z|^2\Big) =& \sum_{j=1}^{\kappa_0}\mu_j|z_j|^4 + \sum_{1\le j<k\le\kappa_0}(\mu_j+\mu_k)|z_jz_k|^2 \\ &+ \sum_{j=1}^{\kappa_0}\mu_j|z_j|^2\cdot(|z|^2 - \sum_{k=1}^{\kappa_0}|z_k|^2). \end{aligned} \tag{3.13}$$

This equation is a trivial formula.

Collecting $z^\alpha\overline{z}^\beta u$ term with $|\alpha|+|\beta| = 3$, we obtain

$$\begin{aligned} &\sum_{j=1}^{\kappa_0}|z_j^2|\cdot 2\mathrm{Re}\Big(\frac{i}{2}\mu_j q^{(1)}(z)\Big) \\ =& \sum_{j=1}^{\kappa_0} 2\mathrm{Re}\Big(\mu_{jj}\overline{z_j^2}\cdot e_{j,jj}z_j\Big) + \sum_{1\le j<k\le\kappa_0} 2\mathrm{Re}\Big(\mu_{jk}\overline{z_jz_k}\cdot\big(e_{j,jk}z_k + e_{k,jk}z_j\big)\Big). \end{aligned} \tag{3.14}$$

We can easily derive from this equation the following

$$-\frac{i}{2}\mu_j\cdot 2i\frac{\mu_{ij}}{\mu_j}e_{1,1j} = \mu_{jj}e_{j,jj},\quad e_{j,jk}\cdot e_{k,jk} = 0 \text{ for } 1 \le j < k \le \kappa_0. \tag{3.15}$$

These equations were achieved before and we do not get new equations.

Consideration of the Degree 5 Terms By considering the degree 5 terms, we get

$$\begin{aligned}&\sum_{j=1}^{\kappa_0}|z_j^2|\cdot 2\mathrm{Re}\Big\{\frac{i}{2}\mu_j(-i|z|^2)\cdot q^{(1)}(z)\Big\}=\sum_{j=1}^{\kappa_0}2\mathrm{Re}\Big\{\mu_{jj}\overline{z_j}^2\cdot e_{j,jj}z_j(i|z|^2)\Big\}\\&+\sum_{1\le j<k\le\kappa_0}2\mathrm{Re}\Big\{\mu_{jk}\overline{z_j}\overline{z_k}\cdot\Big(e_{j,jk}z_j(i|z|^2)+e_{k,kj}z_k(i|z|^2)\Big)\Big\}.\end{aligned}\tag{3.16}$$

Deleting $|z|^2$ from both sides and consider the terms $z_j\overline{z_j}^2$ and $|z_j^2\overline{z_k}|$, respectively, we can easily derive from this equation the following

$$-\frac{i}{2}\mu_j\cdot 2i\frac{\mu_{ij}}{\mu_j}e_{1,1j}=\mu_{jj}e_{j,jj},\quad e_{j,jk}\cdot e_{k,jk}=0 \text{ for } 1\le j<k\le\kappa_0.\tag{3.17}$$

As before, we do not get new equations.

Consideration of the Degree 6 Terms By considering the degree 6 terms, we get

$$\begin{aligned}&\sum_{j=1}^{\kappa_0}|z_j^2|\cdot 2\mathrm{Re}\Big\{\frac{i}{2}\mu_j\cdot E|z|^4-\frac{\mu_j^2}{4}|z|^4\Big\}\\&=\sum_{j=1}^{\kappa_0}|e_{j,jj}z_j|^2\cdot|z|^4+\sum_{1\le j<k\le\kappa_0}|e_{j,jk}z_j+e_{k,kj}z_k|^2\cdot|z|^4.\end{aligned}\tag{3.18}$$

Notice that this equation is equivalent to (3.10).

In conclusion, the map F must be of form (1.1) with relations in (1.2). The arguments above also showed that this map is indeed a map in $\mathrm{Rat}(\mathbb{B}^n,\mathbb{B}^N)$. This completes the proof of Theorem 1.1.

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一类$\mathbb{B}^n$到$\mathbb{B}^N$上全纯逆紧映射的刻画

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摘 要: 本文主要研究单位球$\mathbb{B}^n$到单位球$\mathbb{B}^N$上全纯逆紧映射的问题. 对于$\mathbb{B}^n$到$\mathbb{B}^N$的全纯逆紧映射映射, 当其几何秩为κ_0且$N=n+\frac{(2n-\kappa_0-1)\kappa_0}{2}$时, 给出了其正规化映射的一个刻画.

关键词: 全纯逆紧映射; 全纯等价; 几何秩; 退化秩

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