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BOUNDARY MULTIPLIES AND TOEPLITZ OPERATORS ASSOCIATED WITH ANALYTIC MORREY SPACES

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Abstract: In this paper, by using Hardy space's properties and elementary calculations, we study boundary characterization and boundary multipliers of analytic Moerry space, and Toeplitz operators acting on Hardy space to analytic Moerry space are also investigated. For the above questions, the necessary and sufficient conditions are obtained.

Keywords: boundary multiplies; Toeplitz operators; Hardy spaces; Morrey spaces

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1 Introduction

Denote by $\mathbb{T}$ the boundary of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

Let H^∞ be the space of bounded analytic function on $\mathbb{D}$ consisting of functions $f \in H(\mathbb{D})$ with

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

We refer to [1, 2] for H^p and H^∞ spaces.

For $\lambda \in (0, 1]$, denote by $\mathcal{L}^{2,\lambda}(\mathbb{T})$ the Morrey space of all Lebesgue measurable functions f on $\mathbb{T}$ that satisfy

$$\|f\|_{\mathcal{L}^{2,\lambda}(\mathbb{T})} = \sup_{I\subseteq\mathbb{T}} \left(|I|^{-\lambda} \int_I |f(\zeta) - f_I|^2 |d\zeta| \right)^{\frac{1}{2}} < \infty,$$

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where $|I|$ denotes the length of the arc I and

$$f_I = \frac{1}{|I|}\int_I f(\zeta)|d\zeta|.$$

Clearly, $\mathcal{L}^{2,1}(\mathbb{T})$ coincides with BMO($\mathbb{T}$), the space of functions with bounded mean oscillation on $\mathbb{T}$ (cf. [3, 4]). Similar to a norm on BMO($\mathbb{T}$) given in [4, p. 68], a norm on $\mathcal{L}^{2,\lambda}(\mathbb{T})$ can be defined by

$$|||f|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})} = \left|\int_{\mathbb{T}} f|d\zeta|\right| + \|f\|_{\mathcal{L}^{2,\lambda}(\mathbb{T})}.$$

From Xiao's monograph [5, p. 52],

$$\mathrm{BMO}(\mathbb{T}) \subseteq \mathcal{L}^{2,\lambda_1}(\mathbb{T}) \subseteq \mathcal{L}^{2,\lambda_2}(\mathbb{T}) \subseteq L^2(\mathbb{T}), \quad 0<\lambda_2<\lambda_1<1.$$

It is well known that if $f \in H^2$, then its non-tangential limit $f(\zeta)$ exists almost everywhere for $\zeta \in \mathbb{T}$. For $\lambda \in (0,1]$, the analytic Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$ is the set of $f \in H^2$ with $f(\zeta) \in \mathcal{L}^{2,\lambda}(\mathbb{T})$. It is clear that $\mathcal{L}^{2,1}(\mathbb{D})$ is BMOA, the analytic space of functions with bounded mean oscillation (cf. [3, 4]). For $\lambda \in (0,1]$, $\mathcal{L}^{2,\lambda}(\mathbb{D})$ is located between BMOA and H^2. It is worth mentioning that there exists a isomorphism relation between analytic Morrey spaces and Möbius invariant $\mathcal{Q}_p$ spaces via fractional order derivatives of functions (see [6]). Recall that for $0<p<\infty$, a function $f \in H(\mathbb{D})$ belongs to the space $\mathcal{Q}_p$ if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^p dA(z) < \infty,$$

where dA is the area Lebesgue measure on $\mathbb{D}$ and $\sigma_a(z) = \frac{a-z}{1-\overline{a}z}$ is the Möbius transformation of the unit disk $\mathbb{D}$ interchanging a and 0. See [5, 7] for a general exposition on $\mathcal{Q}_p$ spaces. Recently, the interest in $\mathcal{L}^{2,\lambda}(\mathbb{D})$ spaces grew rapidly (cf. [8–13]).

An important problem of studying function spaces is to characterize the multipliers of such spaces. For a Banach function space X, denote by $M(X)$ the class of all multipliers on X. Namely,

$$M(X) = \{f \in X : fg \in X \text{ for all } \ g \in X\}.$$

Bao and Pau [14] characterized boundary multipliers of $\mathcal{Q}_p$ spaces. Stegenga [15] described multipliers of BMO($\mathbb{T}$) which is equal to $\mathcal{L}^{2,1}(\mathbb{T})$. It is natural to consider multipliers of $\mathcal{L}^{2,\lambda}(\mathbb{T})$ with $\lambda \in (0,1)$ in this paper.

Given a function $\varphi \in L^2(\mathbb{T})$. Let T_φ be the Toeplitz operator on H^2 with symbol φ defined by

$$T_\varphi f(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\varphi(\zeta)f(\zeta)}{1-\overline{\zeta}z}|d\zeta|, \quad f\in H^2, \quad z\in\mathbb{D}.$$

For the study of Toeplitz operators on Hardy spaces and Bergman spaces, see, for example, [16, 17]. We refer to [9] for the results of Toeplitz operators on $\mathcal{L}^{2,\lambda}(\mathbb{D})$ spaces.

The aim of this paper is to consider boundary multiplies and Toeplitz operators associated with analytic Morrey spaces. In Section 2, using a characterization of $\mathcal{L}^{2,\lambda}(\mathbb{T})$ in terms of functions with absolute values, we characterize the multipliers of $\mathcal{L}^{2,\lambda}(\mathbb{T})$. In Section 3, we characterize the boundedness and compactness of Toeplitz operators from Hardy spaces to analytic Morrey spaces.

Throughout this paper, we write $a \lesssim b$ if there exists a constant C such that $a \leq Cb$. Also, the symbol $a \approx b$ means that $a \lesssim b \lesssim a$.

2 Boundary Multiplies of Analytic Morrey Spaces

By the study of certain integral operators on analytic Morrey spaces, Li, Liu and Lou [8] proved that $M(\mathcal{L}^{2,\lambda}(\mathbb{D})) = H^\infty$. In this section, applying a characterization of $\mathcal{L}^{2,\lambda}(\mathbb{T})$ in terms of absolute values of functions, we characterize $M(\mathcal{L}^{2,\lambda}(\mathbb{T}))$, boundary multiplies of analytic Morrey spaces.

Given $f \in L^2(\mathbb{T})$, let $\widehat{f}$ be the Poisson extension of f. Namely,

$$\widehat{f}(z) = \int_{\mathbb{T}} f(\zeta) d\mu_z(\zeta), \ z \in \mathbb{D},$$

where

$$d\mu_z(\zeta) = \frac{1-|z|^2}{2\pi|\zeta - z|^2}|d\zeta|.$$

Let $0 < \lambda < 1$. From [5, p.52], $f \in \mathcal{L}^{2,\lambda}(\mathbb{T})$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\lambda}\int_{\mathbb{D}}|\nabla \widehat{f}(w)|^2(1-|\sigma_z(w)|^2)dA(w) < \infty, \tag{2.1}$$

where ∇ is the Laplace operator. Also, $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ if and only if

$$\|f\|_{\mathcal{L}^{2,\lambda}(\mathbb{D})} = \sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(w)|^2(1-|\sigma_z(w)|^2)dA(w) < \infty. \tag{2.2}$$

We need the following useful inequality (see [18, Lemma 2.5]).

Lemma A Suppose that $s > -1$, r, $t \geq 0$, and $r + t - s > 2$. If $t < s + 2 < r$, then

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\overline{w}z|^r|1-\overline{w}\zeta|^t} dA(w) \lesssim \frac{(1-|z|^2)^{2+s-r}}{|1-\overline{z}\zeta|^t}$$

for all z, $\zeta \in \mathbb{D}$.

Now we characterize $\mathcal{L}^{2,\lambda}(\mathbb{T})$ via absolute values of functions as follows. See [9] for the analytic version of the following result.

Theorem 2.1 Let $0 < \lambda < 1$ and $f \in L^2(\mathbb{T})$. Then $f \in \mathcal{L}^{2,\lambda}(\mathbb{T})$ if and only if

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\int_{\mathbb{T}}|f(\zeta)|^2 d\mu_z(\zeta)|\sigma_a'(z)|^2 dA(z) < \infty.$$

Proof Let $f \in L^2(\mathbb{T})$. It is well known (cf. [19, p. 564]) that

$$\int_{\mathbb{T}}|f(\zeta)|^2 d\mu_z(\zeta) - |\widehat{f}(z)|^2 \approx \int_{\mathbb{D}}|\nabla \widehat{f}(w)|^2(1-|\sigma_z(w)|^2)dA(w)$$

for all $z \in \mathbb{D}$. Combining this with the Fubini theorem, we obtain that for any $a \in \mathbb{D}$,

$$\begin{aligned}
&\int_{\mathbb{D}}\int_{\mathbb{T}} |f(\zeta)|^2 d\mu_z(\zeta)|\sigma_a'(z)|^2 dA(z) - \int_{\mathbb{D}} |\widehat{f}(z)|^2 |\sigma_a'(z)|^2 dA(z) \\
\approx\ & \int_{\mathbb{D}} |\nabla \widehat{f}(w)|^2 \int_{\mathbb{D}} (1-|\sigma_z(w)|^2)|\sigma_a'(z)|^2 dA(z) dA(w) \\
\approx\ & (1-|a|^2)^2 \int_{\mathbb{D}} |\nabla \widehat{f}(w)|^2 (1-|w|^2) \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}w|^2|1-\overline{a}z|^4} dA(z) dA(w).
\end{aligned}$$

By Lemma A and the same argument in [19, p. 563], we get that

$$\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}w|^2|1-\overline{a}z|^4} dA(z) \approx \frac{1}{(1-|a|^2)|1-\overline{a}w|^2}.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}}\int_{\mathbb{T}} |f(\zeta)|^2 d\mu_z(\zeta)|\sigma_a'(z)|^2 dA(z) - \int_{\mathbb{D}} |\widehat{f}(z)|^2 |\sigma_a'(z)|^2 dA(z) \\
\approx\ & \int_{\mathbb{D}} |\nabla \widehat{f}(w)|^2 \frac{(1-|a|^2)(1-|w|^2)}{|1-\overline{a}w|^2} dA(w) \\
\approx\ & \int_{\mathbb{D}} |\nabla \widehat{f}(w)|^2 (1-|\sigma_a(w)|^2) dA(w) \qquad (2.3)
\end{aligned}$$

for all $a \in \mathbb{D}$.

Let

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}}\int_{\mathbb{T}} |f(\zeta)|^2 d\mu_z(\zeta)|\sigma_a'(z)|^2 dA(z) < \infty.$$

From (2.1) and (2.3), we get that $f \in \mathcal{L}^{2,\lambda}(\mathbb{T})$.

On the other hand, let $f \in \mathcal{L}^{2,\lambda}(\mathbb{T})$. Without loss of generality, we may assume that f is real valued. Denote by $\widetilde{f}$ the harmonic conjugate function of $\widehat{f}$. Set $g = \widehat{f} + i\widetilde{f}$. The Cauchy-Riemann equations give $|\nabla \widehat{f}(z)| \approx |g'(z)|$. Thus $g \in \mathcal{L}^{2,\lambda}(\mathbb{D})$. By the growth estimates of functions in $\mathcal{L}^{2,\lambda}(\mathbb{D})$ (cf. [8, Lemma 2]), one gets that

$$|\widehat{f}(z)| \leq |g(z)| \lesssim (1-|z|)^{\frac{\lambda-1}{2}} \|g\|_{\mathcal{L}^{2,\lambda}(\mathbb{D})}$$

for all $z \in \mathbb{D}$. Consequently,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |\widehat{f}(z)|^2 |\sigma_a'(z)|^2 dA(z) \\
\lesssim\ & \|g\|^2_{\mathcal{L}^{2,\lambda}(\mathbb{D})} \sup_{a\in\mathbb{D}} (1-|a|^2)^{3-\lambda} \int_{\mathbb{D}} \frac{(1-|z|)^{\lambda-1}}{|1-\overline{a}z|^4} dA(z) \\
\lesssim\ & \|g\|^2_{\mathcal{L}^{2,\lambda}(\mathbb{D})}.
\end{aligned}$$

Combining this with (2.3), $f \in \mathcal{L}^{2,\lambda}(\mathbb{T})$, we get that

$$\sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}}\int_{\mathbb{T}} |f(\zeta)|^2 d\mu_z(\zeta)|\sigma_a'(z)|^2 dA(z) < \infty.$$

The proof is completed.

Denote by $L^\infty(\mathbb{T})$ the space of essentially bounded measurable functions on $\mathbb{T}$. Using Theorem 2.1, we characterize multipliers of $\mathcal{L}^{2,\lambda}(\mathbb{T})$ as follows.

Theorem 2.2 Let $0<\lambda<1$. Then $M(\mathcal{L}^{2,\lambda}(\mathbb{T}))=L^\infty(\mathbb{T})$.

Proof Let $f\in L^\infty(\mathbb{T})$ and $g\in\mathcal{L}^{2,\lambda}(\mathbb{T})$. From Theorem 2.1, one gets that

$$\begin{aligned}&\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\int_{\mathbb{T}}|f(\zeta)g(\zeta)|^2d\mu_z(\zeta)|\sigma_a'(z)|^2dA(z)\\ \leq\ & \|f\|_{L^\infty(\mathbb{T})}\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\int_{\mathbb{T}}|g(\zeta)|^2d\mu_z(\zeta)|\sigma_a'(z)|^2dA(z)\\ <\ & \infty.\end{aligned}$$

Applying Theorem again, we know that $fg\in\mathcal{L}^{2,\lambda}(\mathbb{T})$. Thus $L^\infty(\mathbb{T})\subseteq M(\mathcal{L}^{2,\lambda}(\mathbb{T}))$.

On the other hand, let $f\in M(\mathcal{L}^{2,\lambda}(\mathbb{T}))$. By the closed graph theorem, there exists a positive constant C such that $|||fg|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}\leq C|||g|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}$ for any $g\in\mathcal{L}^{2,\lambda}(\mathbb{T})$. Set $h=f/C$. Clearly, $h\in\mathcal{L}^{2,\lambda}(\mathbb{T})$. We deduce that $|||h^n|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}\leq|||h|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}$ for all positive integer n. As mentioned in Section 1, $\mathcal{L}^{2,\lambda}(\mathbb{T})\subseteq L^2(\mathbb{T})$. Form the closed graph theorem again, there exists a positive constant C_1 satisfying

$$\left(\int_{\mathbb{T}}|f(\zeta)|^2|d\zeta|\right)^{1/2}\leq C_1|||f|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}$$

for all $f\in\mathcal{L}^{2,\lambda}(\mathbb{T})$. Consequently,

$$\left(\int_{\mathbb{T}}|h^n(\zeta)|^2|d\zeta|\right)^{1/2}\leq C_1|||h^n|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}\leq C_1|||h|||_{\mathcal{L}^{2,\lambda}(\mathbb{T})}<\infty.$$

Since n is arbitrary, we know that $h\in L^\infty(\mathbb{T})$. Hence $f\in L^\infty(\mathbb{T})$. The proof is completed.

3 Toeplitz Operators from Hardy Spaces to Analytic Morrey Spaces

In this section, we characterize the boundedness and compactness of Toeplitz operators from the Hardy space H^p to the analytic Morrey space $\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})$ for $2<p<\infty$. Toeplitz operators on analytic Morrey spaces were investigated in [9]. We refer to [8] for the study of some integral operators from H^p to $\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})$ for $2<p<\infty$.

Following [9], we use a norm of $\mathcal{L}^{2,\lambda}(\mathbb{D})$, $\lambda\in(0,1)$, defined by

$$|||f|||_{\mathcal{L}^{2,\lambda}(\mathbb{D})}=|f(0)|+\sup_{w\in\mathbb{D}}\left((1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}\int_{\mathbb{T}}|f(\zeta)|^2d\mu_z(\zeta)|\sigma_w'(z)|^2dA(z)\right)^{1/2}.$$

The following well-known lemma can be found in [17].

Lemma B Suppose $s>0$ and $t>-1$. Then there exists a positive constant C such that

$$\int_{\mathbb{D}}\frac{(1-|w|^2)^t}{|1-\bar{z}w|^{2+t+s}}dA(w)\leq\frac{C}{(1-|z|^2)^s}$$

for all $z\in\mathbb{D}$.

Applying some well-known results of Toeplitz operators and composition operators on Hardy spaces, we characterize the boundedness of T_φ from Hardy spaces to analytic Morrey spaces as follows.

Theorem 3.3 Let $2 < p < \infty$ and $\varphi \in L^2(\mathbb{T})$. Then the Toeplitz operator T_φ is bounded from H^p to $\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{T})$ if and only if $\varphi \in L^\infty(\mathbb{T})$.

Proof Suppose that T_φ is bounded from H^p to $\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})$. For $b \in \mathbb{D}$, let

$$f_b(z) = \frac{(1-|b|^2)^{1-\frac{1}{p}}}{1-\overline{b}z}, \quad z \in \mathbb{D}.$$

Note that $p > 2$. By the well known estimates in [20, p. 9], one gets that

$$\sup_{b\in\mathbb{D}} \int_0^{2\pi} |f_b(e^{i\theta})|^p d\theta = \sup_{b\in\mathbb{D}} (1-|b|^2)^{p-1} \int_0^{2\pi} \frac{1}{|1-\overline{b}e^{i\theta}|^p} d\theta < \infty.$$

Thus functions f_b belong to H^p uniformly for all $b \in \mathbb{D}$. Consequently,

$$\begin{aligned}
\infty > |||T_\varphi f_b|||^2_{\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})} &\gtrsim (1-|b|^2)^{\frac{2}{p}} \int_{\mathbb{D}} \int_{\mathbb{T}} |T_\varphi f_b(\zeta)|^2 d\mu_z(\zeta) |\sigma_b'(z)|^2 dA(z) \\
&\approx (1-|b|^2)^{\frac{2}{p}} \int_{\mathbb{D}} \int_{\mathbb{T}} |T_\varphi f_b(\zeta)|^2 d\mu_{\sigma_b(z)}(\zeta) dA(z) \\
&\gtrsim (1-|b|^2)^{\frac{2}{p}} \int_{\mathbb{D}} |T_\varphi f_b(\sigma_b(z))|^2 dA(z) \\
&\gtrsim (1-|b|^2)^{\frac{2}{p}} |T_\varphi f_b(b)|^2.
\end{aligned}$$

Note that

$$(1-|b|^2)^{\frac{2}{p}} |T_\varphi f_b(b)|^2 = \frac{1}{4\pi^2} \left| \int_{\mathbb{T}} \frac{\varphi(\zeta)(1-|b|^2)}{(1-\overline{\zeta}b)(1-\overline{b}\zeta)} |d\zeta| \right|^2 = |\hat{\varphi}(b)|^2.$$

Thus $\sup\limits_{b\in\mathbb{D}} |\hat{\varphi}(b)| < \infty$. By [1, p. 5], $\varphi \in L^\infty(\mathbb{T})$.

On the other hand, let $\varphi \in L^\infty(\mathbb{T})$. It is well known that T_φ is bounded on H^p (cf. [21–23]). Namely, $\|T_\varphi g\|_{H^p} \lesssim \|\varphi\|_{L^\infty(\mathbb{T})} \|g\|_{H^p}$ for all $g \in H^p$. Let $f \in H^p$, we deduce that

$$\begin{aligned}
&|||T_\varphi f|||_{\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})} \\
=&|T_\varphi f(0)| + \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{\frac{2}{p}} \int_{\mathbb{D}} \int_{\mathbb{T}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\varphi(\zeta) f(\zeta)}{1-\overline{\zeta}\xi} |d\zeta| \right|^2 d\mu_z(\xi) |\sigma_w'(z)|^2 dA(z) \right)^{1/2} \\
\lesssim&\|\varphi\|_{L^\infty(\mathbb{T})} \|f\|_{H^1} + \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{\frac{2}{p}} \int_{\mathbb{D}} \int_{\mathbb{T}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\varphi(\zeta) f(\zeta)}{1-\overline{\zeta}\sigma_z(\xi)} |d\zeta| \right|^2 |d\xi| |\sigma_w'(z)|^2 dA(z) \right)^{1/2} \\
\lesssim&\|\varphi\|_{L^\infty(\mathbb{T})} \|f\|_{H^p} + \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{\frac{2}{p}} \int_{\mathbb{D}} \|(T_\varphi f) \circ \sigma_z\|^2_{H^2} |\sigma_w'(z)|^2 dA(z) \right)^{1/2} \\
\lesssim&\|\varphi\|_{L^\infty(\mathbb{T})} \|f\|_{H^p} + \sup_{w\in\mathbb{D}} \left((1-|w|^2)^{\frac{2}{p}} \int_{\mathbb{D}} \|(T_\varphi f) \circ \sigma_z\|^2_{H^p} |\sigma_w'(z)|^2 dA(z) \right)^{1/2}.
\end{aligned}$$

By the well-known characterization of composition operators on H^p (cf. [17, Theorem 11.12]), we get that

$$\|(T_\varphi f) \circ \sigma_z\|_{H^p} \le \left(\frac{1+|z|}{1-|z|} \right)^{1/p} \|T_\varphi f\|_{H^p}.$$

Thus,

$$
\begin{aligned}
&|||T_\varphi f|||_{\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})} \\
\lesssim\ & \|\varphi\|_{L^\infty(\mathbb{T})}\|f\|_{H^p} + \sup_{w\in\mathbb{D}}\left((1-|w|^2)^{\frac{2}{p}}\|T_\varphi f\|_{H^p}^2\int_{\mathbb{D}}(1-|z|)^{-2/p}|\sigma_w'(z)|^2dA(z)\right)^{1/2} \\
\lesssim\ & \|\varphi\|_{L^\infty(\mathbb{T})}\|f\|_{H^p} + \|\varphi\|_{L^\infty(\mathbb{T})}\|f\|_{H^p}\sup_{w\in\mathbb{D}}\left((1-|w|^2)^{\frac{2}{p}+2}\int_{\mathbb{D}}\frac{(1-|z|)^{-2/p}}{|1-\overline{w}z|^4}dA(z)\right)^{1/2}.
\end{aligned}
$$

Note that $p>2$. By Lemma B, we get that $|||T_\varphi f|||_{\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})}\lesssim\|\varphi\|_{L^\infty(\mathbb{T})}\|f\|_{H^p}$. The proof is completed.

We characterize the compactness of Toeplitz operators from H^p to $\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{T})$ as follows.

Theorem 3.4 Let $2<p<\infty$ and $\varphi\in L^\infty(\mathbb{T})$. Then the Toeplitz operator T_φ is compact from H^p to $\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})$ if and only if $\varphi=0$.

Proof It suffices to prove the necessity. Let $\{a_n\}_{n=1}^\infty\subseteq\mathbb{D}$ be a sequence such that $|a_n|\to1$ as $n\to\infty$. Set

$$f_n(z)=\frac{(1-|a_n|^2)^{1-\frac{1}{p}}}{1-\overline{a_n}z},\quad z\in\mathbb{T}.$$

As explained in the proof of Theorem 3.3, $\sup\limits_n\|f_n\|_{H^p}<\infty$. Clearly, $f_n\to0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Since T_φ is compact, we get that

$$\lim_{n\to\infty}\|T_\varphi f_n\|_{\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})}=0.$$

By the proof of Theorem 3.3, one gets that $|\hat\varphi(a_n)|\lesssim\|T_\varphi f_n\|_{\mathcal{L}^{2,1-\frac{2}{p}}(\mathbb{D})}$ for all n. Consequently, $|\hat\varphi(a_n)|\to0,\ \ n\to\infty$. Since a_n is arbitrary and $\hat\varphi$ is harmonic, by the maximum principle, $\hat\varphi\equiv0$ on $\mathbb{D}$. Hence $\varphi=0$ on $\mathbb{T}$. We finish the proof.

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边界乘子和Toeplitz 算子关联的解析Morrey 空间

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摘要: 本文利用Hardy空间的性质和初等计算, 研究了解析Moerry空间的边界值与边界值乘子, 也研究了作用在Hardy空间到解析Moerry空间上的Toeplitz算子问题. 对于上述问题, 我们都给出了相应的等价刻画.

关键词: 边界乘子; Toeplitz 算子; Hardy 空间; Moerry 空间

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