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EXISTENCE OF SOLUTIONS AND ERROR BOUNDS FOR A GENERALIZED INVERSE MIXED QUASI-VARIATATIONAL INEQUALITY

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Abstract: In this paper, we introduce and study a new class of generalized inverse mixed quasi-variatational inequalities (GIMQVI) in Hilbert spaces. By making use of the properties of generalized f-projection operator, we obtain the existence and uniqueness results for GIMQVI. Moreover, we also establish the error bounds for GIMQVI according to the residual function, which extend and improve some results in the recent literature.

Keywords: generalized inverse mixed quasi-variatational inequality; generalized f-projection operator; error bound; residual function

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1 Introduction

It has been realized over the years that variational inequalities and their generalizations provide convenient frameworks for the study and applications of many important issues of nonlinear analysis, partial differential equations, optimization, equilibria, control theory, finance, economics, transportation and the engineering sciences. With the development of variational inequality theory with its applications, several classes of inverse (mixed) (quasi-) variatational inequalities were introduced and studied. In this area, the first work owned to He et al. [1, 2] in 2006. They studied a class of inverse variational inequalities and also found their applications in practical world, such as normative flow control problems, which require the network equilibrium state to be in a linearly constrained set, and bipartite market equilibrium problems. Since then, some authors paid more attentions on the inverse variational inequalities and their generalizations. For instance, Yang [3], He and Liu [4], Scrimali [5] studied inverse variational inequalities with their applications. Hu and Fang [6] also studied the well-posedness of inverse variational inequalities. Aussel et al. [7] studied the gap functions and error bounds for inverse quasi-variatational inequality problems. Very

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recently, Li et al. [8, 9] employed generalized f-projection operator to study algorithm, existence, gap functions and error bounds for inverse mixed (quasi-) variatational inequalities. The concept of the generalized f-projection operator was introduced by Wu and Huang [10], which is proved to be a good tool to study inverse mixed quasi-variatational inequalities.

Motivated and inspired by the work mentioned above, in this paper, we introduce and study a new class of generalized inverse mixed quasi-variatational inequalities (GIMQVI) in Hilbert spaces. We firstly make use of the properties of generalized f-projection operator in Hilbert spaces [9, 10] to obtain the existence and uniqueness results for GIMQVI. Then we study error bounds for GIMQVI according to the residual function. The results presented here extend and improve the correspond results Theorem 4.1 and Theorems 5.1 and 5.4 in [9].

2 Preliminaries

Throughout the paper, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, let $K : H \to 2^H$ be a set-valued mapping such that for each $u \in H$, $K(u)$ is a nonempty closed convex subset of H. Let $M, N : H \times H \to H$, $A, B, C, D, h : H \to H$ be nonlinear single-valued mappings, $f : H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous on $K(u)$ for each $u \in H$. We consider the generalized inverse mixed quasi-variatational inequality (GIMQVI) as follows: find a $u \in H$, such that $hu \in K(u)$ and

$$\langle M(Au, Bu) - N(Cu, Du), v - hu\rangle + f(v) - f(hu) \geq 0, \quad \forall\, v \in K(u). \tag{2.1}$$

Let us first see some special cases of GIMQVI (2.1).

(1) If $M(u, v) = N(u, v) = u$ for all $u, v \in H$, then GIMQVI (2.1) is reduced to the following inverse mixed quasi-variational inequality: find a $u \in H$, such that $hu \in K(u)$ and

$$\langle Au - Cu, v - hu\rangle + f(v) - f(hu) \geq 0, \quad \forall\, v \in K(u), \tag{2.2}$$

which is to be a new one.

(2) If $Bu = Cu = Du = N(u, v) = 0, M(u, v) = u$ for all $u, v \in H$, then GIMQVI (2.1) is reduced to the inverse mixed quasi-variational inequality (IMQVI): find a $u \in H$ such that $hu \in K(u)$ and

$$\langle Au, v - hu\rangle + f(v) - f(hu) \geq 0, \quad \forall\, v \in K(u), \tag{2.3}$$

which was introduced and studied by Li and Zou [9].

(3) If $Bu = Cu = Du = N(u, v) = 0, Au = M(u, v) = u, K(u) = \overline{K}$ for all $u, v \in H$, where $\overline{K}$ is a nonempty closed convex subset of H, then GIMQVI (2.1) is equivalent to the inverse mixed variational inequality (IMVI) studied by Li, Li and Huang [8]: find a $u \in H$ such that $hu \in \overline{K}$and

$$\langle u, v - hu\rangle + f(v) - f(hu) \geq 0, \quad \forall\, v \in \overline{K}. \tag{2.4}$$

(4) If $H = R^n, f(u) = Bu = Cu = Du = N(u, v) = 0, Au = M(u, v) = u, K(u) = \overline{K}$for all $u, v \in R^n$, where $\overline{K} \subset R^n$ is a nonempty closed convex subset, then GIMQVI (2.1) is

reduced to the inverse variational inequality (IVI), first proposed by He and Liu [1]: find a $u \in R^n$ such that $hu \in \overline{K}$ and

$$\langle u, v - hu \rangle \geq 0, \quad \forall\, v \in \overline{K}. \tag{2.5}$$

(5) If $H = R^n, f(u) = Bu = Cu = Du = N(u,v) = 0, M(u,v) = u$for all $u, v \in R^n$, then GIMQVI (2.1) reduces to the inverse quasi-variational inequality (IQVI) introduced and studied by Aussel et al. [7]: find a $u \in R^n$ such that $hu \in K(u)$ and

$$\langle A(u), v - hu \rangle \geq 0, \quad \forall\, v \in K(u). \tag{2.6}$$

Moreover, if A is the identify mapping on R^n, then IQVI (2.6) reduces to the following inverse quasi-variational inequality: find a $u \in R^n$ such that $hu \in K(u)$and

$$\langle u, v - hu \rangle \geq 0, \quad \forall\, v \in K(u). \tag{2.7}$$

(6) If $H = R^n, f(u) = Bu = Cu = Du = N(u,v) = 0, M(u,v) = hu = u$ for all $u, v \in R^n$, then GIMQVI (2.1) becomes the classic quasi-variational inequality (QVI): find a $u \in K(u)$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall\, v \in K(u). \tag{2.8}$$

QVI was introduced and investigated at first by Bensoussan and Lions [11, 12]. They introduced these problems in connection with impulse optimal control problems.

In a word, GIMQVI (2.1) is more general, which concludes many new and known inverse mixed (quasi-) variational inequalities, inverse (quasi-) variational inequalities, mixed (quasi-) variational inequalities and (quasi-) variational inequalities as its special cases.

In order to obtain our main results, we need the following definitions and lemmas. Now, we first recall the concept and properties of the generalized f-projection operator, which play an important role in obtaining our main results.

Let $G : H \times K \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(x, \xi) = \|x - \xi\|^2 + 2\rho f(\xi),$$

where $\xi \in K, x \in H$, ρ is a positive number and $f : K \to R \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous function.

Definition 2.1 [10] Let H be a real Hilbert space, and K be a nonempty closed and convex subset of H. We call that $P_K^{f,\rho} : H \to 2^K$ is a generalized f-projection operator if

$$P_K^{f,\rho} x = \{u \in K : G(x, u) = \inf_{\xi \in K} G(x, \xi), \forall\, x \in H\}.$$

From the work of Wu and Huang [10] and Fan et al. [13], we know that the generalized f-projection operator has the following properties.

Lemma 2.2 [10, 13] Let H be a real Hilbert space, and K be a nonempty closed and convex subset of H. Then the following statements hold

(i) $P_K^{f,\rho}x$ is nonempty and $P_K^{f,\rho}$ is a single-valued mapping;

(ii) for all $x \in H, x^* = P_K^{f,\rho}x$ if and only if

$$\langle x^* - x, y - x^* \rangle + \rho f(y) - \rho f(x^*) \geq 0, \quad \forall\, y \in K;$$

(iii) $P_K^{f,\rho}$ is continuous.

In addition, let $K : H \to 2^H$ be a set-valued mapping such that for each $x \in H$, $K(x)$ is a closed convex set in H. Similarly, we can define the generalized f-projection of any $z \in H$ on the set $K(x)$, that is,

$$P_{K(x)}^{f,\rho}z = \arg \inf_{\xi \in K(x)} G(z, \xi).$$

In 2016, Li and Zou applied the basic inequality in Lemma 2.2 to prove the properties (for details, see Theorems 3.1 and 3.3 in [9]) of the operator $P_{K(x)}^{f,\rho}$ in Hilbert spaces.

Definition 2.3 [10, 13] Let H be a real Hilbert space, and $N : H \times H \to H$, $A, B, g : H \to H$ be four single-valued mappings.

(i) g is said to be α-Lipschitz continuous on H, if there exists a constant $\alpha > 0$ such that

$$\|gu - gv\| \leq \alpha \|u - v\|, \forall\, u, v \in H;$$

(ii) N is said to be β-g-strongly mixed monotone with respect to A and B if there exists a constant $\beta > 0$ such that

$$\langle gu - gv, N(Au, Bu) - N(Av, Bv) \rangle \geq \beta \|u - v\|^2, \forall\, u, v \in H;$$

(iii) N is said to be γ-strongly mixed monotone with respect to A and B if there exists a constant $\gamma > 0$ such that

$$\langle u - v, N(Au, Bu) - N(Av, Bv) \rangle \geq \gamma \|u - v\|^2, \forall\, u, v \in H;$$

(iv) N is said to be δ-Lipschitz continuous with respect to A and B if there exists a constant $\delta > 0$ such that

$$\|N(Au, Bu) - N(Av, Bv)\| \leq \delta \|u - v\|, \forall\, u, v \in H;$$

(v) N is said to be υ-g-relaxed Lipschitz with respect to A and B if there exists a constant $\upsilon > 0$ such that

$$\langle gu - gv, N(Au, Bu) - N(Av, Bv) \rangle \geq -\upsilon \|u - v\|^2, \forall\, u, v \in H.$$

Remark 2.4 Note that if $g \equiv I$, identify mapping on H, then g-strong mixed monotonicity of N with respect to A and B reduces to the strong mixed monotonicity of N with respect to A and B. Moreover, if $N(Au, Bu) = Au$ for all $u \in H$, then g-strong mixed monotonicity of N with respect to A and B reduces to the ordinary g-strong monotonicity of A and strong mixed monotonicity of N with respect to A and B reduces to the general strong monotonicity of A.

3 The Existence and Uniqueness Results of GIMQVI

In this section, we give the existence and uniqueness results of GIMQVI (2.1) by the properties of generalized f-projection operator under certain conditions.

From the properties of generalized f-projection operator that $u \in H$ is a solution of GIMQVI (2.1) if and only if u satisfies

$$hu = P^{f,\rho}_{K(u)}[hu - \rho(M(Ax, Bx) - N(Cx, Dx))], \tag{3.1}$$

where $\rho > 0$ is a constant.

Theorem 3.1 Let H be a real Hilbert space, and $K : H \to 2^H$ be a set-valued mapping such that for each $u \in H, K(u) \subset H$ is a closed convex set and $f : H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous on $K(u)$. Let $M, N : H \times H \to H, A, B, C, D, h : H \to H$ be nonlinear single-valued mappings. If the following conditions hold

(i) h is α-Lipschitz continuous;

(ii) M is β-Lipschitz continuous with respect to A and B and N is γ-Lipschitz continuous with respect to C and D;

(iii) M is λ-strongly mixed monotone with respect to A and B;

(iv) M is μ-h-strongly mixed monotone with respect to A and B;

(v) there exists $k > 0$ such that

$$\begin{aligned}&\|P^{f,\rho}_{K(u)}w - P^{f,\rho}_{K(v)}w\| \le k\|u - v\|, \ \forall\, u, v \in H,\\ &w \in \{y : y = hx - \rho[M(Ax, Bx) - N(Cx, Dx)],\ x \in H\};\end{aligned}$$

(vi) $[a(\sqrt{\alpha^2 - 2\rho\mu + \rho^2\beta^2} + k) + \sqrt{1 - 2a\rho\lambda + a^2\rho^2\beta^2} + 2a\rho\gamma] < 1$, where a is a positive constant. Then GIMQVI (2.1) has a unique solution in H.

Proof Let $F : H \to H$ be defined as follows

$$F(u) = u - ahu + aP^{f,\rho}_{K(u)}[hu - \rho(M(Au, Bu) - N(Cu, Du))],\ \forall\, u \in H,$$

where $a > 0$ is a constant. For any $u, v \in H$, we have

$$\begin{aligned}
&\|F(u) - F(v)\|\\
=\ &\|u - ahu + aP^{f,\rho}_{K(u)}[hu - \rho(M(Au, Bu) - N(Cu, Du))]\\
&-(v - ahv + aP^{f,\rho}_{K(v)}[hv - \rho(M(Av, Bv) - N(Cv, Dv))])\|\\
=\ &\|a\{[P^{f,\rho}_{K(u)}(hu - \rho(M(Au, Bu) - N(Cu, Du))) - (hu - \rho(M(Au, Bu) - N(Cu, Du)))]\\
&-[P^{f,\rho}_{K(v)}(hv - \rho(M(Av, Bv) - N(Cv, Dv))) - (hv - \rho(M(Av, Bv) - N(Cv, Dv)))]\}\\
&+u - ahu + a(hu - \rho(M(Au, Bu) - N(Cu, Du)))\\
&-(v - ahv + a(hv - \rho(M(Av, Bv) - N(Cv, Dv))))\|\\
\le\ &a\|hu - \rho(M(Au, Bu) - N(Cu, Du)) - P^{f,\rho}_{K(u)}(hu - \rho(M(Au, Bu) - N(Cu, Du)))\\
&-[hv - \rho(M(Av, Bv) - N(Cv, Dv)) - P^{f,\rho}_{K(v)}(hv - \rho(M(Av, Bv) - N(Cv, Dv)))]\|\\
&+\|u - v - a\rho(M(Au, Bu) - M(Av, Bv))\| + a\rho\|N(Cu, Du) - N(Cv, Dv)\|.
\end{aligned} \tag{3.2}$$

It follows from condition (v) and Theorem 3.3 of [9] that

$$\begin{aligned}
&\|[hu-\rho(M(Au,Bu)-N(Cu,Du))-P^{f,\rho}_{K(u)}(hu-\rho(M(Au,Bu)-N(Cu,Du)))]\\
&-[hv-\rho(M(Av,Bv)-N(Cv,Dv))-P^{f,\rho}_{K(v)}(hv-\rho(M(Av,Bv)-N(Cv,Dv)))]\|\\
\le&\|hu-hv-\rho(M(Au,Bu)-N(Cu,Du)-(M(Av,Bv)-N(Cv,Dv)))\|+k\|u-v\|\\
\le&\|hu-hv-\rho(M(Au,Bu)-M(Av,Bv))\|+\rho\|N(Cu,Du)-N(Cv,Dv)\|+k\|u-v\|.
\end{aligned} \tag{3.3}$$

By conditions (i), (ii), (iv), we get

$$\begin{aligned}
&\|hu-hv-\rho(M(Au,Bu)-M(Av,Bv))\|^2\\
=&\|hu-hv\|^2-2\rho\langle hu-hv,M(Au,Bu)-M(Av,Bv)\rangle\\
&+\rho^2\|M(Au,Bu)-M(Av,Bv)\|^2\\
\le&(\alpha^2-2\rho\mu+\rho^2\beta^2)\|u-v\|^2,
\end{aligned} \tag{3.4}$$

and by conditions (ii) and (iii), we have

$$\begin{aligned}
&\|u-v-a\rho(M(Au,Bu)-M(Av,Bv))\|^2\\
=&\|u-v\|^2-2a\rho\langle u-v,M(Au,Bu)-M(Av,Bv)\rangle\\
&+a^2\rho^2\|M(Au,Bu)-M(Av,Bv)\|^2\\
\le&(1-2a\rho\lambda+a^2\rho^2\beta^2)\|u-v\|^2.
\end{aligned} \tag{3.5}$$

It follows from (3.2)–(3.5) and condition (iv), we have

$$\begin{aligned}
\|F(u)-F(v)\|\le&[a(\sqrt{\alpha^2-2\rho\mu+\rho^2\beta^2}+k+\rho\gamma)\\
&+\sqrt{1-2a\rho\lambda+a^2\rho^2\gamma^2}+a\rho\gamma]\|u-v\|\\
=&[a(\sqrt{\alpha^2-2\rho\mu+\rho^2\beta^2}+k)+\sqrt{1-2a\rho\lambda+a^2\rho^2\gamma^2}+2a\rho\gamma]\|u-v\|\\
=&\theta\|u-v\|,
\end{aligned} \tag{3.6}$$

where $\theta=a(\sqrt{\alpha^2-2\rho\mu+\rho^2\beta^2}+k)+\sqrt{1-2a\rho\lambda+a^2\rho^2\gamma^2}+2a\rho\gamma]$. It follows from condition (vi) that $\theta<1$, therefore F is a contracting mapping in Hilbert space H. So, F has a unique fixed point $u^*\in H$, that is $F(u^*)=u^*$, implying that $u^*\in K(u^*)$ and

$$\langle M(Au^*,Bu^*)-N(Cu^*,Du^*),v-hu^*\rangle+f(v)-f(hu^*)\ge 0,\ \forall\, v\in K(u^*),$$

thus u^* is a unique solution of GIMQVI (2.1). This completes the proof.

If $a=\frac{1}{\rho}$, $M(Au,Bu)=Au$ and $N=0$ for all $u\in H$, then Theorem 3.1 reduces to the following result.

Corollary 3.2 Let H,K,f be same as in Theorem 3.1. $h,A:H\to H$ be Lipschitz continuous with Lipschitz constants α and β, respectively. Assume that

(i) A is λ-strongly monotone and A is μ-h-strongly monotone on H;

(ii) there exists $k > 0$ such that

$$\|P_{K(u)}^{f,\rho}w - P_{K(v)}^{f,\rho}w\| \le k\|u - v\|, \forall\, u, v \in H, w \in \{y : y = hx - \rho Ax,\, x \in H\};$$

(iii) $\sqrt{\beta^2 - 2\rho\mu + \rho^2\alpha^2} + \rho\sqrt{1 - 2\lambda + \alpha^2} < \rho - k.$

Then IMQVI (2.3) has a unique solution in H.

Remark 3.3 Theorem 3.1 extends Theorem 4.1 of [9].

4 Error Bounds for GIMQVI

It is well known that error bounds are closely related to the rate of convergence of algorithms, which play important roles in the study of variational inequality and optimization problems. They allow us to estimate the distance from a feasible element to the solution set even without having computed a single solution of the related variational inequality and optimization problems. In this section, we give two main error bound results for GIMQVI (2.1) by different methods.

By (3.1), let

$$e(u, \rho) = hu - P_{K(u)}^{f,\rho}[hu - \rho(M(Au, Bu) - N(Cu, Du))]$$

denote the residual function. Observe that GIMQVI (2.1) has a solution $\bar{u}$ if and only if $\bar{u}$ is a zero point of $e(u, \rho)$. Now we give the error bounds according to the residual function $e(u, \rho)$.

Theorem 4.1 Let $H, K, M, N, A, B, C, D, h$ be same as in Theorem 3.1 and satisfy conditions (i)–(iv) in Theorem 3.1. If the following conditions hold

(a) N is υ-h-relaxed Lipschitz with respect to A and B;

(b) there exists $0 < k < \frac{\mu+\upsilon}{\beta+\gamma}$ such that for any $\rho > \frac{\alpha k}{\mu+\upsilon-(\beta+\gamma)k}$,

$$\|P_{K(u)}^{f,\rho}w - P_{K(v)}^{f,\rho}w\| \le k\|u - v\|, \forall\, u, v \in H,$$
$$w \in \{y : y = hx - \rho[M(Ax, Bx) - N(Cx, Dx)],\, x \in H\}.$$

If u^* is the solution of GIMQVI (2.1), then for any $u \in H$ and $\rho > \frac{\alpha k}{\mu+\upsilon-(\beta+\gamma)k}$, we have

$$\|u - u^*\| \le \frac{\alpha + \rho(\beta + \gamma)}{(\mu + \upsilon)\rho - [\alpha + \rho(\beta + \gamma)]k}\|e(u, \rho\|.$$

Proof Let $x = P_{K(u^*)}^{f,\rho}[hu - \rho(M(Au, Bu) - N(Cu, Du))]$. Since u^* is the solution of GIMQVI (2.1), then, for all $\rho > 0$, we have

$$\langle \rho(M(Au^*, Bu^*) - N(Cu^*, Du^*)), x - hu^*\rangle + \rho f(x) - \rho f(hu^*) \ge 0. \tag{4.1}$$

From the definition of x and $hu^* \in K(u^*)$, we have

$$\langle x - [hu - \rho(M(Au, Bu) - N(Cu, Du))], hu^* - x\rangle + \rho f(hu^*) - \rho f(x) \ge 0. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}0 \leq &\rho\langle[(M(Au^*,Bu^*)-N(Cu^*,Du^*))-(M(Au,Bu)-N(Cu,Du))]\\&+hu-x,x-hu^*\rangle\\=&\rho\langle[(M(Au^*,Bu^*)-N(Cu^*,Du^*))-(M(Au,Bu)-N(Cu,Du))],x-hu\rangle\\&+\rho\langle[(M(Au^*,Bu^*)-N(Cu^*,Du^*))-(M(Au,Bu)-N(Cu,Du))],hu-hu^*\rangle\\&+\langle hu-x,x-hu\rangle+\langle hu-x,hu-hu^*\rangle.\end{aligned}\tag{4.3}$$

By condition (iv) in Theorem 3.1 and condition (a), we get

$$\begin{aligned}&\langle[(M(Au^*,Bu^*)-N(Cu^*,Du^*))-(M(Au,Bu)-N(Cu,Du))],hu-hu^*\rangle\\=&-\langle M(Au^*,Bu^*)-M(Au,Bu),hu^*-hu\rangle\\&+\langle N(Cu^*,Du^*)-N(Cu,Du))],hu^*-hu\rangle\\\leq&-(\mu+\upsilon)\|u^*-u\|^2.\end{aligned}\tag{4.4}$$

By conditions (i) and (ii) in Theorem 3.1, (4.3) and (4.4) , for any $\rho > \frac{\alpha k}{\mu+\upsilon-(\beta+\gamma)k}$, we obtain that

$$\begin{aligned}&\rho(\mu-\upsilon)\|u^*-u\|^2\\\leq&\rho\langle(M(Au^*,Bu^*)-N(Cu^*,Du^*))-(M(Au,Bu)-N(Cu,Du)),x-hu\rangle\\&+\langle hu-x,hu-hu^*\rangle\\\leq&\rho(\|M(Au^*,Bu^*)-M(Au,Bu)\|+\|N(Cu^*,Du^*)-N(Cu,Du)\|)\|x-hu\|\\&+\|hu-x\|\|hu-hu^*\|\\\leq&[\rho(\beta+\gamma)+\alpha]\|u^*-u\|\|x-hu\|\\=&[\alpha+\rho(\beta+\gamma)]\|u^*-u\|(\|P^{f,\rho}_{K(u^*)}[hu-\rho(M(Au,Bu)-N(Cu,Du))]\\&-P^{f,\rho}_{K(u)}[hu-\rho(M(Au,Bu)-N(Cu,Du))]\|\\&+\|P^{f,\rho}_{K(u)}[hu-\rho(M(Au,Bu)-N(Cu,Du))]-hu\|)\\\leq&[\alpha+\rho(\beta+\gamma)]\|u^*-u\|(k\|u^*-u\|+\|e(u,\rho)\|).\end{aligned}\tag{4.5}$$

Since $(\beta+\gamma)k < \mu+\upsilon$ and $\rho > \frac{\alpha k}{\mu+\upsilon-(\beta+\gamma)k}$, (4.5) implies that

$$\|u-u^*\| \leq \frac{\alpha+\rho(\beta+\gamma)}{(\mu+\upsilon)\rho-[\alpha+\rho(\beta+\gamma)]k}\|e(u,\rho)\|,$$

which is completes the proof.

If $M(Au,Bu)=Au$, $N=0$ for all $u\in H$, from Theorem 4.1, we obtain the following result.

Corollary 4.2 Let H,K,f,h be same as in Theorem 4.1. Assume that

(a1) A is β-Lipschitz continuous and A is μ-h-strongly monotone;

(b1) there exists $0<k<\frac{\mu}{\beta}$ such that for any $\rho > \frac{\alpha k}{\mu-\beta k}$,

$$\|P^{f,\rho}_{K(u)}w-P^{f,\rho}_{K(v)}w\| \leq k\|u-v\|,\ \forall\, u,v\in H, w\in\{y: y=hx-\rho Ax,\ x\in H\}.$$

If u^* is the solution of IMQVI (2.3), then for any $u \in H$ and $\rho > \frac{\alpha k}{\mu - \beta k}$, we have

$$\|u-u^*\| \le \frac{(\alpha+\rho\beta)}{\rho\mu-(\alpha+\rho\beta)k}\|e(u,\rho)\|.$$

If $H = R^n, M(Au, Bu) = Au, N = 0$ and $f(u) = 0$ for all $u \in R^n$, then Theorem 4.1 reduces to the following.

Corollary 4.3 Let $h, A : R^n \to R^n$ be Lipschitz continuous with Lipschitz constants α and β, respectively. Let $K : R^n \to 2^{R^n}$ be set-valued mapping such that for each $u \in R^n, K(u) \subset R^n$ is a nonempty closed convex set. Assume that

(a2) A is μ-h-strongly monotone;

(b2) there exists $0 < k < \frac{\mu}{\beta}$ such that for any $\rho > \frac{\alpha k}{\mu - \beta k}$,

$$\|P^{f,\rho}_{K(u)}w - P^{f,\rho}_{K(v)}w\| \le k\|u-v\|, \forall\, u, v \in R^n, w \in \{y : y = hx - \rho Ax,\, x \in R^n\}.$$

If u^* is the solution of (2.6), then for any $u \in R^n$ and $\rho > \frac{\alpha k}{\mu - \beta k}$, we have

$$\|u-u^*\| \le \frac{(\alpha+\rho\beta)}{\mu\rho-(\alpha+\rho\beta)k}\|e(u,\rho)\|,$$

where $e(u,\rho) = hu - P_{K(u)}[hu - \rho Au]$, $P_{K(\cdot)}w$ is the general projection of w onto the nonempty closed convex subset $K(\cdot)$ of R^n.

Remark 4.4 Theorem 4.1 extends Theorems 5.1 and 5.2 in [9].

For any $u \in H$, by Theorme 3.3 of [9], we know that $P^{f,\rho}_{K(u)}$ is a firmly expansive on H. Applying this property of $P^{f,\rho}_{K(u)}$, we prove another error bound result for GIMQVI (2.1).

Theorem 4.5 Let $H, K, M, N, A, B, C, D, h$ be same as in Theorem 3.1 and satisfy conditions (i)–(iv) in Theorem 3.1 and condition (a) in Theorem 4.1. If the following condition holds

(b3) there exists $0 < k < \frac{\mu+v}{\beta+\gamma}$ such that for any $\rho > \frac{\alpha(\alpha+8k)}{4[\mu+v-k(\beta+\gamma)]}$,

$$\begin{aligned}&\|P^{f,\rho}_{K(u)}w - P^{f,\rho}_{K(v)}w\| \le k\|u-v\|, \forall\, u, v \in H,\\ &w \in \{y : y = hx - \rho[M(Ax, Bx) - N(Cx, Dx)],\, x \in H\}.\end{aligned}$$

If u^* is the solution of GIMQVI (2.1), then for any $u \in H$ and $\rho > \frac{\alpha(\alpha+8k)}{4[\mu+v-k(\beta+\gamma)]}$, we have

$$\|u-u^*\| \le \frac{4(\beta+\gamma)\rho}{4\rho[\mu+v-k(\beta+\gamma)]-\alpha(\alpha+8k)}\|e(u,\rho)\|.$$

Proof Let

$$\begin{aligned}&x = hu - \rho y,\, y = M(Au, Bu) - N(Cu, Du),\\ &x^* = hu^* - \rho y^*,\, y^* = M(Au^*, Bu^*) - N(Cu^*, Du^*).\end{aligned}$$

From the definition of $e(u,\rho)$, we have

$$\begin{aligned}
&\langle e(u,\rho), y-y^*\rangle\\
=&\langle e(u,\rho)-e(u^*,\rho), y-y^*\rangle\\
=&\langle P^{f,\rho}_{K(u^*)}x^*-P^{f,\rho}_{K(u)}x, y-y^*\rangle+\langle hu-hu^*, y-y^*\rangle\\
=&\frac{1}{\rho}\langle P^{f,\rho}_{K(u^*)}x^*-P^{f,\rho}_{K(u)}x, hu-hu^*\rangle+\frac{1}{\rho}\langle P^{f,\rho}_{K(u^*)}x^*-P^{f,\rho}_{K(u)}x, x^*-x\rangle\\
&+\langle hu-hu^*, y-y^*\rangle\\
=&\frac{1}{\rho}\langle P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u)}x, hu-hu^*\rangle-\frac{1}{\rho}\langle P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u^*)}x^*, hu-hu^*\rangle\\
&+\frac{1}{\rho}\langle P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u)}x, x^*-x\rangle-\frac{1}{\rho}\langle P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u^*)}x^*, x^*-x\rangle\\
&+\langle hu-hu^*, y-y^*\rangle.
\end{aligned}\tag{4.6}$$

By conditions (i), (ii), (iv) in Theorem 3.1 and condition (a) in Theorem 4.1, it follows from (4.6) and Theorem 3.1 of [9] that

$$\begin{aligned}
&\langle e(u,\rho), y-y^*\rangle\\
\geq&\frac{1}{\rho}\langle P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u)}x, hu-hu^*\rangle+\frac{1}{\rho}\|P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u)}x\|^2\\
&+(\mu+\upsilon)\|u-u^*\|^2-\frac{1}{\rho}\|P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u^*)}x^*\|\|hu-hu^*\|\\
&-\frac{1}{\rho}\|P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u^*)}x^*\|\|x^*-x\|\\
=&\frac{1}{\rho}\|P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u)}x+\frac{1}{2}(hu^*-hu)\|^2-\frac{1}{4\rho}\|hu-hu^*\|^2+(\mu+\upsilon)\|u-u^*\|^2\\
&-\frac{1}{\rho}\|P^{f,\rho}_{K(u)}x^*-P^{f,\rho}_{K(u^*)}x^*\|^2(\|hu-hu^*\|+\|x^*-x\|)\\
\geq&(\mu+\upsilon)\|u-u^*\|^2-\frac{\alpha^2}{4\rho}\|u^*-u\|^2-\frac{1}{\rho}k(2\|hu-hu^*\|+\rho\|y-y^*\|)\\
\geq&(\mu+\upsilon)\|u-u^*\|^2-\frac{\alpha^2}{4\rho}\|u^*-u\|^2-\frac{1}{\rho}k(2\alpha+\rho(\beta+\gamma))\|u-u^*\|^2
\end{aligned}\tag{4.7}$$

for any $\rho>\frac{\alpha(\alpha+8k)}{4[\mu+\upsilon-k(\beta+\gamma)]}$.

On the other hand, by condition (ii) in Theorem 3.1 again, we have

$$\langle e(u,\rho), y-y^*\rangle\leq\|e(u,\rho)\|\|y-y^*\leq\|(\beta+\gamma)\|e(u,\rho)\|\|u-u^*\|.\tag{4.8}$$

Since $(\beta+\gamma)k<\mu+\upsilon$ and $\rho>\frac{\alpha(\alpha+8k)}{4[\mu+\upsilon-k(\beta+\gamma)]}$, it follows from (4.7) and (4.8) that

$$\|u-u^*\|\leq\frac{4(\beta+\gamma)\rho}{4\rho[\mu+\upsilon-k(\beta+\gamma)]-\alpha(\alpha+8k)}\|e(u,\rho)\|.$$

This completes the proof.

If $M(Au, Bu) = Au$, $N = 0$ for all $u \in H$, from Theorem 4.5, we have the following corollary.

Corollary 4.6 Let H, K, f, h be the same as in Corollary 4.2 and satisfy condition (a1) in Corollary 4.2. Assume that

(b4) there exists $0 < k < \frac{\mu}{\beta}$ such that for any $\rho > \frac{\alpha k}{\mu - \beta k}$,

$$\|P^{f,\rho}_{K(u)} w - P^{f,\rho}_{K(v)} w\| \leq k\|u - v\|, \forall\, u, v \in R^n, w \in \{y : y = hx - \rho Ax,\, x \in R^n\}.$$

If u^* is the solution of IMQVI (2.3), then for any $u \in H$ and $\rho > \frac{\alpha k}{\mu - \beta k}$, we have

$$\|u - u^*\| \leq \frac{(\alpha + \rho\beta)}{\mu\rho - k(\alpha + \rho\beta)} \|e(u, \rho)\|.$$

If $H = R^n$, $M(Au, Bu) = Au, N = 0$ and $f(u) = 0$ for all $u \in R^n$, h is an identity mapping on R^n, then Theorem 4.5 reduces to the following result.

Corollary 4.7 Let H, K, A be the same as in Corollary 4.3. Assume that

(a3) h is μ-strongly monotone on R^n;

(b5) there exists $0 < k < \mu$ such that for any $\rho > \frac{\alpha(8k+\alpha)}{4(\mu-k)}$,

$$\|P^{f,\rho}_{K(u)} w - P^{f,\rho}_{K(v)} w\| \leq k\|u - v\|, \forall\, u, v \in R^n, w \in \{y : y = hx - \rho x,\, x \in R^n\}.$$

If u^* is the solution of (IQVI)(2.6), then for any $u \in R^n$ and any $\rho > \frac{\alpha(\alpha+8k)}{4(\mu-k)}$, we have

$$\|u - u^*\| \leq \frac{4\rho}{4\rho(\mu - k) - \alpha(\alpha + 8k)} \|e(u, \rho)\|,$$

where $e(u, \rho) = hu - P_{K(u)}(hu - \rho u)$, $P_{K(\cdot)} w$ is the general projection of w onto the nonempty closed convex subset $K(\cdot)$ of R^n.

Remark 4.8 Theorem 4.2 extends Theorems 5.3 and 5.4 in [9].

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广义逆混合拟变分不等式解的存在性与误差界

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摘 要: 本文引入并研究希尔伯特空间中一类新的广义逆混合拟变分不等式问题 (GIMQVI). 利用广义投影算子的性质, 得到了GIMQVI解的存在性和唯一性结果, 而且得到了利用剩余函数刻画的GIMQVI 的误差界. 本文得到的结果推广和改进了近期文献的一些结果.

关键词: 广义逆混合拟变分不等式; 广义投影算子; 误差界; 剩余函数

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