Vol. 39 ( 2019 )
No. 1

数 学 杂 志
J. of Math. (PRC)

# FORCED OSCILLATION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH DAMPING TERM

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**Abstract:** In this paper, we study the forced oscillation of a fractional partial differential equation with damping term subject to Robin boundary condition. Using an integration average technique and the properties of the Riemann-Liouville calculus, we obtain some new oscillation criteria for the fractional partial differential equations, which are the generalization of some classical results involving partial differential equations. Two examples are given to show the applications of our main results.

**Keywords:** fractional partial differential equations; forced oscillation; Riemann-Liouville fractional calculus

**2010 MR Subject Classification:** 35B05; 35R11

**Document code:** A **Article ID:** 0255-7797(2019)01-0111-10

## 1 Introduction

Fractional differential equations (FDE) played an important role in the modeling of many phenomena in various fields such as viscoelasticity, electroanalytical chemistry, control theory, many physics problems, etc. In the past few years, many articles investigated some aspects of fractional differential equations, such as the existence, the uniqueness and stability of solutions, the methods for explicit and numerical solutions, see for example, the books [1–5]. Recently, the research on oscillation of various fractional differential equations was a hot topic, see [6–9]. However, to author's knowledge, very little is known regarding the oscillatory behavior of fractional partial differential equations up to now, see [10–14]. In [13], by using the generalized Riccati transformation and the properties of fractional calculus, the author considered the forced oscillation of a fractional partial differential equation of the

* **Received date:** 2017-05-17 **Accepted date:** 2017-10-18

**Foundation item:** Supported by National Natural Science Foundation of China (11601496; 41630643).

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form

$$\begin{aligned}&\frac{\partial}{\partial t}\left(D^{\alpha}_{0+,t}u(x,t)\right)+p(t)D^{\alpha}_{0+,t}u(x,t)\\=&a(t)\Delta u(x,t)-q(x,t)u(x,t)+g(x,t),\ (x,t)\in\Omega\times R_+\equiv G\end{aligned}$$

with two boundary conditions

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial N}=\psi(x,t),\ \ (x,t)\in\partial\Omega\times R_+,\\&u(x,t)=0,\ \ (x,t)\in\partial\Omega\times R_+,\end{aligned}$$

where $R_+=[0,\infty),\alpha\in(0,1)$ is a constant, $D^{\alpha}_{0+,t}u(x,t)$ is the Riemann-Liouville fractional derivative of order $\alpha$ with respect to $t$ of a function $u(x,t)$.

In this paper, we use only the properties of fractional calculus without the generalized Riccati transformation to consider the forced oscillation of the fractional partial differential equation with damping term of this form

$$\begin{aligned}&\frac{\partial}{\partial t}\left(r(t)D^{\alpha}_{0+,t}u(x,t)\right)+p(t)D^{\alpha}_{0+,t}u(x,t)+q(x,t)f(u(x,t))\\=&a(t)\Delta u(x,t)+\tilde{g}(x,t),\ (x,t)\in D\end{aligned}\tag{1.1}$$

with the boundary condition

$$\frac{\partial u(x,t)}{\partial N}+\beta(x,t)u(x,t)=0,\ \ (x,t)\in\tilde{D},\tag{1.2}$$

where $\Omega$ is a bounded domain in $R^n$ with a piecewise smooth boundary $\partial\Omega,\Delta$ is the Laplacian in $R^n,N$ is a unit exterior normal vector to $\partial\Omega,\beta(x,t)$ is a continuous nonnegative function on $\tilde{D}$; $\alpha\in(0,1)$ is a constant, $\tilde{g}(x,t)$ is the forced term of the equation.

Set $R^+=(0,\infty),D=\Omega\times R^+,\tilde{D}=\partial\Omega\times R^+,\overline{D}=\overline{\Omega}\times R^+$.

We assume throughout this paper that

A1) $r(t)\in C^1(R^+,R^+),\ a(t)\in C(R^+,R^+),\ p(t)\in C(R^+,R)$;

A2) $\tilde{g}(x,t)\in C(D,R),\ q(x,t)\in C(\overline{D},R^+)$ and $\min\limits_{x\in\overline{\Omega}}q(x,t)=Q(t)$;

A3) $f(u)\in C(R,R)$ for all $u\neq0,\ \frac{f(u)}{u}\geq k,k$ is a positive constant.

**Definition 1.1** By a solution of problem (1.1)–(1.2), we mean a function $u(x,t)$ which satisfies (1.1) and the boundary condition (1.2).

**Definition 1.2** A solution of problem (1.1)–(1.2) is said to be oscillatory in $D$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory.

## 2 Preliminaries

In this section, we introduce the definitions and properties of fractional integrals and derivatives, which are useful throughout this paper. There are several kinds of definitions of fractional integrals and derivatives [2]. In this paper, we use Riemann-Liouville definition.

**Definition 2.1** The Riemann-Liouville fractional partial derivative of order $\alpha \in (0,1)$ with respect to $t$ of a function $u(x,t)$ is defined by

$$(D^{\alpha}_{0+,t}u)(x,t) = \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(x,s)ds,\ t>0, \tag{2.1}$$

provided the right hand side is pointwise defined on $R^+$, where $\Gamma(z)$ is the Gamma function defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$$

for $z>0$.

**Definition 2.2** The Riemann-Liouville fractional integral of order $\alpha \in R^+$ of a function $y(t)$ is defined by

$$(I^{\alpha}_{0+}y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,\ t>0, \tag{2.2}$$

provided the right hand side is pointwise defined on $R^+$.

**Definition 2.3** The Riemann-Liouville fractional derivative of order $\alpha \in R^+$ of a function $y(t)$ is defined by

$$(D^{\alpha}_{0+}y)(t) = \frac{d^n}{dt^n}(I^{n-\alpha}_{0+}y)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds,\ t>0 \tag{2.3}$$

with $n=[\alpha]+1$, where $[\alpha]$ means the integer part of $\alpha$.

**Lemma 2.4** Let $\alpha>0, m\in\mathbb{N}$ and $D=d/dx$. If the fractional derivatives $(D^{\alpha}_{0+}y)(t)$ and $(D^{\alpha+m}_{0+}y)(t)$ exist, then

$$(D^m D^{\alpha}_{0+}y)(t) = (D^{\alpha+m}_{0+}y)(t). \tag{2.4}$$

**Lemma 2.5** Let

$$F(t) = \int_0^t (t-s)^{-\alpha}y(s)ds,\ \ \alpha\in(0,1),\ \ t>0, \tag{2.5}$$

then

$$F'(t) = \Gamma(1-\alpha)(D^{\alpha}_{0+}y)(t).$$

## 3 Main Results

For convenience, we introduce the following notations

$$v(t) = \int_{\Omega} u(x,t)dx,\ \ \omega(t) = \exp\int_{t_0}^t \frac{r'(s)+p(s)}{r(s)}ds,\ \ G(t) = \int_{\Omega}\tilde{g}(x,t)dx.$$

**Lemma 3.1** If $u(x,t)$ is a positive solution of problems (1.1)–(1.2) in the domain $D$, then $v(t)$ satisfies the fractional differential inequality

$$\frac{d}{dt}(D^{\alpha}_{0+}v(t)\omega(t)) \le \frac{\omega(t)}{r(t)}(G(t)-kQ(t)v(t)). \tag{3.1}$$

**Proof** Let $u(x,t)$ is a positive solution of problem (1.1)–(1.2) in the domain $D$, then there exists $t_0 > 0$, such that $u(x,t) > 0$ in $\Omega \times [t_0, \infty)$. Integrating (1.1) with respect to $x$ over $\Omega$ yields

$$\begin{aligned}&\frac{d}{dt}(r(t)\int_\Omega D^\alpha_{0+,t}u(x,t)dt) + p(t)\int_\Omega D^\alpha_{0+,t}u(x,t)dx + \int_\Omega q(x,t)f(u(x,t))dx\\ =&a(t)\int_\Omega \Delta u(x,t)dx + \int_\Omega \tilde{g}(x,t)dx.\end{aligned}$$

From A2) and A3), it is easy to see that

$$\int_\Omega q(x,t)f(u(x,t))dx \geq kQ(t)\int_\Omega u(x,t)dx = kQ(t)v(t).$$

Green's formula and the boundary condition (1.2) yield

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega}\frac{\partial u}{\partial N}dS = -\int_{\partial\Omega}\beta(x,t)u(x,t)ds \leq 0,$$

where $dS$ is the surface element on $\partial\Omega$. It shows that $v(t)$ satisfies the inequality

$$(r(t)D^\alpha_{0+}v(t))' + p(t)D^\alpha_{0+}v(t) + kQ(t)v(t) \leq G(t). \tag{3.2}$$

Using Lemma 2.4 and inequality (3.2), we obtain

$$\begin{aligned} r(t)(D^\alpha_{0+}v(t)\omega(t))' &= r(t)D^{1+\alpha}_{0+}v(t)\omega(t) + D^\alpha_{0+}v(t)\omega(t)(r'(t)+p(t))\\ &= \omega(t)\{r(t)D^{1+\alpha}_{0+}v(t) + r'(t)D^\alpha_{0+}v(t) + p(t)D^\alpha_{0+}v(t)\}\\ &= \omega(t)\{(r(t)(D^\alpha_{0+}v(t))' + p(t)D^\alpha_{0+}v(t)\}\\ &\leq \omega(t)\{G(t) - kQ(t)v(t)\},\end{aligned}$$

which shows that $v(t)$ is a positive solution of inequality (3.1). The proof is completed.

**Lemma 3.2** If $u(x,t)$ is a negative solution of problems (1.1)–(1.2) in the domain $D$, then $v(t)$ satisfies the fractional differential inequality

$$\frac{d}{dt}(D^\alpha_{0+}v(t)\omega(t)) \geq \frac{\omega(t)}{r(t)}(G(t) - kQ(t)v(t)). \tag{3.3}$$

**Proof** Let $u(x,t)$ is a negative solution of problems (1.1)–(1.2) in the domain $D$, then there exists $\bar{t}_0 > 0$, such that $u(x,t) < 0$ in $\Omega \times [\bar{t}_0, \infty)$. Integrating (1.1) with respect to $x$ over $\Omega$ yields

$$\begin{aligned}&\frac{d}{dt}(r(t)\int_\Omega D^\alpha_{0+,t}u(x,t)dt) + p(t)\int_\Omega D^\alpha_{0+,t}u(x,t)dx + \int_\Omega q(x,t)f(u(x,t))dx\\ =&a(t)\int_\Omega \Delta u(x,t)dx + \int_\Omega \tilde{g}(x,t)dx.\end{aligned}$$

From A2) and A3), it is easy to see that

$$\int_\Omega q(x,t)f(u(x,t))dx \leq kQ(t)\int_\Omega u(x,t)dx = kQ(t)v(t).$$

Green's formula and the boundary condition (1.2) yield

$$\int_{\Omega} \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} dS = -\int_{\partial\Omega} \beta(x,t)u(x,t)ds \geq 0,$$

where $dS$ is the surface element on $\partial\Omega$. It shows that $v(t)$ satisfies the inequality

$$(r(t)D_{0+}^{\alpha}v(t))' + p(t)D_{0+}^{\alpha}v(t) + kQ(t)v(t) \geq G(t). \tag{3.4}$$

Using Lemma 2.4 and inequality (3.4), we obtain

$$\begin{aligned} r(t)(D_{0+}^{\alpha}v(t)\omega(t))' &= r(t)D_{0+}^{1+\alpha}v(t)\omega(t) + D_{0+}^{\alpha}v(t)\omega(t)(r'(t)+p(t)) \\ &= \omega(t)\{r(t)D_{0+}^{1+\alpha}v(t) + r'(t)D_{0+}^{\alpha}v(t) + p(t)D_{0+}^{\alpha}v(t)\} \\ &= \omega(t)\{(r(t)(D_{0+}^{\alpha}v(t))' + p(t)D_{0+}^{\alpha}v(t)\} \\ &\geq \omega(t)\{G(t) - kQ(t)v(t)\}, \end{aligned}$$

which shows that $v(t)$ is negative solution of inequality (3.3). The proof is completed.

**Theorem 3.3** If inequality (3.1) has no eventually positive solutions and the inequality (3.3) has no eventually negative solutions, then every solution of problems (1.1)–(1.2) is oscillatory in $D$.

**Proof** Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of problems (1.1)–(1.2). It is obvious that there exists $\tilde{t}_0$ such that $u(x,t) > 0$ or $u(x,t) < 0$ for $t \geq \tilde{t}_0$.

If $u(x,t) > 0, t \geq \tilde{t}_0$, by using Lemma 3.1, we obtain that $v(t) > 0$ is a solution of inequality (3.1), which is a contradiction.

If $u(x,t) < 0, t \geq \tilde{t}_0$, by using Lemma 3.2, we obtain that $v(t) < 0$ is a solution of inequality (3.3), which is a contradiction. The proof is completed.

**Lemma 3.4** If

$$\liminf_{t\to+\infty} \int_{t_1}^{t} \frac{M + \int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds}{\omega(\rho)} d\rho = -\infty, \tag{3.5}$$

then inequality (3.1) has no eventually positive solutions.

**Proof** Suppose to the contrary that (3.1) has a positive solution $v(t)$, then there exists $t_1 \geq t_0$ such that $v(t) > 0,\ t \geq t_1$. Integrating both sides of (3.1) from $t_1$ to $t$, we obtain

$$\begin{aligned} (D_{0+}^{\alpha}v(t))\omega(t) &\leq (D_{0+}^{\alpha}v(t_1))\omega(t_1) + \int_{t_1}^{t} \frac{\omega(s)}{r(s)}(G(s) - kQ(s)v(s))ds \\ &< (D_{0+}^{\alpha}v(t_1))\omega(t_1) + \int_{t_1}^{t} \frac{\omega(s)}{r(s)} G(s)ds \\ &= M + \int_{t_1}^{t} \frac{\omega(s)}{r(s)} G(s)ds, \end{aligned}$$

where $M=(D_{0+}^{\alpha}v(t_1))\omega(t_1)$. Using Lemma 2.5, we have

$$\frac{F'(t)}{\Gamma(1-\alpha)}=D_{0+}^{\alpha}v(t)\le\frac{M+\displaystyle\int_{t_1}^{t}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(t)},\ \ t\ge t_1. \tag{3.6}$$

Integrating (3.6) from $t_1$ to $t$, we obtain

$$F(t)\le F(t_1)+\Gamma(1-\alpha)\int_{t_1}^{t}\frac{M+\displaystyle\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho. \tag{3.7}$$

Taking $t\to\infty$, from (3.7), we have

$$\liminf_{t\to+\infty}F(t)=-\infty,$$

which contradicts the conclusion that $v(t)>0$. The proof is completed.

**Lemma 3.5** If

$$\limsup_{t\to+\infty}\int_{t_1}^{t}\frac{M+\displaystyle\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho=+\infty, \tag{3.8}$$

then inequality (3.3) has no eventually negative solutions.

Using Theorem 3.3, Lemma 3.4 and Lemma 3.5, we immediately obtain the following theorem.

**Theorem 3.6** If (3.5) and (3.8) hold, then every solution of problems (1.1)–(1.2) is oscillatory in $D$.

## 4 Example

**Example 4.1** Consider the fractional partial differential equation

$$\frac{\partial}{\partial t}\left(\sin tD_{0+,t}^{\frac{1}{2}}u(x,t)\right)+\cos tD_{0+,t}^{\frac{1}{2}}u(x,t)+t^2ue^u=t^3\Delta u(x,t)+\frac{e^t\sin x}{4},$$
$$(x,t)\in(0,\pi)\times(0,+\infty) \tag{4.1}$$

with the boundary condition

$$u_x(0,t)=u_x(\pi,t)=0,\ \ t>0, \tag{4.2}$$

here

$$n=1,\alpha=\frac{1}{2},r(t)=\sin t,p(t)=\cos t,Q(t)=q(x,t)=t^2,$$
$$f(u)=ue^u,a(t)=t^3,\tilde{g}(x,t)=\frac{e^t\sin x}{4}.$$

Set $t_0 = t_1 = \frac{\pi}{4}$, it is obvious that

$$\omega(t) = 2\sin^2 t, G(t) = \frac{1}{2}e^t, \omega(t_1) = 1,$$
$$\int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds = \int_{t_1}^{\rho} e^s \sin s ds = \frac{1}{\sqrt{2}} e^{\rho} \sin(\rho - \frac{\pi}{4}).$$

Hence

$$\begin{aligned}
\int_{t_1}^{t} \frac{M + \int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds}{\omega(\rho)} d\rho &= \int_{t_1}^{t} \frac{M + \frac{1}{\sqrt{2}} e^{\rho} \sin(\rho - \frac{\pi}{4})}{2\sin^2 \rho} d\rho \\
&= \frac{M}{2} \int_{t_1}^{t} \frac{1}{\sin^2 \rho} d\rho + \frac{1}{2\sqrt{2}} \int_{t_1}^{t} \frac{e^{\rho} \sin(\rho - \frac{\pi}{4})}{\sin^2 \rho} d\rho \\
&= \frac{M}{2}(1 - \cot t) + \frac{e^t}{4\sin t} - \frac{\sqrt{2}}{4} e^{\frac{\pi}{4}}.
\end{aligned}$$

Select sequence $\{t_k\} = \{2k\pi + \frac{\pi}{4}\}$, then

$$\lim_{k\to\infty} \int_{t_1}^{t_k} \frac{M + \int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds}{\omega(\rho)} d\rho = \lim_{k\to\infty} \frac{\sqrt{2}}{4} \left( e^{(2k+\frac{1}{4})\pi} - e^{\frac{\pi}{4}} \right) = +\infty. \tag{4.3}$$

Similarly, select sequence $\{t_j\} = \{2j\pi - \frac{\pi}{4}\}$,

$$\lim_{j\to\infty} \int_{t_1}^{t_j} \frac{M + \int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds}{\omega(\rho)} d\rho = \lim_{j\to\infty} \left( M - \frac{\sqrt{2}}{4} \left( e^{(2j-\frac{1}{4})\pi} + e^{\frac{\pi}{4}} \right) \right) = -\infty. \tag{4.4}$$

From (4.3), (4.4), we have

$$\liminf_{t\to+\infty} \int_{t_1}^{t} \frac{M + \int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds}{\omega(\rho)} d\rho = -\infty,$$
$$\limsup_{t\to+\infty} \int_{t_1}^{t} \frac{M + \int_{t_1}^{\rho} \frac{\omega(s)}{r(s)} G(s)ds}{\omega(\rho)} d\rho = +\infty,$$

which shows that all the conditions of Theorem 3.6 are fulfilled. Then every solution of problems (4.1)–(4.2) is oscillatory in $(0, \pi) \times R^+$.

**Example 4.2** Consider the fractional partial differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left( D_{0+,t}^{\frac{1}{2}} u(x,t) \right) - D_{0+,t}^{\frac{1}{2}} u(x,t) + t^2 u e^u = t^3 \Delta u(x,t) + e^{2t} \sin t \sin x, \\
&(x,t) \in (0,\pi) \times (0,+\infty)
\end{aligned} \tag{4.5}$$

with the boundary condition (4.2), where

$$n=1,\alpha=\frac{1}{2},r(t)=1,p(t)=-1,Q(t)=q(x,t)=t^2,$$
$$f(u)=ue^u,a(t)=t^3,\tilde{g}(x,t)=e^{2t}\sin t\sin x.$$

Set $t_0=t_1=\frac{\pi}{4}$, it is obvious that

$$\omega(t)=e^{\frac{\pi}{4}-t},\ G(t)=2e^{2t}\sin t,\ \omega(t_1)=1,$$
$$\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds=2e^{\frac{\pi}{4}}\int_{t_1}^{\rho}e^s\sin sds=\sqrt{2}e^{\frac{\pi}{4}}e^{\rho}\sin(\rho-\frac{\pi}{4}).$$

Hence

$$\begin{aligned}&\int_{t_1}^{t}\frac{M+\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho=\int_{t_1}^{t}\frac{M+\sqrt{2}e^{\frac{\pi}{4}}e^{\rho}\sin(\rho-\frac{\pi}{4})}{e^{\frac{\pi}{4}-\rho}}d\rho\\ =&Me^{-\frac{\pi}{4}}\int_{t_1}^{t}e^{\rho}d\rho+\sqrt{2}\int_{t_1}^{t}e^{2\rho}\sin(\rho-\frac{\pi}{4})d\rho\\ =&Me^{-\frac{\pi}{4}}(e^t-e^{\frac{\pi}{4}})+\frac{\sqrt{2}}{5}[2\sin(t-\frac{\pi}{4})e^{2t}-\cos(t-\frac{\pi}{4})e^{2t}+e^{\frac{\pi}{2}}].\end{aligned}$$

Select sequence $\{t_k\}=\{2k\pi+\frac{3\pi}{4}\}$, then

$$\begin{aligned}&\lim_{k\to\infty}\int_{t_1}^{t_k}\frac{M+\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho\\ =&\lim_{k\to\infty}\left(Me^{-\frac{\pi}{4}}(e^{(2k+\frac{3}{4})\pi}-e^{\frac{\pi}{4}})+\frac{\sqrt{2}}{5}(2e^{(4k+\frac{3}{2})\pi}+e^{\frac{\pi}{2}})\right)\\ =&+\infty.\end{aligned}\tag{4.6}$$

Similarly, select sequence $\{t_j\}=\{2j\pi+\frac{\pi}{4}\}$,

$$\begin{aligned}&\lim_{j\to\infty}\int_{t_1}^{t_j}\frac{M+\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho\\ =&\lim_{j\to\infty}\left(Me^{-\frac{\pi}{4}}(e^{(2j+\frac{1}{4})\pi}-e^{\frac{\pi}{4}})+\frac{\sqrt{2}}{5}(-e^{(4j+\frac{1}{2})\pi}+e^{\frac{\pi}{2}})\right)\\ =&-\infty.\end{aligned}\tag{4.7}$$

From (4.6), (4.7), we have

$$\liminf_{t\to+\infty}\int_{t_1}^{t}\frac{M+\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho=-\infty,$$
$$\limsup_{t\to+\infty}\int_{t_1}^{t}\frac{M+\int_{t_1}^{\rho}\frac{\omega(s)}{r(s)}G(s)ds}{\omega(\rho)}d\rho=+\infty,$$

which shows that all the conditions of Theorem 3.6 are fulfilled. Then every solution of problem (4.5) with (4.2) is oscillatory in $(0,\pi)\times R^+$.

**Remark** In this paper, we did not mention oscillation of fractional partial differential equation with time delay. Actually, we have considered the following equation

$$\begin{aligned}&\frac{\partial}{\partial t}\left(r(t)D^{\alpha}_{0+,t}u(x,t)\right)+p(t)D^{\alpha}_{0+,t}u(x,t)+q(x,t)f(u(x,t-\tau))\\=&a(t)\Delta u(x,t-\delta)+\tilde{g}(x,t),\ (x,t)\in D\end{aligned}\tag{4.8}$$

with the boundary condition (1.2), where $\tau,\delta$ are nonnegative constants and conditions A1)–A3) are satisfied. The conclusion is that if (3.5) and (3.8) hold, then every solution of problem (4.8) with (1.2) is oscillatory in $D$. That means time delays $\tau$ and $\delta$ have no effect on oscillatory property.

However, we have not studied the fractional partial differential equations with time delays which are on $D^{1+\alpha}_{0+,t}u(x,t)$ or $D^{\alpha}_{0+,t}u(x,t)$, since it is more complicated than discussion in this paper. In the future, we would like to discuss this case and hope to acquire desired results.

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# 带阻尼项的分数阶偏微分方程的强迫振动性

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**摘 要**: 本文研究了一类带阻尼项的分数阶偏微分方程在Robin边界条件下的强迫振动性. 利用积分平均值方法和Riemann-Liouville微积分的一些特殊性质, 得到了强迫振动新的准则, 推广了偏微分方程强迫振动的一些经典结论.

**关键词**: 分数阶微分方程; 强迫振动; Riemann-Liouville分数阶微积分

MR(2010)**主题分类号**: 35B05; 35R11 **中图分类号**: O175