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# 一维线性 Klein-Gordon 方程 Neumann 边值问题的高阶差分格式

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**摘 要**: 本文主要研究非线性 Klein-Gordon 方程 Neumann 边值问题的高阶差分格式. 利用边界条件及非线性 Klein-Gordon 方程, 得到其在空间上的三阶与五阶导数的边界值, 进而分别在内点和边界点建立三点和两点紧差分格式. 借助能量估计、Gronwall 和 Schwarz 不等式、数学归纳法等技巧进行分析, 得到截断误差是关于时间和空间上的二阶和四阶收敛. 通过理论分析差分格式的收敛性和稳定性以及数值算例, 验证了理论分析结果.

**关 键 词**: 非线性 Klein-Gordon 方程; 紧差分格式; 收敛性; 稳定性; 高精度

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## 1 引言

本文研究如下一维非线性 Klein-Gordon 方程 Neumann 的边值问题的数值解

$$\frac{\partial^2 u}{\partial t^2}-\alpha\frac{\partial^2 u}{\partial x^2}+g(u)=f(x,t),\quad a<x<b,\ 0<t\le T, \tag{1.1}$$

$$u(x,0)=\varphi(x),\ \frac{\partial u}{\partial t}(x,0)=\psi(x),\quad a\le x\le b, \tag{1.2}$$

$$\frac{\partial u}{\partial x}(a,t)=0,\ \frac{\partial u}{\partial x}(b,t)=0,\quad 0\le t\le T, \tag{1.3}$$

其中 $\alpha>0$ 为常数, $f(x,t)$, $g(u)$ 为满足相容性条件的光滑函数.

Klein-Gordon 方程是相对论量子力学和量子场论中用于描述零自旋粒子的自由运动方程, 关于它的数值解法已有不少研究结果. 文献 [1] 基于样条基函数提出了一个数值格式; 四阶紧格式在文献 [2] 中进行了研究; 基于变分迭代法的数值格式及边界元方法可参见文献 [3, 4]; 在文献 [5] 中导出了以三层样条差分格式逼近非线性 Klein-Gordon 方程; 无界域上的问题的数值研究可参见文献 [6]; 文献 [7] 中提出了一个基于有限差分和匹配法的新的数值格式; 而文献 [8] 提出了微分积分法. 所有这些文献中的研究都是针对 Dirichlet 边界条件, 对于 Neumann 边界条件下的高阶差分格式还没有很好的结果. 近年来, 具有 Neumann 边界条件的热方程的高阶差分格式已有一些研究结果如文献 [9–11]. 文献 [12] 研究了 Cahn-Hilliard 方程 Neumann 边界条件下的三层线性化紧格式; 文献 [13] 中建立了哈密尔顿非线性波方

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程 Neumann 边界条件下的高阶显格式, 该方法空间方向基于紧格式, 时间方向基于 Runge-Kutta-Nystrom 方法. 通过分析以上文献, 了解到 Klein-Gordon 方程 Neumann 边值问题的无条件稳定的高阶差分格式, 目前还没有这方面的结果, 其主要困难是边界点的处理. 本文利用 Klein-Gordon 方程及边界条件可得到在边界处的三阶导数和五阶导数的函数值, 从而建立边界点和内点处的两点和三点紧差分格式, 构造一个紧格式, 并证明差分格式关于时间 2 阶收敛, 关于空间 4 阶收敛.

## 2 记号及引理

设问题 (1.1)–(1.3) 存在光滑解 $u$, 记 $\Omega=[a,b]\times[0,T]$, 本文假设

H1: $u\in C^{(4,3)}(\Omega)$, 且存在常数 $C_0$, 使 $\forall(x,t)\in(\Omega)$ 有

$$\max(|\frac{\partial^4 u}{\partial t^4}|,|\frac{\partial^4 u}{\partial x^2\partial t^2}|,|\frac{\partial^4 u}{\partial x\partial t^3}|,|\frac{\partial^3 u}{\partial x^3}|,|u|)\le C_0.$$

H2: 函数 $g$ 一阶可导, 且存在正常数 $C$, 使得当 $|s|\le C_0+1$ 时, 有

$$\max\{|g(s)|,|g'(s)|,|g''(s)|\}\le C.$$

H3: $\forall(x,t)\in\Omega$ , 且存在常数 $C_1$, 使 $\max|\frac{\partial f}{\partial x}(x,t)|=C_1$ .

由 (1.1) 式中的方程可得

$$\alpha\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}+g(u)-f(x,t). \tag{2.1}$$

将 (2.1) 式两边关于 $x$ 的 $k$ ($k=3,4,5$) 阶导数, 且由边界条件 (1.3) 可得

$$\begin{aligned}
&\frac{\partial^3 u}{\partial x^3}(a,t)=-\frac{1}{\alpha}\frac{\partial f}{\partial x}(a,t),\ \frac{\partial^3 u}{\partial x^3}(b,t)=-\frac{1}{\alpha}\frac{\partial f}{\partial x}(b,t),\\
&\frac{\partial^5 u}{\partial x^5}(a,t)=-\left[\frac{1}{\alpha}\frac{\partial^3 f}{\partial x^3}(a,t)+\frac{g'(u)}{\alpha^2}\frac{\partial f}{\partial x}(a,t)+\frac{1}{\alpha^2}\frac{\partial^3 f}{\partial t^2\partial x}(a,t)\right],\\
&\frac{\partial^5 u}{\partial x^5}(b,t)=-\left[\frac{1}{\alpha}\frac{\partial^3 f}{\partial x^3}(b,t)+\frac{g'(u)}{\alpha^2}\frac{\partial f}{\partial x}(b,t)+\frac{1}{\alpha^2}\frac{\partial^3 f}{\partial t^2\partial x}(b,t)\right].
\end{aligned}$$

取正整数 $m,n$, 记空间和时间步长分别为 $h=\dfrac{b-a}{m}$, $\tau=\dfrac{T}{n}$, 记 $x_i=a+ih$, $0\le i\le m$, $t_k=k\tau$, $0\le k\le n$. 定义

$$\Omega_h=\{x_i|0\le i\le m\},\ \ \Omega_{h\tau}=\{(x_i,t_k)|0\le i\le m,0\le k\le n\},$$

称 $(x_i,t_k)$ 为节点, 并设 $\{v_i^k|0\le i\le m,0\le k\le n\}$ 为 $\Omega_{h\tau}$ 上的网格函数, 引进下面的记号

$$\begin{aligned}
&v_i^{k+1/2}=\frac{1}{2}(v_i^k+v_i^{k+1}),\quad v_i^{\bar k}=\frac{1}{2}(v_i^{k+1}+v_i^{k-1}),\qquad \delta_t v_i^{k+1/2}=\frac{1}{\tau}(v_i^{k+1}-v_i^k),\\
&\delta_x v_{i+1/2}^k=\frac{1}{h}(v_{i+1}-v_i^k),\quad \delta_x^2 v_i^k=\frac{1}{h^2}(v_{i+1}^k-2v_i^k+v_{i-1}^k),\quad \delta_t^2 v_i^k=\frac{1}{\tau^2}(v_i^{k+1}-2v_i^k+v_i^{k-1}),\\
&D_{\hat t}v_i^k=\frac{1}{2\tau}(v_i^{k+1}-v_i^{k-1}).
\end{aligned}$$

记 $V_h=\{v|v=\{v_i|0\le i\le m\}$ 为 $\Omega_h$ 上的网格函数$\}$, 设 $v\in V_h$, 引进下列网格函数的模与半模

$$\|v\|=\sqrt{\frac{h}{2}(v_0^2+2\sum_{i=1}^{m-1}v_i^2+v_m^2)},\ |v|_1=\sqrt{\frac{1}{h}\sum_{i=1}^{m-1}(v_i-v_{i-1})^2},\ \|v\|_\infty=\max_{0\le i\le m}|v_i|.$$

**引理 2.1** [3] 记 $\zeta(s)=(1-s)^3[5-3(1-s)^2], s\in[0,1]$.

(1) 若 $g(x)\in C^6[x_0,x_1]$, 则有

$$\begin{aligned}&\frac{1}{6}[5g''(x_0)+g''(x_1)]-\frac{2}{h}\left[\frac{g(x_1)-g(x_0)}{h}-g'(x_0)\right]\\&=-\frac{h}{6}g'''(x_0)+\frac{h^3}{90}g^{(5)}(x_0)+\frac{h^4}{180}\int_0^1 g^{(6)}(x_0+sh)\zeta(s)ds.\end{aligned}$$

(2) 若 $g(x)\in C^6[x_{m-1},x_m]$, 则有

$$\begin{aligned}&\frac{1}{6}[g''(x_{m-1})+5g''(x_m)]-\frac{2}{h}\left[g'(x_m)-\frac{g(x_m)-g(x_{m-1})}{h}\right]\\&=-\frac{h}{6}g'''(x_m)+\frac{h^3}{90}g^{(5)}(x_m)+\frac{h^4}{180}\int_0^1 g^{(6)}(x_m-sh)\zeta(s)ds.\end{aligned}$$

(3) 若 $g(x)\in C^6[x_{i-1},x_{i+1}], 1\le i\le m-1$, 则有

$$\begin{aligned}&\frac{1}{12}[g''(x_{i-1})+10g''(x_i)+g''(x_{i+1})]-\frac{1}{h^2}[g(x_{i-1})-2g(x_i)+g(x_{i+1})]\\=&\frac{h^4}{360}\int_0^1[g^{(6)}(x_i+sh)+g^{(6)}(x_i-sh)]\zeta(s)ds.\end{aligned}$$

**引理 2.2** [5] 设 $v\in V_h$, 则有 $|v|_1^2\le\frac{4}{h^2}\|v\|^2$, 且对任意 $\varepsilon>0$, 有

$$\|v\|_\infty^2\le\varepsilon|v|_1^2+\left(\frac{1}{\varepsilon}+\frac{1}{b-a}\right)\|v\|^2.$$

**引理 2.3**[14] 设 $h$ 和 $c$ 为两个常数, 且 $h>0$, 若 $g(x)\in C^2[c-h,c+h]$, 则有

$$g(c)=\frac{1}{2}[g(c-h)+g(c+h)]-\frac{h^2}{2}g''(\xi_0),\xi_0\in(c-h,c+h).$$

**引理 2.4** [15] 设 $\{v_i^k|0\le i\le m,0\le k\le n\}$ 为 $\Omega_{h\tau}$ 上的网格函数, 则

$$\|v^k\|^2\le 2k\tau^2\sum_{l=0}^{k-1}\|\delta_t v^{l+1/2}\|^2+2\|v^0\|^2.$$

**引理 2.5** 设 $\{v_i^k|0\le i\le m,0\le k\le n\}$ 为 $\Omega_{h\tau}$ 上的网格函数, 记

$$E^k=\frac{5}{6}\|\delta_t v^{k+1/2}\|^2+\frac{h}{6}\sum_{i=0}^{m-1}(\delta_t v_i^{k+1/2})(\delta_t v_{i+1}^{k+1/2})+\frac{\alpha}{2}(|v^{k+1}|_1^2+|v^k|_1^2),\ 0\le k\le n-1,$$

则当 $0 \le k \le n-1$ 时, 有

$$\frac{2}{3}\|\delta_t v^{k+1/2}\|^2+\frac{\alpha}{2}(|v^{k+1}|_1^2+|v^k|_1^2) \le E^k \le \|\delta_t v^{k+1/2}\|^2+\frac{\alpha}{2}(|v^{k+1}|_1^2+|v^k|_1^2).$$

**证** 由不等式

$$h\sum_{i=0}^{m-1}\left|(\delta_t v_i^{k+\frac{1}{2}})(\delta_t v_{i+1}^{k+\frac{1}{2}})\right| \le \frac{h}{2}\sum_{i=0}^{m-1}\left[(\delta_t v_i^{k+\frac{1}{2}})^2+(\delta_t v_{i+1}^{k+\frac{1}{2}})^2\right]=\|\delta_t v^{k+\frac{1}{2}}\|^2,$$

易得引理 2.5.

## 3 差分格式的建立

记 $U_i^k=u(x_i,t_k)$, $0\le i\le m$, $0\le k\le n$, 则 $U=[U_i^k|0\le i\le m, 0\le k\le n]$ 为定义在 $\Omega_{h\tau}$ 上的网格函数. 再设 $g\in V_h$, 定义算子 $\mathscr{P}$:

$$(\mathscr{P}g)_i=\begin{cases}\frac{1}{12}(g_{i-1}+10g_i+g_{i+1}), 1\le i\le m-1,\\ \frac{1}{6}(5g_0+g_1), \quad i=0,\\ \frac{1}{6}(g_{m-1}+5g_m), \quad i=m.\end{cases}$$

令 $v=\frac{\partial^2 u}{\partial x^2}$, 记 $V_i^k=v(x_i,t_k), 0\le i\le m$, $0\le k\le n$ 处考虑方程 (1.1), 则有

$$\frac{\partial^2 u}{\partial t^2}(x,t)-\alpha v(x,t)+gu(x,t)=f(x,t), \quad a<x<b,\ 0<t\le T, \tag{3.1}$$

在结点 $(x_i,t_k)$ 处考虑方程 (3.1), 并对两边作用算子 $\mathscr{P}$, 且 $1\le i\le m-1$, $0\le k\le n-1$, 可得

$$\mathscr{P}\frac{\partial^2 u}{\partial t^2}(x_i,t_k)-\alpha\mathscr{P}V(x_i,t_k)+\mathscr{P}gU(x_i,t_k)=\mathscr{P}f(x_i,t_k). \tag{3.2}$$

根据引理 2.1, 边界条件及作用算子 $\mathscr{P}$ 可得

$$\begin{aligned}\mathscr{P}V_0^k=&\frac{2}{h}\delta_x U_{1/2}^k+\frac{h}{6\alpha}\frac{\partial f}{\partial x}(x_0,t_k)-\frac{h^3}{90\alpha}\frac{\partial f^3}{\partial x^3}(x_0,t_k)+\frac{1}{\alpha}g'(U_0^k)\frac{\partial f}{\partial x}(x_0,t_k)\\&+\frac{1}{\alpha}\frac{\partial f^3}{\partial t^2\partial x}(x_0,t_k)+O(h^4),\ 0\le k\le n-1,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathscr{P}V_m^k=&-\frac{2}{h}\delta_x U_{m-1/2}^k-\frac{h}{6\alpha}\frac{\partial f}{\partial x}(x_m,t_k)+\frac{h^3}{90\alpha}\frac{\partial f^3}{\partial x^3}(x_m,t_k)+\frac{1}{\alpha}g'(U_m^k)\frac{\partial f}{\partial x}(x_m,t_k)\\&+\frac{1}{\alpha}\frac{\partial f^3}{\partial t^2\partial x}(x_m,t_k)+O(h^4),\ 0\le k\le n-1,\end{aligned} \tag{3.4}$$

$$\mathscr{P}V_i^k=\delta_x U_i^k+O(h^4), \quad 0\le i\le m,\ 1\le k\le n-1. \tag{3.5}$$

由 Taylor 展开知

$$\frac{\partial^2 u}{\partial t^2}=\delta_t^2 U_i^k-\frac{\tau^2}{12}\frac{\partial^4}{\partial t^4}(x_i,\eta_{ik}), t_{k-1}<\eta_{ik}<t_{k+1}, \tag{3.6}$$

$$\frac{\partial^2 u}{\partial t^2}(x_i,t_k)=\frac{1}{2}[\frac{\partial^2 u}{\partial x^2}(x_i,t_{k-1})+\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+1})]-\frac{\tau^2}{2}\frac{\partial^4 u}{\partial x^2\partial t^2}(x_i,\xi_{ik}), t_{k-1}<\xi_{ik}<t_{k+1}. \tag{3.7}$$

将 (3.6)–(3.7) 式带入到 (3.2) 式, 且 $0 \le i \le m-1, 0 \le k \le n-1$, 得到

$$\mathscr{P}\frac{\partial^2 u}{\partial x^2}(x_i,t_k) - \frac{\alpha}{2}(\mathscr{P}V_i^{k+1} + \mathscr{P}V_i^{k-1}) + \mathscr{P}g(U_i^k) = \mathscr{P}f(x_i,t_k). \tag{3.8}$$

由 (3.3)–(3.5) 式求得 $\mathscr{P}V_i^{k+1}, \mathscr{P}V_i^{k-1}$, 将其带入到 (3.8) 式, 结合 (3.7)–(3.8) 式和引理 2.3 得

$$\mathscr{P}\delta_t^2 U_i^k - \alpha\delta_x^2 U_i^{\bar{k}} + \mathscr{P}g(U_i^k) = \mathscr{P}f(x_i,t_k) + R_i^k, 0 \le i \le m-1, 0 \le k \le n-1, \tag{3.9}$$

$$\begin{aligned}&\mathscr{P}\delta_t^2 U_0^k - \frac{2\alpha}{h}\delta_x U_{1/2}^{\bar{k}} - \frac{h}{6}\frac{\partial f}{\partial x}(x_0,t_k) + \frac{h^3}{90}[\frac{\partial^3 f}{\partial x^3}(x_0,t_k) + \frac{1}{\alpha}g'(U_0^k)\frac{\partial f}{\partial x}(x_0,t_k) + \frac{1}{\alpha}\frac{\partial^3 f}{\partial t^2 \partial x}(x_0,t_k)]\\ &+ \mathscr{P}g(U_0^k) = \mathscr{P}f(x_0,t_k) + R_0^k, 0 \le k \le n-1,\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\mathscr{P}\delta_t^2 U_m^k - \frac{2\alpha}{h}\delta_x U_{1/2}^{\bar{k}} + \frac{h}{6}\frac{\partial f}{\partial x}(x_m,t_k) - \frac{h^3}{90}[\frac{\partial^3 f}{\partial x^3}(x_m,t_k) + \frac{1}{\alpha}g'(U_0^k)\frac{\partial f}{\partial x}(x_m,t_k) + \frac{1}{\alpha}\frac{\partial^3 f}{\partial t^2 \partial x}(x_m,t_k)]\\ &+ \mathscr{P}g(U_m^k) = \mathscr{P}f(x_m,t_k) + R_m^k, 0 \le k \le n-1,\end{aligned} \tag{3.11}$$

其中 $|R_i^k| \le M_1(\tau^2 + h^4), 0 \le i \le m-1, 0 \le k \le n-1, M_1$ 为正常数, 由初始条件有 $u(x_i,0) = \varphi(x_i), 0 \le i \le m$, 并由 (1.1) 式且根据带积分余项的 Taylor 展开式以及初始条件得到

$$U_i^1 = \varphi(x_i) + \tau\psi(x_i) + \frac{\tau^2}{2}[\alpha\varphi''(x_i) + f(x_i,0) - g(\varphi(x_i)] + r_i^1, 0 \le i \le m, \tag{3.12}$$

其中

$$r_i^1 = \frac{1}{2}\int_0^\tau (\tau - t)^2 \frac{\partial^3 u}{\partial t^3}(x_i,t)dt, 0 \le i \le m.$$

易知 $|r_i^1| \le M_2\tau^3, M_2$ 为正常数, 令

$$\begin{aligned}&G(U_i^k, x_i, t_k) = \frac{h}{6}\frac{\partial f}{\partial x}(x_i,t_k) - \frac{h^3}{90}\frac{\partial^3 f}{\partial x^3}(x_i,t_k) + \frac{1}{\alpha}g'(U_i^k)\frac{\partial f}{\partial x}(x_i,t_k) + \frac{1}{\alpha}\frac{\partial^3 f}{\partial t^2 \partial x}(x_i,t_k),\\ &0 \le i \le m, 0 \le k \le n.\end{aligned}$$

在 (3.9)–(3.12) 式中分别略去小量项 $R_i^k, r_i^1$, 并用 $u_i^k$ 代替 $U_i^k$, 可得如下差分格式

$$\mathscr{P}\delta_t^2 u_i^k - \alpha\delta_x^2 u_i^{\bar{k}} + \mathscr{P}g(u_i^k) = \mathscr{P}f(x_i,t_k), \quad 1 \le i \le m-1, 1 \le k \le n-1, \tag{3.13}$$

$$\mathscr{P}\delta_t^2 u_0^k - \frac{2\alpha}{h}\delta_x u_{1/2}^{\bar{k}} + \mathscr{P}g(u_0^k) = \mathscr{P}f(x_0,t_k) + G(u_0^k, x_0, t_k), \quad 1 \le k \le n-1, \tag{3.14}$$

$$\mathscr{P}\delta_t^2 u_m^k + \frac{2\alpha}{h}\delta_x u_{m-1/2}^{\bar{k}} + \mathscr{P}g(u_m^k) = \mathscr{P}f(x_m,t_k) + G(u_m^k, x_m, t_k), \quad 1 \le k \le n-1, \tag{3.15}$$

$$u_i^0 = \varphi(x_i), \quad 1 \le i \le m, \tag{3.16}$$

$$u_i^1 = \varphi(x_i) + \tau\psi(x_i) + \frac{\tau^2}{2}[\alpha\varphi''(x_i) + f(x_i,0) - g(\varphi(x_i))], \quad 1 \le i \le m. \tag{3.17}$$

## 4 差分格式的收敛性

**定理 1** 设 $u(x_i,t_k)$ 是问题 (1.1)–(1.3) 式的解, $\{u_i^k | 0 \le i \le m, 0 \le k \le n\}$ 是差分格式的解 (3.13)–(3.17) 的解, 记 $e_i^k = u(x_i,t_k) - u_i^k$ , 则当 $\tau \le \sqrt{\frac{C_0+1}{2\tilde{c}}}, h \le \sqrt{\frac{C_0+1}{2\tilde{c}}}$ 时, 有

$$\|e^k\|_\infty \le \tilde{c}(\tau^2 + h^4), 1 \le k \le n, \tag{4.1}$$

其中

$$\tilde{c}=\max\{\sqrt{\varepsilon M_3^2+M_2^2(\frac{1}{\varepsilon}+\frac{1}{b-a})},\sqrt{e^{MT}(1+\frac{\alpha}{2}+2T)M^2[\frac{2\varepsilon}{\alpha}+3T^2(\frac{1}{\varepsilon}+\frac{1}{b-a})]}\}.$$

**证** 将 (3.9)–(3.12) 式分别与 (3.13)–(3.17) 式相减, 且 $1\le k\le n-1$, 得到误差方程

$$\mathscr{P}\delta_t^2e_i^k-\alpha\delta_x^2e_i^{\overline{k}}=R_i^k-\mathscr{P}g(U_i^k)-\mathscr{P}g(u_i^k),\quad 1\le i\le m-1, \tag{4.2}$$

$$\mathscr{P}\delta_t^2e_0^k-\frac{2\alpha}{h}\delta_xe_{1/2}^{\overline{k}}=R_0^k-\frac{h^3}{90\alpha}[g'(U_0^k)-g'(u_0^k)\frac{\partial f}{\partial x}(x_0,t_k)-\mathscr{P}[g(U_0^k)-g(u_0^k)], \tag{4.3}$$

$$\mathscr{P}\delta_t^2e_m^k+\frac{2\alpha}{h}\delta_xe_{m-1/2}^{\overline{k}}=R_m^k-\frac{h^3}{90\alpha}[g'(U_m^k)-g'(u_m^k)\frac{\partial f}{\partial x}(x_m,t_k)-\mathscr{P}[g(U_m^k)-g(u_m^k)], \tag{4.4}$$

$$e_i^0=\varphi(x_i),e_i^1=r_i^1,\quad 1\le i\le m. \tag{4.5}$$

令

$$M_3=\frac{1}{6}\max\{|\frac{\partial^4u}{\partial x\partial t^3}|,a\le x\le b,0\le t\le T\},$$

则有

$$\delta_xe_{i+1/2}^l\le M_3\tau^3,0\le i\le m-1,$$

从而得到$|e_i^1|_1^2\le c_1^2(b-a)\tau^6$, 由$|r_1^1|\le M_2(\tau^2+h^4)$可知 $\|e^1\|^2\le M_2^2(b-a)\tau^6$ . 由引理 2.2 得到

$$\|e^1\|_\infty^2\le\varepsilon|e|_1^2+(\frac{1}{\varepsilon}+\frac{1}{b-a})\|e^k\|^2\le[\varepsilon|e^k|_1^2+M_2^2(\frac{1}{\varepsilon}+\frac{1}{b-a})](b-a)\tau^6. \tag{4.6}$$

显然, (4.1) 式对于 $k=1$ 成立, 现假设 (4.1) 式对于 $1\le k\le l$ 成立, 其中 $1\le l\le n-1$. 下面证明 (4.2) 式对 $k=l+1$ 也成立, 用 $2hD_{\hat{t}}e_i^k,hD_{\hat{t}}e_0^k,hD_{\hat{t}}e_m^k$, 分别乘方程 (4.2)–(4.4) 三式, 并关于 $i$ 从 1 到 $m-1$ 求和, 且将三式相加可得 $J_1+J_2=J_3$, 其中

$$\begin{aligned}J_1=&2h\sum_{i=1}^{m-1}(\mathscr{P}\delta_t^2e_i^k)(D_{\hat{t}}e_i^k)+h(\mathscr{P}\delta_t^2e_0^k)(D_{\hat{t}}e_0^k)+h(\mathscr{P}\delta_t^2e_m^k)(D_{\hat{t}}e_m^k)\\ \le&\frac{5}{6\tau}(\|\delta_te^{k+1/2}\|^2-\|\delta_te^{k-1/2}\|^2)+\frac{h}{6\tau}\sum_{i=0}^{m-1}(\delta_te_i^{k+1/2}\delta_te_{i+1}^{k+1/2}-\delta_te_i^{k-1/2}\delta_te_{i+1}^{k-1/2}),\\ J_2=&-2h\sum_{i=1}^{m-1}\alpha(\delta_x^2e_i^{\bar{k}})D_{\hat{t}}e_i^k-2\alpha\delta_xe_{1/2}^{\bar{k}}D_{\hat{t}}e_m^k+2\alpha\delta_xe_{m-1/2}^{\bar{k}}D_{\hat{t}}e_m^k\le\frac{\alpha}{2\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2),\\ J_3=&2h\sum_{i=1}^{m-1}R_i^kD_{\hat{t}}e_i^k+hR_0^kD_{\hat{t}}e_0^k+hR_m^kD_{\hat{t}}e_m^k+2h\sum_{i=1}^{m-1}a_i^k(D_{\hat{t}}e_0^k)+hb_i^k(D_{\hat{t}}e_0^k)+hc_i^k(D_{\hat{t}}e_m^k)\\ \le&(\frac{1}{2}+\beta)(\|\delta_te^{k+1/2}\|^2+\|\delta_te^{k-1/2}\|^2)+2\beta\|e^k\|+\|R^k\|.\end{aligned}$$

上述 $\beta$ 为常数, 其中 $a_i^k,b_i^k,c_i^k$ 如下

$$\begin{aligned}a_i^k&=-\mathscr{P}[g(U_i^k)-g(u_i^k)],\\ b_i^k&=-\frac{h^3}{90\alpha}[g'(U_0^k)-g'(u_0^k)]\frac{\partial f}{\partial x}(x_0,t_k)-\mathscr{P}[g(U_0^k)-g(u_0^k)],0\le k\le n-1,\\ c_i^k&=\frac{h^3}{90\alpha}[g'(U_m^k)-g'(u_m^k)]\frac{\partial f}{\partial x}(x_m,t_k)-\mathscr{P}[g(U_m^k)-g(u_m^k)],0\le k\le n-1.\end{aligned}$$

令

$$E^k=\frac{5}{6}\|\delta_t e^{k+1/2}\|^2+\frac{h}{6}\sum_{i=0}^{m-1}\delta_t e_i^{k+1/2}\delta_t e_{i+1}^{k+1/2}+\frac{\alpha}{2}(|e^{k+1}|_1^2+|e^k|_1^2),$$

则 $J_1+J_2=J_3$ 可写为

$$E^k-E^{k-1}\leq(\frac{3}{4}+\frac{3\beta}{2})(E^k+E^{k-1})+3T\beta\tau^2\sum_{l=0}^{k-1}E^l+\tau\|R^k\|^2. \tag{4.7}$$

记 $\lambda=\frac{3}{4}+\frac{3\beta}{2}$ , 且当 $\lambda\tau\leq\frac{1}{2}$ 时, 则 (4.7) 式可表示为

$$E^k\leq\frac{1+\lambda\tau}{1-\lambda\tau}E^{k-1}+\frac{3T\beta\tau^2}{1-\lambda\tau}\sum_{l=0}^{k-1}E^l+2\tau\|R^k\|^2.$$

令 $F^k=\max(E^k,E^{k-1},E^{k-2},\cdots,E^1,E^0)$ , 则有

$$F^k\leq[1+2(2\lambda+3kT^2\beta)\tau]F^{k-1}+2\tau\|R^k\|^2.$$

记 $M_4=2(2\lambda+3kT^2\beta), M_4$ 为正整数, 故 $F^k\leq(1+M_4\tau)F^{k-1}+2\tau\|R^k\|^2$, 由 Granwall 不等式可得到

$$F^k\leq e^{M_4k\tau}(F^0+2\tau\sum_{l=1}^{k-1}\|F^k\|^2).$$

由 $F^k$ 的定义可知 $F^0=E^0$, 则有

$$E^k\leq e^{M_4k\tau}(E^0+2\tau\sum_{l=1}^{k-1}\|R^l\|^2).$$

取 $M=\max(M_1,M_2,M_3,M_4)$, 并由引理 2.5 可得

$$\begin{aligned}E^k&\leq e^{MT}[\frac{5}{6}\|\delta_t e^{1/2}\|^2+\frac{h}{6}\sum_{i=0}^{m-1}\delta_t e_i^{1/2}\delta_t e_{i+1}^{1/2}+\frac{\alpha}{2}(|e^1|_1^2+|e^0|_1^2)+2\tau\sum_{l=1}^{k-1}\|R^k\|^2]\\&\leq e^{MT}(1+\frac{\alpha}{2}+2T)M^2(\tau^2+h^4)^2.\end{aligned}$$

由此可得

$$E^k\leq e^{MT}(1+\frac{\alpha}{2}+2T)M^2(\tau^2+h^4)^2. \tag{4.8}$$

由引理 2.4 和引理 2.5 可知

$$\|e^{k+1}\|^2\leq\frac{3T^2}{2}e^{MT}(1+\frac{\alpha}{2}+2T)M^2(\tau^2+h^4)^2, 0\leq k\leq n-1. \tag{4.9}$$

由 (4.8)–(4.9) 两式和引理 2.2 可知, 对于任意的 $\varepsilon>0$, 有

$$\|e^{k+1}\|_\infty^2\leq\varepsilon|e^{k+1}|_1^2+(\frac{1}{\varepsilon}+\frac{1}{b-a})\|e^{k+1}\|^2\leq\widetilde{c}(\tau^2+h^4)^2,$$

其中 $\widetilde{c}$ 在定理中已经说明, 因此上述结论对于 $l=k+1$ 也成立由归纳原理知 (4.1) 式成立.

## 5 差分格式的稳定性

类似讨论差分格式的收敛性可以得到差分格式 (3.13)–(3.17) 式关于初值的稳定性.

**定理 2** 设 $u_i^k, v_i^k$ 分别是差分格式 (3.13)–(3.17) 的解, 记 $\varepsilon_i^k = u_i^k - v_i^k$, 假设条件 H1, H2, H3 成立, 则当 $h,\tau$ 充分小时, 且 $1 \le k \le n-1$ 时, 有

$$\begin{aligned}&\frac{2}{3}\|\delta_t\varepsilon^{k+1/2}\|^2+\frac{\alpha}{2}(|\varepsilon^{k+1}|_1^2+|\varepsilon^k|_1^2)\\ \le\ & e^{B_1k\tau}[\|\delta_t\varepsilon^{1/2}\|^2+\frac{\alpha}{2}(|\varepsilon^1|_1^2+|\varepsilon^0|_1^2)+B_2\|\varepsilon^0\|^2+2\tau\sum_{l=1}^{k-1}\|\varepsilon^l\|^2],\end{aligned}$$

且 $B_1, B_2$ 是与 $h,\tau$ 无关的正的常数, 证明略.

## 6 数值试验

本节利用构造的差分格式 (3.13)–(3.17) 计算下面的定解问题

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}+u^2=-[(x^2-1)^2+4x]\cos t+(x^2-1)^4\cos^2 t, -1<x<1, 0<t<1,\\ u(x,0)=(x^2-1)^2,\ \dfrac{\partial u}{\partial t}(x,0)=0,\quad 0\le x\le 1,\\ \dfrac{\partial u}{\partial x}(-1,t)=0,\ \dfrac{\partial u}{\partial x}(1,t)=0,\quad 0\le t\le 1.\end{cases}$$

该问题的精确解为 $u(x,t)=(x^2-1)^2\cos t$. 计算结果见表 6.1 和表 6.2. 表 6.1 给出了在不同步长时某些节点处的误差, 表 6.2 给出了不同步长时数值解的最大误差和误差比, 其中

$$E_i^k = u(x_i,t_k)-u_i^k,\ \ E_\infty(h,\tau)=\max_{0\le i\le m, 0\le k\le n}\left|u(x_i,t_k)-u_i^k\right|.$$

表 6.1: 部分结点处数值解的误差

| $(x,t)$ | $\lvert E_i^k\rvert(h=1/100,\tau=1/100)$ | $\lvert E_i^k\rvert(h=1/200,\tau=1/200)$ |
|---|---|---|
| $(0.5,0.1)$ | $1.9701741e-7$ | $5.2290328e-8$ |
| $(0.5,0.2)$ | $7.7788723e-7$ | $1.9984994e-7$ |
| $(0.5,0.3)$ | $1.5563009e-6$ | $3.9531242e-7$ |
| $(0.5,0.4)$ | $2.2190456e-6$ | $5.5957991e-7$ |
| $(0.5,0.5)$ | $2.3321701e-6$ | $5.8274421e-7$ |
| $(0.5,0.6)$ | $1.5174309e-6$ | $3.7159848e-7$ |
| $(0.5,0.7)$ | $4.8622903e-8$ | $1.3832075e-9$ |
| $(0.5,0.8)$ | $1.6602792e-6$ | $4.2667309e-7$ |
| $(0.5,0.9)$ | $3.3172045e-6$ | $8.4037174e-7$ |
| $(0.5,1.0)$ | $4.7377242e-6$ | $1.1942151e-6$ |

表 6.2: 不同步长时数值解的最大误差和误差比

| $h$ | $\tau$ | $E_\infty(h,\tau)$ | $E_\infty(2h,4\tau)/E_\infty(h,\tau)$ |
|---|---|---|---|
| 1/10 | 1/10 | $4.2911507e-3$ | * |
| 1/20 | 1/40 | $2.7169988e-4$ | 15.794 |
| 1/40 | 1/160 | $1.7039251e-5$ | 15.946 |
| 1/80 | 1/640 | $1.065908e-6$ | 15.986 |

由表 6.2 可以看出, 差分格式在无穷范数下的收敛阶为 $O(\tau^2+h^4)$, 这和理论分析结果一致.

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# A HIGH ORDER ACCURACY DIFFERENCE SCHEME FOR THE NONLINEARKLEIN-GORDON EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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**Abstract:** This paper is devoted to the study of high-order accuracy difference methods for the Klein-Gordon equation with Neumann boundary conditions. By using the boundary values of three-order and five-order derivatives, the three points scheme at inside points and two points scheme at boundary points are established respectively. The truncation error of difference scheme is second order in time and fourth order in space. Convergence and stability of difference scheme are analyzed by using energy estimate. Numerical results are conducted to illustrate the theoretical results of the presented scheme in this paper.

**Keywords:** nonlinear Klein-Gordon equation; compact difference scheme; convergence; stability; high order accuracy

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