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# 一类 Kuramoto Sivashinsky 方程的显式解

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**摘 要**: 本文研究了一类 KS 方程的初边值问题. 利用一致变换方法, 并结合 Green 公式和 Jordan 引理, 在半直线上得到了这类方程的显式解公式. 所得结论将为该类方程适定性和数值计算的研究提供新的思路.

**关键词**: KS 方程; 一致变换方法; 显式解; Green 公式; Jordan 引理
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## 1 引言

在研究空间三维 Belousov-Zhabotinski 型反应扩散系统的角相位湍流时, Kuramoto[1] 导出了如下一类四阶 Kuramoto Sivashinsky (KS) 方程

$$u_t + u_{xxxx} + \lambda u_{xx} + u u_x = 0, \tag{1.1}$$

其中 $u = u(x,t)$ 是未知函数, $\lambda > 0$ 被称为反扩散常数. 这个方程也被 Sivashinsky[2] 用来研究空间二维层焰面的微热扩散不稳定性问题. 此外, 该方程在其他领域还有广泛的应用 [3]. 近年来, 该方程的很多性质被广泛研究 [4–6], 如非平凡解的分岔与稳定性、吸收集半径、渐近吸因子、零解的可控性、时间离散化与稳定性区域等.

本文将在半无界区域上研究如下一类 KS 方程的初边值问题

$$\begin{cases} u_t + u_{xxxx} + u_{xx} = f, \\ u(x,0) = \varphi(x), \qquad x \geqslant 0, \\ u(0,t) = \psi_0(t), \quad u_x(0,t) = \psi_1(t). \end{cases} \tag{1.2}$$

KS 方程 Cauchy 问题的研究已有很多结果 [7–9]. 而初边值问题的适定性和数值计算至今尚未解决. 本文将采用近年来在相关领域内提出的新的一致变换方法 (UTM) 来研究方程 (1.2) 的显式解, 这个方法的最新进展参见文献 [11–13]. 利用 UTM 方法得到的显式解的公式, 可以为后续 KS 方程初边值问题的适定性和数值计算的研究提供新的思路.

本文的主要结论由下面的定理给出.

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**定理 1** 初边值问题 (1.2) 的显式解是

$$
\begin{aligned}
u(x,t) = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\widehat{\varphi}(k)\mathrm{d}k - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ikx+\omega(k)(\tau-t)}\widehat{f}(k,\tau)\mathrm{d}\tau\mathrm{d}k \\
& + \frac{1}{2\pi}\int_{\partial D_1^+} e^{ikx-\omega(k)t}G_1(k,t)\mathrm{d}k + \frac{1}{2\pi}\int_{\partial D_2^+} e^{ikx-\omega(k)t}G_2(k,t)\mathrm{d}k,
\end{aligned} \tag{1.3}
$$

其中 $D_i^+$ 由下文中的图 1 给出, $G_1(k,t), G_2(k,t)$ 分别由 (2.6), (2.7) 式决定.

## 2 显式解的推导过程

假定 $v(x,t) = e^{ikx-\omega(k)t}$ 是方程 $v_t + v_{xxxx} + v_{xx} = 0$ 的一个解, 代入方程中得到 KS 方程的象征关系式 $\omega(k) = k^4 - k^2$, $\omega(k)$ 在显式解的求解过程中起着关键的作用.

首先把方程写成散度形式, 这样做的目的是便于使用 Green 公式. 设 $u(x,t)$ 是方程 (1.2) 的解, 则

$$
\begin{aligned}
\left(e^{-ikx+\omega(k)t}u\right)_t &= \omega(k)e^{-ikx+\omega(k)t}u + e^{-ikx+\omega(k)t}u_t \\
&= \omega(k)e^{-ikx+\omega(k)t}u + e^{-ikx+\omega(k)t}\left(-u_{xxxx} - u_{xx} + f\right) \\
&= \left(\omega(k)u + f\right)e^{-ikx+\omega(k)t} - \left[e^{-ikx+\omega(k)t}\left(u_{xxx} + u_x\right)\right]_x \\
&\quad - \left[ike^{-ikx+\omega(k)t}\left(u_{xx} + u\right)\right]_x + k^2 e^{-ikx+\omega(k)t}\left(u_{xx} + u\right) \\
&= e^{-ikx+\omega(k)t}f - \left[e^{-ikx+\omega(k)t}\left(u_{xxx} + iku_{xx}\right)\right]_x \\
&\quad + \left[e^{-ikx+\omega(k)t}\left((k^2-1)u_x + (ik^3 - ik)u\right)\right]_x.
\end{aligned} \tag{2.1}
$$

(2.1) 式就是和 (1.2) 式等价的散度形式的方程, 把它称为方程 (1.2) 的局部关系等式.

将 (2.1) 式在区域 $D = \{x \geqslant 0, 0 < t \leqslant T\}$ 上作二重积分, 得到

$$
\begin{aligned}
& \int\int_D \left(e^{-ikx+\omega(k)t}u(x,t)\right)_t \mathrm{d}t\mathrm{d}x + \int\int_D e^{-ikx+\omega(k)t}f(x,t)\mathrm{d}t\mathrm{d}x \\
& + \int\int_D \left(e^{-ikx+\omega(k)t}\left(u_{xxx} + iku_{xx} - (k^2-1)u_x - (ik^3 - ik)u\right)\right)_x \mathrm{d}t\mathrm{d}x = 0.
\end{aligned}
$$

使用 Green 公式, 有

$$
\begin{aligned}
& \int_{\partial D} e^{-ikx+\omega(k)t}u(x,t)\mathrm{d}x + \int\int_D e^{-ikx+\omega(k)t}f(x,t)\mathrm{d}t\mathrm{d}x \\
& - \int_{\partial D} e^{-ikx+\omega(k)t}\left(u_{xxx} + iku_{xx} - (k^2-1)u_x - (ik^3 - ik)u\right)\mathrm{d}t = 0,
\end{aligned}
$$

即

$$
\begin{aligned}
& -\int_0^\infty e^{-ikx}\varphi(x)\mathrm{d}x + \int_0^\infty e^{-ikx+\omega(k)T}u(x,T)\mathrm{d}x + \int_0^\infty\int_0^T e^{-ikx+\omega(k)t}f(x,t)\mathrm{d}t\mathrm{d}x \\
= & -\int_0^T e^{\omega(k)t}\left(u_{xxx}(0,s) + iku_{xx}(0,s) - (k^2-1)u_x(0,s) - (ik^3 - ik)u(0,s)\right)\mathrm{d}t.
\end{aligned}
$$

定义在半直线上函数 $u(x,t)$ 的 Fourier 变换为

$$\widehat{u}(k,t)=\int_0^\infty e^{-ikx}u(x,t)\mathrm{d}x,\quad k\in\mathbb{C}^-=\{k\in\mathbb{C}:\mathrm{Im}\ k\leqslant 0\},$$

这里 $\mathrm{Re}(-ikx)=x\mathrm{Im}k\leqslant 0$. 另外, 记

$$g_i(k,t)=\int_0^t e^{k\tau}\partial_x^i u(0,\tau)\mathrm{d}\tau,\quad k\in\mathbb{C},\quad i=0,1,2,3.$$

则下列整体关系等式成立

$$\begin{aligned}&\widehat{\varphi}(k)+g_3(\omega(k),T)+ikg_2(\omega(k),T)-(k^2-1)g_1(\omega(k),T)-(ik^3-ik)g_0(\omega(k),T)\\=&e^{\omega T}\widehat{u}(k,T)-\int_0^T e^{\omega(k)t}\widehat{f}(k,t)\mathrm{d}t.\end{aligned}\tag{2.2}$$

把上式中的 $T$ 替换为 $t$, 并令

$$G(k,t)=g_3(\omega(k),t)+ikg_2(\omega(k),t)-(k^2-1)g_1(\omega(k),t)-(ik^3-ik)g_0(\omega(k),t),\tag{2.3}$$

则有

$$e^{\omega(k)t}\widehat{u}(k,t)=\widehat{\varphi}(k)+G(k,t)-\int_0^t e^{\omega(k)\tau}\widehat{f}(k,\tau)\mathrm{d}\tau.\tag{2.4}$$

定义区域

$$D=\{k:\mathrm{Re}\,(\omega(k))<0\},\quad D^+=D\cap\mathbb{C}^+,\quad D^-=D\cap\mathbb{C}^-.$$

由于

$$\mathrm{Re}\,(\omega(k))=(\mathrm{Re}k)^4-6(\mathrm{Re}k)^2(\mathrm{Im}k)^2+(\mathrm{Im}k)^4-(\mathrm{Re}k)^2+(\mathrm{Im}k)^2,$$

故 $D$ 的图形如图 1 所示, 共由四部分组成, 分别是 $D_i^+,D_j^-,i=1,2;j=1,2$.

由 Fourier 逆变换公式, 得到

$$\begin{aligned}u(x,t)=&\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx-\omega(k)t}\widehat{\varphi}(k)\mathrm{d}k+\frac{1}{2\pi}\int_{-\infty}^\infty\int_0^t e^{ikx+\omega(k)(\tau-t)}\widehat{f}(k,\tau)\mathrm{d}\tau\mathrm{d}k\\&+\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx-\omega(k)t}G(k,t)\mathrm{d}k+\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx-\omega(k)t}G(k,t)\mathrm{d}k.\end{aligned}$$

利用与 Fokas[14] 文中的第一章命题 1.1 证明类似的方法, 可以证明

$$\begin{aligned}u(x,t)=&\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx-\omega(k)t}\widehat{\varphi}(k)\mathrm{d}k+\frac{1}{2\pi}\int_{-\infty}^\infty\int_0^t e^{ikx+\omega(k)(\tau-t)}\widehat{f}(k,\tau)\mathrm{d}\tau\mathrm{d}k\\&+\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}G(k,t)\mathrm{d}k,\end{aligned}\tag{2.5}$$

这里 $\partial D_i^+$ 取逆时针方向.

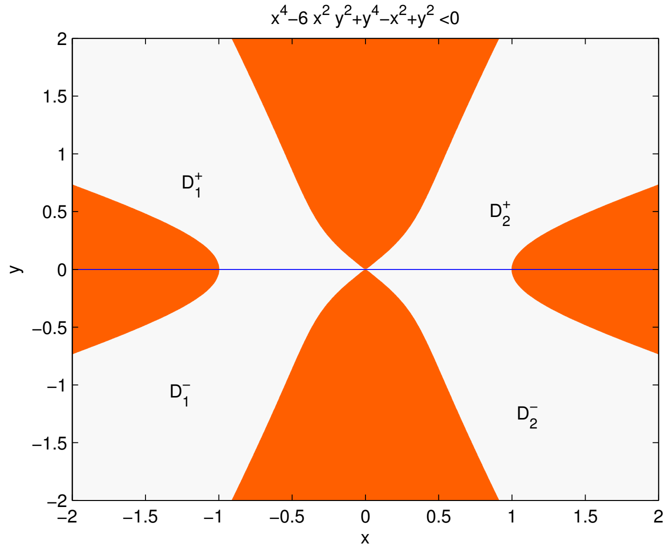


图 1

在上式中, $\varphi(x)$, $f(x,t)$ 是已知的, 因此第一、二项直接可得. 在后两项中, 积分路径已知, 被积函数 $G(k,t)$ 中存在未知项 $g_2, g_3$. 下面用 $g_0, g_1$ 来表示 $g_2, g_3$. 解方程 $k^4-k^2=\mu^4(k)-\mu^2(k)$, 除 $\mu(k)=k$ 之外, 还有三个根

$$\mu_1(k)=-\sqrt{1-k^2},\ \ \mu_2(k)=-k,\ \ \mu_3(k)=\sqrt{1-k^2},$$

且满足

$$\mu_1(k): D_2^+ \mapsto D_1^+,\ \ \mu_2(k): D_2^+ \mapsto D_1^-,\ \ \mu_3(k): D_2^+ \mapsto D_2^-.$$

如果 $k\in D_1^+$, 那么 $\mu_1(k)\in D_1^-$, $\mu_2(k)\in D_2^-$, 把 $\mu_1(k), \mu_2(k)$ 代入整体关系式 (2.2), 有

$$\begin{aligned}&\widehat{\varphi}(\mu_1)+\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_1,\tau)\mathrm{d}\tau+g_3(\omega,t)+i\mu_1 g_2(\omega,t)\\&-(\mu_1^2-1)g_1(\omega,t)-(i\mu_1^3-i\mu_1)g_0(\omega,t)-e^{\omega t}\widehat{u}(\mu_1,t)=0,\\&\widehat{\varphi}(\mu_2)+\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_2,\tau)\mathrm{d}\tau+g_3(\omega,t)+i\mu_2 g_2(\omega,t)\\&-(\mu_2^2-1)g_1(\omega,t)-(i\mu_2^3-i\mu_2)g_0(\omega,t)-e^{\omega t}\widehat{u}(\mu_2,t)=0.\end{aligned}$$

联立以上两式, 解得

$$\begin{aligned}g_3(\omega,t)\ =\ &\frac{-\mu_2\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_1(k),\tau)\mathrm{d}\tau+\mu_1\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_2(k),\tau)\mathrm{d}\tau}{\mu_2-\mu_1}\\&+e^{\omega t}\frac{\widehat{u}(\mu_1,t)\mu_2-\widehat{u}(\mu_2,t)\mu_1}{\mu_2-\mu_1}-\frac{\widehat{\varphi}(\mu_1)\mu_2-\widehat{\varphi}(\mu_2)\mu_1}{\mu_2-\mu_1}\\&+(1-\mu_1\mu_2)g_1(\omega,t)-i(\mu_1+\mu_2)\mu_1\mu_2 g_0(\omega,t),\end{aligned}$$

$$
\begin{aligned}
g_2(\omega,t) \quad = \quad & \frac{-\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_1(k),\tau)\mathrm{d}\tau+\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_2(k),\tau)\mathrm{d}\tau}{i(\mu_1-\mu_2)}-\frac{\widehat{\varphi}(\mu_2)-\widehat{\varphi}(\mu_1)}{i(\mu_2-\mu_1)} \\
&+e^{\omega t}\frac{\widehat{u}(\mu_1,t)-\widehat{u}(\mu_2,t)}{i(\mu_1-\mu_2)}-i(\mu_1+\mu_2)g_1(\omega,t) \\
&+(\mu_1^2+\mu_1\mu_2+\mu_2^2-1)g_0(\omega,t).
\end{aligned}
$$

将 $g_2(\omega,t),g_3(\omega,t)$ 代入 (2.3) 式, 就可以计算 (2.5) 式的第三项

$$
\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}G(k,t)\mathrm{d}k,
$$

但其中有两部分包含未知函数 $u$, 即

$$
\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}\frac{\widehat{u}(\mu_1,t)\mu_2-\widehat{u}(\mu_2,t)\mu_1}{\mu_2-\mu_1}\mathrm{d}k,\quad \frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}\frac{\widehat{u}(\mu_1,t)-\widehat{u}(\mu_2,t)}{i(\mu_1-\mu_2)}\mathrm{d}k.
$$

由于被积函数项在 $\mathbb{C}^+$ 上是有界解析的, 且当 $k\to\infty$ 时, 它们一致收敛到 0, 利用 Jordan 引理可知, 上面两个积分均为 0. 令

$$
\begin{aligned}
&g_{21}(k,t)=g_2(\omega,t)-e^{\omega t}\frac{\widehat{u}(\mu_1,t)-\widehat{u}(\mu_2,t)}{i(\mu_1-\mu_2)}, \\
&g_{31}(k,t)=g_3(\omega,t)-e^{\omega t}\frac{\widehat{u}(\mu_1,t)\mu_2-\widehat{u}(\mu_2,t)\mu_1}{\mu_2-\mu_1}, \\
&G_1(k,t)=g_{31}(\omega,t)+ikg_{21}(\omega,t)-(k^2-1)g_1(\omega,t)-(ik^3-ik)g_0(\omega,t),
\end{aligned}
\tag{2.6}
$$

则

$$
\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}G(k,t)\mathrm{d}k=\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}G_1(k,t)\mathrm{d}k.
$$

如果 $k\in D_2^+$, 那么 $\mu_2(k)\in D_1^-$, $\mu_3(k)\in D_2^-$, 把 $\mu_2(k),\mu_3(k)$ 代入 (2.2) 式并联立方程组, 可解得

$$
\begin{aligned}
g_3(\omega,t)=&\frac{-\mu_3\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_2,\tau)\mathrm{d}\tau+\mu_2\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_3,\tau)\mathrm{d}\tau}{\mu_3-\mu_2} \\
&+e^{\omega t}\frac{\widehat{u}(\mu_2,t)\mu_3-\widehat{u}(\mu_3,t)\mu_2}{\mu_3-\mu_2}-\frac{\widehat{\varphi}(\mu_2)\mu_3-\widehat{\varphi}(\mu_3)\mu_2}{\mu_3-\mu_2} \\
&+(1-\mu_2\mu_3)g_1(\omega,t)-i(\mu_2+\mu_3)\mu_2\mu_3g_0(\omega,t), \\
g_2(\omega,t)=&\frac{-\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_2,\tau)\mathrm{d}\tau+\int_0^t e^{\omega(k)\tau}\widehat{f}(\mu_3,\tau)\mathrm{d}\tau}{i(\mu_2-\mu_3)}-\frac{\widehat{\varphi}(\mu_3)-\widehat{\varphi}(\mu_2)}{i(\mu_3-\mu_2)} \\
&+e^{\omega t}\frac{\widehat{u}(\mu_2,t)-\widehat{u}(\mu_3,t)}{i(\mu_2-\mu_3)}-i(\mu_2+\mu_3)g_1(\omega,t) \\
&+(\mu_2^2+\mu_2\mu_3+\mu_3^2-1)g_0(\omega,t).
\end{aligned}
$$

同理, 令

$$
\begin{aligned}
&g_{22}(k,t)=g_2(\omega,t)-e^{\omega t}\frac{\widehat{u}(\mu_2,t)-\widehat{u}(\mu_3,t)}{i(\mu_2-\mu_3)},\\
&g_{32}(k,t)=g_3(\omega,t)-e^{\omega t}\frac{\widehat{u}(\mu_2,t)\mu_3-\widehat{u}(\mu_3,t)\mu_2}{\mu_3-\mu_2},\\
&G_2(k,t)=g_{32}(\omega,t)+ikg_{22}(\omega,t)-(k^2-1)g_1(\omega,t)-(ik^3-ik)g_0(\omega,t),
\end{aligned}
\tag{2.7}
$$

则

$$
\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}G(k,t)\mathrm{d}k=\frac{1}{2\pi}\int_{\partial D_1^+}e^{ikx-\omega(k)t}G_1(k,t)\mathrm{d}k.
$$

定理得到证明.

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# THE EXPLICIT SOLUTION OF KURAMOTO SIVASHINSKY EQUATION

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**Abstract:** In this paper, we study the initial boundary value problem of KS equation. By means of the unified transform method, we obtain the explicit solution on the half line with Green formula and Jordan lemma, which provides a new way of studying for the well-posedness and numerical calculation of this kind of equation.

**Keywords:** KS equation; unified transform method; explicit solution; Green formula; Jordan lemma

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