

ON MULTILINEAR COMMUTATORS OF THE LITTLEWOOD-PALEY OPERATORS IN VARIABLE EXPONENT LEBESGUE SPACES

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Abstract: In this paper, we study the boundedness of multilinear commutators of the Littlewood-Paley operators in variable exponent Lebesgue spaces. Based on the atomic decomposition and generalization of the BMO norms, we also prove some boundedness results for such multilinear commutators in variable exponent Herz-type Hardy spaces, which essentially extend some known results.

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1 Introduction

Let ψ be a function on \mathbb{R}^n such that there exist positive constants C and γ satisfying

- (a) $\psi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x)dx = 0$;
- (b) $|\psi(x)| \leq C(1 + |x|)^{-n-1}$;
- (c) $|\psi(x+y) - \psi(x)| \leq C|y|^\gamma/(1 + |x|)^{n+\gamma+1}$ for $2|y| \leq |x|$.

For this ψ and $\mu > 1$, the Littlewood-Paley's g_μ^* function is defined by

$$g_\mu^*(f)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |(\psi_t * f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad (1.1)$$

where $\mathbb{R}_+^{n+1} = \{(y, t) : y \in \mathbb{R}^n, t > 0\}$ and $\psi_t(x) = t^{-n}\psi(x/t)$.

Given a positive integer m and a vector $\vec{b} = (b_1, b_2, \dots, b_m)$ of locally integrable functions, motivated by the work of Pérez and Trujillo-González [1] on multilinear operators, we define multilinear commutators of the Littlewood-Paley's g_μ^* function as follows:

$$g_{\mu, \vec{b}}^*(f)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \left| \int_{\mathbb{R}^n} \psi_t(y - z) \prod_{i=1}^m (b_i(x) - b_i(z)) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \quad (1.2)$$

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In the case of $m = 1$, we usually denote $g_{\mu, \vec{b}}^*$ by $[b, g_\mu^*]$.

A locally integrable function b is said to be a BMO function, if it satisfies

$$\|b\|_* := \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B|} \int_B |b(y) - b_B| dy < \infty,$$

where and in the sequel B is ball centered at x and radius of r ,

$$b_B = \frac{1}{|B|} \int_B b(t) dt$$

and $\|b\|_*$ is the norm in $\text{BMO}(\mathbb{R}^n)$. For $b_i \in \text{BMO}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$, Xue and Ding [2] established the weighted L^p and weighted weak $L(\log L)$ -type estimates for the multilinear commutators $g_{\mu, \vec{b}}^*$. Zhang et al. [3] obtained some boundedness results for $g_{\mu, \vec{b}}^*$ on certain classical Hardy and Herz-Hardy spaces. We refer to [4–6] for an extensive study of multilinear operators.

In recent years, following the fundamental work of Kováčik and Rákosník [7], function spaces with variable exponent, such as variable exponent Lebesgue and Herz-type Hardy spaces etc., have attracted a great attention in connection with problems of the boundedness of classical operators on those spaces, which in turn were motivated by the treatment of recent problems in fluid dynamics, image restoration and differential equations with $p(x)$ -growth, see [8–16] and the references therein.

Karlovich and Lerner in [17] showed that $[b, T]$, the commutator of a standard Calderón-Zygmund singular integral operator T and a BMO function b , is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, which improved a celebrated result by Coifman et al. in [18]. Recently, Xu [19] made a further step and proved that the multilinear commutators $T_{\vec{b}}$, a generalization of the commutator $[b, T]$, enjoy the same $L^{p(\cdot)}(\mathbb{R}^n)$ estimates when $b_i \in \text{BMO}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$. These results inspire us to ask whether the multilinear commutators $g_{\mu, \vec{b}}^*$ have the similar mapping properties in variable exponent spaces $L^{p(\cdot)}(\mathbb{R}^n)$? Our first result (see Theorem 3.1 below) will give an affirmative answer to this question.

The variable exponent Herz spaces $\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)$ were first studied by Izuki [20, 21]. Simultaneously, he gave some basic lemmas on generalization of the BMO norms to get the boundedness of classical operators on such spaces. On the other hand, the variable exponent Herz-type Hardy spaces, as well as their atomic decomposition characterizations, were intensively studied by a significant number of authors [22, 23]. Using these decompositions, they also established the boundedness results for some singular integral operators. Motivated by the results mentioned above, another purpose of this article is to study the boundedness of $g_{\mu, \vec{b}}^*$ in variable exponent Herz-type Hardy spaces, which improves the corresponding main result in classical case (see [3, Theorem 2]).

In general, we denote cubes in \mathbb{R}^n by Q . If E is a subset of \mathbb{R}^n , $|E|$ denotes its Lebesgue measure and χ_E denotes its characteristic function. For $l \in \mathbb{Z}$, we define $B_l = \{x \in \mathbb{R}^n : |x| \leq 2^l\}$. $p'(\cdot)$ denotes the conjugate exponent defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. By $\mathcal{S}'(\mathbb{R}^n)$,

we denote the space of tempered distributions. We use $x \approx y$ if there exist constants c_1, c_2 such that $c_1 x \leq y \leq c_2 x$, C stands for a positive constant, which may vary from line to line.

2 Preliminaries and Lemmas

We begin with a brief and necessarily incomplete review of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, see [24, 25] for more information.

Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a measurable function. We assume that

$$1 \leq p_- \leq p(x) \leq p_+ < \infty,$$

where and in the sequel

$$p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \text{ and } p_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the set of all measurable functions f on \mathbb{R}^n such that

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

This is a Banach space with the norm (the Luxemburg-Nakano norm)

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\mu > 0 : \varrho_{p(\cdot)}(f/\mu) \leq 1\}.$$

Given an open set $\Omega \subset \mathbb{R}^n$, the space $L_{\operatorname{loc}}^{p(\cdot)}(\Omega)$ is defined by

$$L_{\operatorname{loc}}^{p(\cdot)}(\Omega) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \Omega\}.$$

For the sake of simplicity, we use the notation

$$\begin{aligned} \mathcal{P}(\mathbb{R}^n) &:= \{p(\cdot) : p_- > 1 \text{ and } p_+ < \infty\}, \\ \mathcal{B}(\mathbb{R}^n) &:= \{p(\cdot) \in \mathcal{P}(\mathbb{R}^n) : M \text{ is bounded on } L^{p(\cdot)}(\mathbb{R}^n)\}, \end{aligned}$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

When $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the generalized Hölder inequality holds in the form

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \quad (2.1)$$

with $r_p = 1 + 1/p_- - 1/p_+$, see [7, Theorem 2.1].

We say a measurable function $\phi : \mathbb{R}^n \rightarrow [1, \infty)$ is globally log-Hölder continuous if it satisfies

$$|\phi(x) - \phi(y)| \leq \frac{-C}{\log(|x - y|)}, \quad |x - y| \leq 1/2, \quad (2.2)$$

$$|\phi(x) - \phi(y)| \leq \frac{C}{\log(e + |x|)}, \quad |y| \geq |x| \quad (2.3)$$

for any $x, y \in \mathbb{R}^n$. The set of $p(\cdot)$ satisfying (2.2) and (2.3) is denoted by $LH(\mathbb{R}^n)$. It is well-known that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, thus we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, see [24].

Lemma 2.1 (see [21]) Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we have

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.2 (see [21]) Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we have for all measurable subsets $E \subset B$,

$$\frac{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|E|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_E\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|E|}{|B|} \right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Remark 2.1 We would like to stress that everywhere below the constants δ_1 and δ_2 are always the same as in Lemma 2.2.

Lemma 2.3 (see [24]) Suppose $p_i(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$, so that

$$\frac{1}{p(x)} = \sum_{i=1}^m \frac{1}{p_i(x)},$$

where $m \in \mathbb{N}$. Then for all $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, we have

$$\left\| \prod_{i=1}^m f_i \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.4 (see [25]) Suppose $p(\cdot) \in LH(\mathbb{R}^n)$ and $0 < p_- \leq p(x) \leq p_+ < \infty$.

(i) For all balls (or cubes) $|B| \leq 2^n$ and any $x \in B$, we have

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B|^{1/p(x)}.$$

(ii) For all balls (or cubes) $|B| \geq 1$, we have

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B|^{1/p_\infty},$$

where $p_\infty := \lim_{x \rightarrow \infty} p(x)$.

Combining Lemma 2.3, Lemma 2.4 and Lemma 3 in [21, page 464], a simple computation shows that

Lemma 2.5 Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $b_i \in \text{BMO}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$, $k > j$ ($k, j \in \mathbb{N}$), then we have

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\| \prod_{i=1}^m (b_i - (b_i)_B) \chi_B \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \prod_{i=1}^m \|b_i\|_*$$

and

$$\left\| \prod_{i=1}^m (b_i - (b_i)_{B_j}) \chi_{B_k} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k-j)^m \prod_{i=1}^m \|b_i\|_* \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 2.2 We note that Lemma 2.5 generalizes the well known properties for $BMO(\mathbb{R}^n)$ spaces (see [26]), and is also a generalization of Lemma 3 in [21].

3 Boundedness on Variable Exponent Lebesgue Spaces

We first recall some pointwise estimates for sharp maximal functions, the duality and density in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ can be endowed with the Orlicz type norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^0 := \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| dx : g \in L^{p'(\cdot)}(\mathbb{R}^n), \quad \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1 \right\}.$$

This norm, as pointed out in [7], is equivalent to the Luxemburg-Nakano norm, that is

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^0 \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad (3.1)$$

where $r_p = 1 + 1/p_- - 1/p_+$.

By L_c^∞ we denote the set of all bounded functions f with compact support. From [7, Theorem 2.11] (see also [17, Lemma 2.2]), we get the following result.

Proposition 3.1 Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then L_c^∞ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ and in $L^{p'(\cdot)}(\mathbb{R}^n)$. For $\delta > 0$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, we define

$$M_\delta(f)(x) = M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\delta dy \right)^{1/\delta}.$$

Given a function $f \in L_{\text{loc}}^\delta(\mathbb{R}^n)$, set also

$$f_\delta^\sharp(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \left(\frac{1}{|Q|} \int_Q |f(y) - c|^\delta dy \right)^{1/\delta},$$

where the supremums are taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

The non-increasing rearrangement of a measurable function f on \mathbb{R}^n is defined as

$$f^*(t) := \inf \left\{ a > 0 : |\{s \in \mathbb{R}^n : |f(s)| > a\}| \leq t \right\}, \quad t > 0,$$

and for a fixed $\lambda \in (0, 1)$, the local sharp maximal function $M_\lambda^\sharp f$ is given by

$$M_\lambda^\sharp(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|).$$

The next lemma is due to Karlovich and Lerner [17, Proposition 2.3].

Lemma 3.1 Suppose $\lambda \in (0, 1)$, $\delta > 0$ and $f \in L_{\text{loc}}^\delta(\mathbb{R}^n)$, then we have

$$M_\lambda^\sharp(f)(x) \leq (1/\lambda)^{1/\delta} f_\delta^\sharp(x), \quad x \in \mathbb{R}^n.$$

A function Φ defined on $[0, \infty)$ is said to be a Young function, if Φ is a continuous, nonnegative, strictly increasing and convex function with $\lim_{t \rightarrow 0^+} \Phi(t)/t = \lim_{t \rightarrow 0^+} t/\Phi(t) = 0$. We define the Φ -average of a function f over a cube Q by

$$\|f\|_{\Phi, Q} = \inf \left\{ \eta > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|f(y)|}{\eta} \right) dy \leq 1 \right\}.$$

Associated to this Φ -average, we define the maximal operator M_Φ by

$$M_\Phi(f)(x) := \sup_{Q \ni x} \|f\|_{\Phi, Q}.$$

When $\Phi(t) = t \log^r(e + t)$ ($r \geq 1$), we denote M_Φ by $M_{L(\log L)^r}$. It is well-known that if $m \in \mathbb{N}$, then $M_{L(\log L)^m} \approx M^{m+1}$, the $m + 1$ iterations of the Hardy-Littlewood maximal operator M , see [1].

Lemma 3.2 (see [2]) Let $0 < \delta < 1$. Then there exists a positive constant C , independent of f and x , such that $(g_\mu^*(f))_\delta^\sharp(x) \leq CMf(x)$, $x \in \mathbb{R}^n$ holds for all bounded function f with compact support.

In fact, there holds a similar pointwise estimate for the multilinear commutators $g_{\mu, \vec{b}}^*$. To state it, we first introduce some notations.

As in [1], given any positive integer m , for all $1 \leq j \leq m$, we denote by C_j^m the family of all finite subset $\sigma = \{\sigma(1), \sigma(2), \dots, \sigma(j)\}$ of $\{1, 2, \dots, m\}$ of j different elements. For any $\sigma \in C_j^m$, we associate the complementary sequence σ' given by $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$.

Suppose $\vec{b} = (b_1, b_2, \dots, b_m)$ and $\sigma = \{\sigma(1), \sigma(2), \dots, \sigma(j)\} \in C_j^m$. Denote

$$\vec{b}_\sigma = \{b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(j)}\}, \quad b_\sigma = b_{\sigma(1)} b_{\sigma(2)} \cdots b_{\sigma(j)}$$

and $\|b_\sigma\| = \prod_{j \in \sigma} \|b_j\|_*$. If $\sigma = \{1, 2, \dots, m\}$, then we denote $\|b_\sigma\|$ by $\|\vec{b}\|$.

For any $\sigma = \{\sigma(1), \sigma(2), \dots, \sigma(j)\} \in C_j^m$, we define

$$g_{\mu, \vec{b}_\sigma}^*(f)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \left| \int_{\mathbb{R}^n} \psi_t(y - z) \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(z)) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

In the case $\sigma = \{1, 2, \dots, m\}$, we understand $g_{\mu, \vec{b}_\sigma}^* = g_{\mu, \vec{b}}^*$ and $g_{\mu, \vec{b}_{\sigma'}}^* = g_\mu^*$.

We now mention an immediate consequence of Proposition 2.4 in [2].

Lemma 3.3 Suppose $\mu > 2$ and $0 < \delta < \varepsilon < 1$. Then for any $f \in L_c^\infty$, there exists a constant $C > 0$, depending only on δ and ε , such that

$$(g_{\mu, \vec{b}}^*(f))_\delta^\sharp(x) \leq C \left\{ \|\vec{b}\| M_{L(\log L)^m} f(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \|b_\sigma\| M_\varepsilon(g_{\mu, \vec{b}_{\sigma'}}^* f)(x) \right\}.$$

We also need the following result from Lerner [27, Theorem 1].

Lemma 3.4 Suppose $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and let f be a measurable function with $f^*(+\infty) = 0$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq c_n \int_{\mathbb{R}^n} M_{\lambda_n}^\sharp f(x)Mg(x)dx,$$

where constants $0 < \lambda_n < 1$ and c_n depend only on dimension n .

To prove Theorem 3.1, we first prove the following result which has its independent role.

Lemma 3.5 Suppose $\mu > 2$ and $0 < \gamma < \min\{(\mu - 2)n/2, 1\}$. If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then g_μ^* is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

Proof Let $f \in L_c^\infty$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. Since g_μ^* is of weak type $(1, 1)$ provided that $\mu > 2$ and $0 < \gamma < \min\{(\mu - 2)n/2, 1\}$ (see [2, Theorem 1.1]), from Lemmas 3.4, 3.1, 3.2 and the generalized Hölder inequality (2.1), we get that

$$\begin{aligned} \int_{\mathbb{R}^n} |g_\mu^*(f)(x)g(x)|dx &\leq C_n \int_{\mathbb{R}^n} M_{\lambda_n}^\sharp (g_\mu^*(f))(x)Mg(x)dx \\ &\leq C_n \int_{\mathbb{R}^n} (1/\lambda_n)^{1/\delta} (g_\mu^*(f))_\delta^\sharp(x)Mg(x)dx \leq C_n \int_{\mathbb{R}^n} Mf(x)Mg(x)dx \\ &\leq C_n r_p \|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|Mg\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (3.2)$$

Here for the last inequality we have used the fact that if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, see [21, Proposition 2]. Thus we have

$$\|g_\mu^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|g_\mu^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^0 \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

By Proposition 3.1, this concludes the proof of Lemma 3.5.

We now state the main result of this section.

Theorem 3.1 Suppose $\mu > 2$, $0 < \gamma < \min\{(\mu - 2)n/2, 1\}$ and $b_i \in \text{BMO}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$. If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then $g_{\mu, \vec{b}}^*$ are bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

Proof Let $f \in L_c^\infty$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. We show Theorem 3.1 by induction on m . For $m = 1$, by Theorem 1.2 in [2, page 1850], $[b, g_\mu^*]f$ satisfies the assumptions of Lemma 3.4. Thus, as argued in (3.2), we deduce that

$$\begin{aligned} \int_{\mathbb{R}^n} |[b, g_\mu^*](f)(x)g(x)|dx &\leq C_n \int_{\mathbb{R}^n} M_{\lambda_n}^\sharp ([b, g_\mu^*]f)(x)Mg(x)dx \\ &\leq C_n \int_{\mathbb{R}^n} (1/\lambda_n)^{1/\delta} ([b, g_\mu^*]f)_\delta^\sharp(x)Mg(x)dx \\ &\leq C_n \|b\|_* \int_{\mathbb{R}^n} \left(M_{L(\log L)} f(x) + M_\varepsilon(g_\mu^*(f))(x) \right) Mg(x)dx \\ &\leq C_n r_p (\|M^2 f\|_{L^{p(\cdot)}} + \|M_\varepsilon(g_\mu^*(f))\|_{L^{p(\cdot)}}) \|Mg\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ &\leq C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (3.3)$$

This together with (3.1) yields

$$\|[b, g_\mu^*](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|[b, g_\mu^*](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^0 \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Suppose now that the Theorem 3.1 is true for $m - 1$. We will show that it is true for m . Once again by Theorem 1.2 in [2], according to Lemmas 3.4, 3.1, 3.2 and the generalized Hölder inequality (2.1), we have

$$\begin{aligned}
 & \int_{\mathbb{R}^n} |(g_{\mu, \vec{b}}^* f)(x)g(x)|dx \\
 & \leq C \int_{\mathbb{R}^n} M_{\lambda_n}^\sharp(g_{\mu, \vec{b}}^* f)(x)Mg(x)dx \\
 & \leq C \int_{\mathbb{R}^n} (1/\lambda_n)^{1/\delta} (g_{\mu, \vec{b}}^* f)_\delta^\sharp(x)Mg(x)dx \\
 & \leq C \int_{\mathbb{R}^n} \left\{ \|\vec{b}\| M_{L(\log L)^m} f(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \|b_\sigma\| M_\varepsilon(g_{\mu, \vec{b}_\sigma}^* f)(x) \right\} Mg(x)dx \quad (3.4) \\
 & \leq C \int_{\mathbb{R}^n} \left\{ \|\vec{b}\| M_{L(\log L)^m} f(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \|b_\sigma\| M_{L(\log L)^{m-j}} f(x) \right\} Mg(x)dx \\
 & \leq C \prod_{j=1}^m \|b\|_* \int_{\mathbb{R}^n} \sum_{j=1}^m M_{L(\log L)^{m-j}} f(x)Mg(x)dx \\
 & \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
 \end{aligned}$$

Now we obtain from (3.4) and (3.1) that

$$\|g_{\mu, \vec{b}}^* f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|g_{\mu, \vec{b}}^* f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^0 \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

By Proposition 3.1, this concludes the proof of Theorem 3.1.

4 Boundedness on Variable Exponent Herz-Type Hardy Spaces

The main purpose of this section is to further study the mapping properties of the multilinear commutators $g_{\mu, \vec{b}}^*$ in variable exponent Herz-type Hardy spaces. Before stating the main result, we give some definitions.

Let

$$B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, \quad R_k = B_k \setminus B_{k-1}$$

and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$.

Definition 4.1 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$. The homogeneous variable exponent Herz space $\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ satisfying

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)} := \left(\sum_{k \in \mathbb{Z}} 2^{\alpha k q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty$$

with the usual modification when $q = \infty$.

For $x \in \mathbb{R}$, we denote by $[x]$ the largest integer less than or equal to x .

Definition 4.2 Suppose $\alpha \geq n\delta_2$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and non-negative integer $s \geq [\alpha - n\delta_2]$. Let b_i ($i = 1, 2, \dots, m$) be alocally integrable function and $\vec{b} = (b_1, b_2, \dots, b_m)$. A function $a(x)$ on \mathbb{R}^n is said to be a central $(\alpha, p(\cdot), s; \vec{b})$ -atom, if it satisfies

- (i) $\text{supp } a \subset \tilde{B} := \{x \in \mathbb{R}^n : |x| < r\}$.
- (ii) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq |\tilde{B}|^{-\frac{\alpha}{n}}$.
- (iii) $\int_{\tilde{B}} x^\beta a(x) \prod_{i \in \sigma} b_i(x) dx = 0$ for $|\beta| \leq s, \sigma \in C_j^m, j = 0, 1, \dots, m$.

Remark 4.1 It is easy to see that if $p(x) \equiv p$ is constant, then taking $\delta_2 = 1 - 1/p$, we can get the classical case, see [28].

A temperate distribution f is said to belong to $\dot{H}\dot{K}_{p(\cdot), \vec{b}}^{\alpha, q, s}(\mathbb{R}^n)$, if it can be written as

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where a_j is a central $(\alpha, p(\cdot), s; \vec{b})$ -atom with support contained in B_j , $\lambda_j \in \mathbb{R}$ and

$$\sum |\lambda_j|^q < \infty.$$

Moreover,

$$\|f\|_{\dot{H}\dot{K}_{p(\cdot), \vec{b}}^{\alpha, q, s}(\mathbb{R}^n)} \approx \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f .

Our main result in this section can be stated as follows.

Theorem 4.1 Suppose $0 < \gamma < 1$ and $\mu > 3 + 2/n + 2\gamma/n$. If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $0 < q < \infty$ and $n\delta_2 \leq \alpha < n\delta_2 + \gamma$, where δ_2 is the constant appearing in Lemma 2.2. Then the multilinear commutators $g_{\mu, \vec{b}}^*$ map $\dot{H}\dot{K}_{p(\cdot), \vec{b}}^{\alpha, q, 0}(\mathbb{R}^n)$ into $\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)$.

Proof Let a_j be a central $(\alpha, p(\cdot), 0; \vec{b})$ -atom with support contained in B_j . We first restrict $0 < q \leq 1$. In this case, it suffices to show that

$$\|g_{\mu, \vec{b}}^* a_j\|_{\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)} \leq C.$$

We write

$$\begin{aligned} \|g_{\mu, \vec{b}}^* a_j\|_{\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)}^q &= \sum_{k=-\infty}^{j+2} 2^{k\alpha q} \|\chi_k g_{\mu, \vec{b}}^* a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q + \sum_{k=j+3}^{\infty} 2^{k\alpha q} \|\chi_k g_{\mu, \vec{b}}^* a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ &:= I + J. \end{aligned}$$

For I , by the boundedness of $g_{\mu, \vec{b}}^*$ in $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain

$$I \leq C \sum_{k=-\infty}^{j+2} 2^{k\alpha q} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \leq C \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha q} \leq C.$$

We proceed now to estimate J . If $x \in R_k$, $y \in B_j$ and $k \geq j + 3$, then $2|y| < |x|$. By the vanishing condition of a_j , we get that

$$\begin{aligned} g_{\mu, \vec{b}}^*(a_j)(x) &= \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \left| \int_{B_j} \psi_t(y - z) \prod_{i=1}^m (b_i(x) - b_i(z)) a_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \left(\int_{B_j} |\psi_t(y - z) - \psi_t(y)| \prod_{i=1}^m |b_i(x) - b_i(z)| |a_j(z)| dz \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \frac{t^{1-n} dy dt}{(t + |y|)^{2(n+\gamma+1)}} \right)^{1/2} \int_{B_j} |z|^\gamma |a_j(z)| \prod_{i=1}^m |b_i(x) - b_i(z)| dz. \end{aligned} \quad (4.1)$$

If $\mu > 3 + 2/n + 2\gamma/n$, using the same estimates in [3, page 5], then we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \frac{t^{1-n} dy}{(t + |y|)^{2(n+\gamma+1)}} \\ &\leq C \sum_{k=1}^{\infty} 2^{k(3n+2+2\gamma-n\mu)} \frac{t}{(t + |x|)^{2(n+\gamma+1)}} \\ &\leq C \frac{t}{(t + |x|)^{2(n+\gamma+1)}}. \end{aligned} \quad (4.2)$$

Combining (4.1) and (4.2), we arrive at the estimate

$$\begin{aligned} g_{\mu, \vec{b}}^*(a_j)(x) &\leq C \left(\int_0^\infty \frac{t dt}{(t + |x|)^{2(n+\gamma+1)}} \right)^{1/2} \int_{B_j} |z|^\gamma |a_j(z)| \prod_{i=1}^m |b_i(x) - b_i(z)| dz \\ &\leq C |x|^{-n-\gamma} \int_{B_j} |z|^\gamma |a_j(z)| \prod_{i=1}^m |b_i(x) - b_i(z)| dz \\ &\leq C 2^{(j-k)\gamma-kn} \int_{B_j} |a_j(z)| \prod_{i=1}^m |b_i(x) - b_i(z)| dz. \end{aligned} \quad (4.3)$$

Let $\lambda_i = (b_i)_{B_j}$. An application of (4.3), (2.1), Lemmas 2.5, 2.1 and 2.2 give

$$\begin{aligned} &\|\chi_k g_{\mu, \vec{b}}^* a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{(j-k)\gamma-kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{i=0}^m \sum_{\sigma \in C_i^m} \|(b(\cdot) - \vec{\lambda})_\sigma \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b(\cdot) - \vec{\lambda})_{\sigma'} \chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ &\leq C (k - j)^m \prod_{i=1}^m \|b_i\|_* 2^{(j-k)\gamma-j\alpha} 2^{-kn} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ &\leq C (k - j)^m 2^{(j-k)\gamma-j\alpha} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\ &\leq C (k - j)^m 2^{(j-k)(\gamma+n\delta_2)-j\alpha}. \end{aligned} \quad (4.4)$$

Consequently, by the condition $\gamma + n\delta_2 - \alpha > 0$, we have

$$J \leq C \sum_{k=j+3}^{\infty} (k-j)^{mq} 2^{(j-k)(\gamma+n\delta_2-\alpha)q} \leq C.$$

Now let $1 < q < \infty$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$. For convenience below we put $\sigma = \gamma + n\delta_2 - \alpha$, then we have $\sigma > 0$. From (4.4) and the $L^{p(\cdot)}(\mathbb{R}^n)$ -boundedness of the multilinear commutators $g_{\mu, \vec{b}}^*$, it follows that

$$\begin{aligned} \|g_{\mu, \vec{b}}^* f\|_{\dot{K}_{p(\cdot)}^{\alpha, q}(\mathbb{R}^n)} &\leq \left\{ \sum_{k=-\infty}^{\infty} 2^{\alpha k q} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|\chi_k g_{\mu, \vec{b}}^* a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &\quad + \left\{ \sum_{k=-\infty}^{\infty} 2^{\alpha k q} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|\chi_k g_{\mu, \vec{b}}^* a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &\leq C \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| (k-j)^m 2^{(j-k)\sigma} \right)^q \right\}^{1/q} \\ &\quad + C \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^q \right\}^{1/q} \\ &\leq C \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^q 2^{(j-k)\sigma q/2} \right) \left(\sum_{j=-\infty}^{k-3} (k-j)^{mq'} 2^{(j-k)\sigma q'/2} \right)^{q/q'} \right\}^{1/q} \\ &\quad + C \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{\infty} |\lambda_j|^q 2^{(k-j)\alpha q/2} \right) \left(\sum_{j=k-2}^{\infty} 2^{(j-k)\alpha q'/2} \right)^{q/q'} \right\}^{1/q} \\ &\leq C \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^q \left(\sum_{k=j+3}^{\infty} 2^{(j-k)\sigma q/2} \right) \right\}^{1/q} + C \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^q \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha q/2} \right) \right\}^{1/q} \\ &\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^q \right)^{1/q}. \end{aligned}$$

This completes the proof of Theorem 4.1.

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变指数Lebesgue空间上Littlewood-Paley算子的多线性交换子

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摘要: 本文研究了Littlewood-Paley算子的多线性交换子在变指数Lebesgue空间上的有界性. 基于原子分解和广义BMO范数, 证明了这类多线性交换子在变指数Herz型Hardy空间上的有界性, 推广了一些已知结果.

关键词: Littlewood-Paley算子; 变指数; Herz型Hardy空间; 交换子

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