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DISTRIBUTIVE LATTICES WITH A HOMOMORPHIC OPERATION

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Abstract: In this paper, we initiate an investigation into the class **eD** of extended bounded distributive lattices, namely, bounded distributive lattices endowed with an unary operation, which is an endomorphism. By using the set of fixed points and congruences, we characterise the (finitely) subdirectly irreducible algebras in $\mathbf{e}_{p,q}\mathbf{D}$. In particular, we show that for every algebra in $\mathbf{e}_{p,q}\mathbf{D}$, the properties of being finitely subdirectly irreducible and subdirectly irreducible are equivalent. Some results obtained by Balbes and Dwinger on semilattices, distributive lattices and Boolean algebras are generalized.

Keywords: extended bounded distributive lattice; congruence; subdirectly irreducible

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1 Introduction

An algebra L is said to be subdirectly irreducible (s.i) if it has a smallest non-trivial congruence; i.e., a congruence α, such that $\theta \geq \alpha$ for all $\theta \in \mathrm{Con}L$ with $\theta \neq \omega$. A particularly important case of a subdirectly irreducible algebra is a simple algebra, namely, one for which the lattice of congruence is the two-element chain $\{\omega, \iota\}$. Associated with the notion of a subdirectly irreducible algebra is that of a finitely subdirectly irreducible (f.s.i) algebra, this being defined as an algebra in which the intersection of two non-trivial principal congruences is non-trivial. Clearly, every s.i. algebra is f.s.i. algebra and f.s.i. semilattices, distributive lattices and Boolean algebras are s.i. Many algebras arising in logic have (distributive) lattice reducts and share the same property; for example, De Morgan algebras [1], the Post algebras of order n, the n-valued Lukasiewicz algebras [2] and demi-p-lattices [3]. In this paper, we consider the class of bounded distributive lattices endowed with an unary operation, which is an endomorphism f with $f(0) = 0, f(1) = 1$ (the class will be denoted by **eD**), characterise finitely subdirectly irreducible algebras and the subdirectly irreducible algebras in **eD**.

2 Preminaries

Let $(L, f) \in \mathbf{eD}$ and let $f^0 = id$ and define f^n recursively by $f^n(x) = f(f^{n-1}(x))$ for $n \geq 1$, then for $p, q \in \mathbb{N}$ with $p > q \geq 0$, we define the subclass $\mathbf{e}_{p,q}\mathbf{D}$ of **eD** by adjoining the

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equation $f^p = f^q$. It follows that the smallest nontrivial subclass $\mathbf{e}_{p,q}\mathbf{D}$ is the class $\mathbf{e}_{1,0}\mathbf{D}$, which is determined by the equation $f = id$. We define $\mathbf{e}_{2,0}\mathbf{D}$ to be the class of symmetric extended distributive lattices, see [4].

A congruence on L is an equivalence relation that has the substitution property for both the lattice operations and for the unary operation f. It follows that every congruence is in particular a lattice congruence and it is essential to distinguish these two types. In order to do so, we shall use the subscript 'lat' to denote a lattice congruence.

Lemma 2.1 (see [5]) Let $(L, f) \in \mathbf{eD}$. If $a, b \in L$ with $a \leq b$ in L, then

$$\theta(a, b) = \bigvee_{n \geq 0} \theta_{\text{lat}}(f^n(a), f^n(b)).$$

Corollary 2.1 Let $(L, f) \in \mathbf{e}_{2,0}\mathbf{D}$. Then

$$\theta(a, b) = \theta_{\text{lat}}(a, b) \vee \theta_{\text{lat}}(f(a), f(b)).$$

For an algebra $(L; f) \in \mathbf{eD}$, consider now, for every $n \in \mathbb{N}$, the relation Φ_n on L defined by $(x, y) \in \Phi_n \Leftrightarrow f^n(x) = f^n(y)$.

It is clear that Φ_n is a congruence on L. Moveover, the subset $f^n(L) = \{f^n(x) | x \in L\}$ is a subalgebra of L.

We now consider some basic results concerning these congruences. Of especial importance in this is the congruence $\Phi\omega = \bigvee\limits_{i \geq 0} \Phi_i$.

For every non-trivial algebra $(L; f) \in \mathbf{eD}$, it is clear that we have

$$\omega = \Phi_0 \leq \Phi_1 \leq \Phi_2 \leq \cdots \leq \Phi_i \leq \Phi_{i+1} \leq \cdots \leq \Phi_\omega < \iota$$

and with $\leq$ meaning "is a subalgebra of",

$$\{0, 1\} \leq \ldots \leq f^{i+1}(L) \leq f^i(L) \leq \ldots \leq f(L) \leq f^0(L) = L.$$

It is readily seen that $[x]\Phi_i \to f^i(x)$ describes an algebra isomorphism in $\mathbf{eD}$. We shall denote by writing $L/\Phi_i \backsimeq f^i(L)$.

The following result is therefore clear.

Lemma 2.2 If $(L; f) \in \mathbf{e}_{p,q}\mathbf{D}$, then, for $n \leq q$,

$$L/\Phi_n \backsimeq f^n(L) \in \mathbf{e}_{p,q-n}\mathbf{D}.$$

The following two lemmas are an extension to $\mathbf{eD}$-algebras of Blyth for Ockham algebras [6].

Lemma 2.3 If $(L; f) \in \mathbf{e}_{p,q}\mathbf{D}$, then

$$\omega = \Phi_0 \leq \Phi_1 \leq \cdots \leq \Phi_q = \Phi_{q+1} = \cdots = \Phi_\omega.$$

Moreover, if $(L; f)$ belongs to the subclass $\mathbf{e}_{p,q}\mathbf{D}$, then each of the above is equivalent to $(L; f) \in \mathbf{e}_{p-q,0}\mathbf{D}$.

Lemma 2.4 If $(L;f) \in \mathbf{e}_{p,q}\mathbf{D}$, $a, b \in L$ are such that $a \prec b$ and $f(a) = f(b)$ then $\theta(a,b)$ is an atom of Con$L$.

3 Subdirectly Irreducible Algebras

Given an algebra $(L;f) \in \mathbf{eD}$, consider now for each $i \geq 1$ the subset

$$T_i(L) = \{x \in L | f^i(x) = x\}.$$

In particular, $T_1(L)$ is the set of fixed points of f. Of course, $T_1(L)$ is never empty, for it clearly contains 0 and 1. It is readily seen that every subset $T_n(L)$ is a subalgebra of L; in fact $T_n(L)$ is the largest $\mathbf{e}_{n,0}\mathbf{D}$-subalgebra of L.

Consider now the subset

$$T(L) = \{x \in L | \text{ there exists a positive integer } \ m_x \text{ such that } \ f^{m_x}(x) = x\}.$$

Given $x, y \in T(L)$, let $m = lcm\{m_x, m_y\}$. Then, m being positive integer, we have

$$f^m(x \vee y) = f^m(x) \vee f^m(y) = x \vee y,$$

and similarly $f^m(x \wedge y) = x \wedge y$. Since $x \in T(L)$ clearly implies $f(x) \in T(L)$, it follows that $T(L)$ is also a subalgebra of L.

Theorem 3.1 Let $(L;f) \in \mathbf{eD}$ be finitely subdirectly irreducible. Then $T_1(L) = \{0, 1\}$.

Proof Suppose that $T_1(L)$ contains at least three elements. Then it contains a 3-element chain $0 < a < 1$. Then, by Lemma 2.1, we have $\theta(0,a) = \theta_{\text{lat}}(0,a)$ and $\theta(a,1) = \theta_{\text{lat}}(a,1)$, whence we have the contradiction $\theta(0,a) \wedge \theta(a,1) = \omega$.

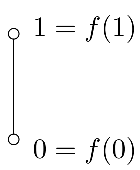
Consider now the particular case where L is a finitely subdirectly irreducible symmetry extended distributive lattice.

Lemma 3.1 Let $L \in \mathbf{e}_{2,0}\mathbf{D}$. Then for $x \in L \setminus \{0,1\}$, either $x || f(x)$, or $x = f(x)$.

Corollary 3.1 Let $L \in \mathbf{e}_{2,0}\mathbf{D}$ be finitely subdirectly irreducible and $|L| \geq 2$. Then for $x \in L \setminus \{0,1\}$, $x || f(x)$.

Proof If $x \in L \setminus \{0,1\}$, then $x \neq f(x)$ according to Theorem 3.1 and so $x || f(x)$ by Lemma 3.1.

Theorem 3.2 In the class $\mathbf{e}_{2,0}\mathbf{D}$ of symmetric extended distributive lattices there are only two (finitely) subdirectly irreducible algebras, each of which is simple, namely, the algebras.



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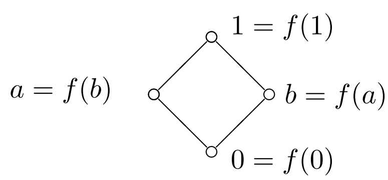

(M)

Proof If $|L| = 2$, then $L = \{0, 1\}$. If $|L| = 3$, let $L = \{0, a, 1\}$, then $a \notin \{0, 1\}$. We have $a||f(a)$ according to Corollary 3.1 and so $|L| \neq 3$, which is absurd. If $|L| = 4$, let $L = \{0, a, b, 1\}$, then $a, b \notin \{0, 1\}$ and $a||f(a)$, $b||f(b)$. Thus $f(a) = b$, $f(b) = a$. Thus L is M in the above figures. If $|L| \geq 5$, then L has 5 distinct points $0, a, b, c, 1$. If $a \wedge b \neq 0$, then $f(a \wedge b) \neq 0$ and $a \wedge b \wedge f(a \wedge b) = 0, (a \wedge b) \vee f(a \wedge b) = 1$ by Theorem 3.1. Since $a \wedge f(a) = 0$ and $b \wedge f(b) = 0, (a \wedge b) \wedge f(a \vee b) = 0$ and $(a \wedge b) \vee f(a \vee b) = 1$, $f(a \vee b) = f(a \wedge b)$ and so $f(a) = f(b), a = b$, a contradiction. Thus $a \wedge b = 0$ and so $a \wedge (f(a) \vee b) = 0, a \vee (f(a) \vee b) = 1$ and therefore

$$f(a) \vee b = f(a), \text{i.e.}, f(a) \geq b.$$

Likely, $b \wedge (f(b) \vee a) = 0, b \vee (f(b) \vee a) = 1$ and therefore $f(b) \vee a = f(b)$ and so $f(b) \geq a$ and also $b \geq f(a)$. Hence we have $f(a) = b$ and so $a \vee b = 1$. Also, $a \wedge c = 0, a \vee c = 1$. By distributive $b = c$, a contradiction.

Thus if L is a finitely subdirectly irreducible symmetry extended distributive lattice, then L is only of the two kinds in the above figures.

It is readily seen that for these we have Con$B$ =Con$M \simeq 2$. So these algebras are indeed subdirectly irreducible; in fact, they are simple.

Theorem 3.3 If $L \in \mathbf{eD}$ is such that $T_1(L) = \{0, 1\}$ and if $a, b \in T(L)$ are such that $a < b$ then $\theta(a, b) = \iota$.

Proof For every $x \in T(L)$, let m_x be the least positive integer such that $f^{m_x}(x) = x$. Consider the elements

$$\alpha(x) = x \wedge f(x) \wedge \cdots \wedge f^{m_x - 1}(x), \beta(x) = x \vee f(x) \vee \cdots \vee f^{m_x - 1}(x).$$

Observe that $f(\alpha(x)) = \alpha(x)$ and $f(\beta(x)) = \beta(x)$, so $\alpha(x), \beta(x) \in T_1(L) = \{0, 1\}$. Now, let $a, b \in T(L)$ be such that $a < b$. Consider the sublattice A that is generated by

$$\{f^i(a), f^j(b) | 0 \leq i \leq m_a - 1, 0 \leq j \leq m_b - 1\}.$$

Clearly, A is finite with the smallest element $\alpha(a) = 0$ and the greatest element $\beta(b) = 1$. Let p be an atom of A and since every atom of A is of the form $\bigwedge_{i \neq j} f^i(a)$ for some j, it follows that $f(p)$ is also an atom of A and $f^{m_a}(p) = p$, and so

$$p \in T(L), \beta(p) = p \vee f(p) \vee \cdots \vee f^{m_a - 1}(p) = 1.$$

Consequently, A is boolean.

Let c be an atom of A with $c \not\leq a$ and $c \leq b$. Then $(0, c) = (a \wedge c, b \wedge c) \in \theta(a, b)$. It follows that $(0, f(c)) \in \theta(a, b)$ and so $(0, \beta(c)) \in \theta(a, b)$, i.e., $(0, 1) \in \theta(a, b)$ and therefore $\theta(a, b) = \iota$.

Theorem 3.4 For an algebra $L \in \mathbf{eD}$ the following are equivalent:

(1) $T_1(L) = \{0, 1\}$;

(2) the subalgebra $T(L)$ is simple;

(3) all symmetric extended distributive sublattices of L are simple.

Proof (1) $\Rightarrow$ (2) If (1) holds, then by Theorem 3.3 every non-trivial principal congruence on $T(L)$ coincides with ι. Since every congruence is the supremum of the principal congruences that it contains, it follows that $T(L)$ is simple.

(2) $\Leftrightarrow$ (3) $T_2(L)$ is the largest symmetric extended distributive sublattice of L.

(3) $\Rightarrow$ (1) If (3) holds, then $T_2(L)$ is simple. But $T_2(L)$ is a symmetric extended distributive lattice and by Theorem 3.2 there are only two non-isomorphic simple symmetric extended distributive lattices, in each of which $T_1(T_2(L)) = \{0, 1\}$. Since $T_2(L)$ and L have the same fixed points, (1) follows.

Theorem 3.5 If $L \in \mathbf{eD}$ is finitely subdirectly irreducible, then every Φ_1-class in L contains at most two elements.

Proof Suppose that a Φ_1-class contains at least three elements. Then it contains a 3-element chain $x < y < z$ with $f(x) = f(y) = f(z)$. Then, by Lemma 2.1, we have $\theta(x, y) = \theta_{\text{lat}}(x, y)$ and $\theta(y, z) = \theta_{\text{lat}}(y, z)$, whence we have the contradiction $\theta(x, y) \wedge \theta(y, z) = \omega$.

Theorem 3.6 If $L \in \mathbf{e}_{p,q}\mathbf{D}$ then the following statements are equivalent:

(1) L is finitely subdirectly irreducible;

(2) L is subdirectly irreducible.

Proof (1) $\Rightarrow$ (2) Since $L \in \mathbf{e}_{p,q}\mathbf{D}$, for every $x \in L$, we have $f^p(x) = f^q(x)$. If $\Phi_1 = \omega$, then f is injective and $x = f^{p-q}(x)$, whence $x \in T(L)$. Thus $L = T(L)$ and it follows by Theorem 3.1 and Theorem 3.4 that L is simple, hence subdirectly irreducible.

On the other hand, if $\Phi_1 \neq \omega$, then by Theorem 3.5 there is a two-element Φ_1-class $\{a, b\}$, and by Lemma 2.4, $\theta(a, b)$ is an atom in the interval $[\omega, \Phi_1]$ of Con$L$. If now $\alpha \in$ Con$L$ with $\alpha \neq \omega$ then, since α is the supremum of the non-trivial principal congruences which it contains and since Con$L$ satisfies the infinite distributive law $\beta \wedge \vee\gamma_i = \vee(\beta \wedge \gamma_i)$, it follows by the hypothesis that L is finitely subdirectly irreducible that $\theta(a, b) \wedge \alpha = \theta(a, b)$ and hence $\theta(a, b) \leq \alpha$. Thus $\theta(a, b)$ is the smallest non-trivial congruence on L, so L is subdirectly irreducible.

(2) $\Rightarrow$ (1) This is clear.

The following theorem is an extension to **eD**-algebras of Blyth for Ockham algebras, see [6].

Theorem 3.7 $L \in \mathbf{e}_{p,q}\mathbf{D}$ is subdirectly irreducible if and only if Con L reduces to the finite chain

$$\omega = \Phi_0 \prec \Phi_1 \prec \cdots \prec \Phi_q \prec l.$$

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带有同态运算的分配格

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摘要: 本文研究了扩展的有界分配格 (即带有一个自同态运算的有界分配格)的次直不可约问题. 利用不动点集和同余的方法, 刻画了$\mathbf{e}_{p,q}\mathbf{D}$代数类中次直不可约代数, 获得了此类代数中有限次直不可约性和次直不可约性两者等价的结果, 推广了Balbes以及Dwinger关于半格、分配格和布尔代数的相关结果.

关键词: 扩展有界分配格; 同余关系; 次直不可约

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