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COMMON LIMIT DIRECTIONS OF JULIA SETS OF ENTIRE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS AND THEIR DERIVATIVES

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Abstract: In this paper, we mainly investigate the limit directions of Julia sets of solutions of linear differential equations. By using Nevanlinna theory, we obtain the lower bound on the limit directions of Julia sets of non-trivial solutions to such equations in some additional conditions on coefficients, which improves some results of concerned literature.

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1 Introduction and Main Results

In this paper, we shall use the standard notations of Nevanlinna theory, its usual notations and basic results come mainly from [1–4]. Now let f be a meromorphic function in the whole complex plane. We use $\lambda(f)$ and $\mu(f)$ to denote the order and the lower order of f, respectively, which are defined as [5, Definition 1.6]

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Define f^n, $n \in \mathbb{N}$ as the nth iterate of f, that is, $f^1 = f, \cdots, f^n = f \circ (f^{n-1})$. The Fatou set $F(f)$ of f is the subset of $\mathbb{C}$ where $\{f^n(z)\}_{n=1}^{\infty}$ forms a normal family, and its complement $J(f) = \mathbb{C} \setminus F(f)$ is called the Julia set of f. It is well known that $F(f)$ is open and completely invariant under f, and $J(f)$ is closed and non-empty. For an introduction to the dynamics of meromorphic functions, we refer the reader to see Bergweiler's paper [6] and Zheng's book [7].

Assuming $0 < \alpha < \beta < 2\pi$, we denote

$$\Omega(\alpha,\beta) = \{z \in \mathbb{C} | \arg z \in (\alpha,\beta)\}.$$

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Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded for every $\varepsilon > 0$, we say the radial $\arg z = \theta$ is a limit direction of $J(f)$. Define

$$\Delta(f) = \{\theta \in [0, 2\pi) : \text{ the radial } \quad \arg z = \theta \quad \text{is a limit direction of } \quad J(f)\}.$$

Clearly, $\Delta(f)$ is closed and measurable, and we use $\text{mes}\Delta(f)$ to denote its linear measure. The research on the limit directions was initially studied by Baker (see [8]), where Baker proved that, for a transcendental entire function f, $J(f)$ cannot lie in finitely many rays emanating from the origin. In [9], Qiao considered the limit directions of Julia sets of transcendental entire functions with finite lower order and obtained the following.

Theorem A Let $f(z)$ be a transcendental entire function with $\mu(f) < \infty$. Then $\text{mes}\Delta(f) = 2\pi$ if $\mu(f) < 1/2$ and $\text{mes}\Delta(f) \geq \pi/\mu(f)$ if $\mu(f) \geq 1/2$.

Naturally, a question arise here.

Question 1 What can we say about the limit directions of Julia set of entire functions with infinite lower order?

Baker (see [8]) constructed an entire function, for every $M > 0$, of infinite lower order satisfying

$$J(f) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| < M, \operatorname{Re} z > 0\}.$$

Thus $\Delta(f) = \{0\}$. Recently, Huang and Wang (see [10]) investigated the limit directions of Julia sets of products of the solution base of the following equation (1.1).

Theorem B Let $\{f_1, f_2, \cdots, f_n\}$ be a solution base of

$$f^{(n)} + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is a transcendental entire function with finite order, and denote $E = f_1 f_2 \cdots f_n$. Then

$$\text{mes}\Delta(E) \geq \min\{2\pi, \pi/\sigma(A)\}.$$

Actually, $E(z)$ can be of infinite lower order in some cases. For example, for the equation $f'' - (e^{2z} + e^z)f = 0$, we have $\mu(E(z)) = \infty$ (see [11, pp.394]). Later, Huang and Wang considered the limit directions of Julia sets of solutions of linear differential equations directly.

Theorem C (see [12]) Let $A_i(z)$ $(i = 0, 1, \cdots, n-1)$ be entire functions of finite lower order such that A_0 is transcendental and $m(r, A_i) = o(m(r, A_0))$ $(i = 1, 2, \cdots, n-1)$ as $r \to \infty$. Then every non-trivial solution f of the equation

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_0 f = 0 \tag{1.2}$$

satisfies $\text{mes}\Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$.

Clearly, by the lemma of logarithmic derivatives, each non-trivial solution $f(z)$ in Theorem C must have infinite lower order. Theorems B and C, therefore, obtain some results about the limit directions of Julia sets for some classes of entire functions of infinite lower order.

In this paper, we continue to discuss Question 1. Moreover, we will investigate the common limit directions of transcendental entire functions with infinite lower order and their derivatives.

Theorem 1.1 Let $A_0(z), \cdots, A_{n-1}(z)$, $A_0(z) \not\equiv 0$ be entire functions such that for real constants a, b, c, θ_1, θ_2, where $0 \le a < b$, $c > 0$ and $\theta_1 < \theta_2$, we have

$$|A_0(z)| \ge e^{b|z|^c} \tag{1.3}$$

and

$$|A_k(z)| \le e^{a|z|^c}, \quad k = 1, \cdots, n-1 \tag{1.4}$$

as $z \to \infty$ with $\theta_1 \le \arg z \le \theta_2$. If f is a non-trivial solution of equation (1.2), then $\mathrm{mes}(\Delta(f) \cap \Delta(f^{(k)})) \ge \theta_2 - \theta_1$.

Before we prove Theorem 1.1, we need to prove the next result.

Theorem 1.2 Under the hypothesis of Theorem 1.1, every solution $f(\not\equiv 0)$ of equation (1.2) satisfies $\mathrm{mes}\Delta(f) \ge \theta_2 - \theta_1$.

Remark 1 Clearly from Lemma 2.4 in Section 2, each non-trivial solution of (1.2) has infinite order. Next, we will give an example to show that the entire solutions of (1.2) can be of infinite lower order in some cases. In addition, we note that both Theorems B and C require that there is a dominant coefficient whose growth of order is greater than the other coefficients, while all the coefficients in our theorems may have the same order .

Example Consider the differential equation

$$f''' - (3 + 6e^z)f'' + (2 + 6e^z + 11e^{2z})f' - 6e^{3z}f = 0. \tag{1.5}$$

As we see, all the coefficients of this equation have the same order 1. In addition, for $z = re^{i\theta}$, $r \to +\infty$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{4}$, we have

$$\begin{aligned} |A_0(z)| &= \ |-6\,e^{3z}| = 6e^{3r\cos\theta} > e^{3\frac{\sqrt{2}}{2}r}, \\ |A_1(z)| &= |2 + 6e^z + 11e^{2z}| \le 19e^{2r\cos\theta} \le 19e^{\sqrt{3}\,r} < e^{2r}, \\ |A_2(z)| &= |-(3 + 6e^z)| \le 9e^{r\cos\theta} \le 9e^{\frac{\sqrt{3}}{2}r} < e^{2r}. \end{aligned}$$

Clearly, the three linearly independent functions

$$f_1(z) = e^{e^z}, \ \ f_2(z) = e^{2e^z}, \ \ f_3(z) = e^{3e^z}$$

are solutions of (1.5) with $\mu(f_1) = \mu(f_2) = \mu(f_3) = \infty$ and $\sigma(A_0) = \sigma(A_1) = \sigma(A_2)$.

Remark 2 Some results about common limit directions of transcendental entire functions and their derivatives were obtained by Qiao (see [13]). He proved that for transcendental entire functions of finite lower order and their derivatives, there exist a large amount of common limit directions. Wang [14] obtained some similar results for the case of transcendental meromorphic functions with finite lower order. By using the method of [12, 14], Sun (see [15]) and Zhang (see [16, 17]) obtained some results for solutions to some special classes of linear differential equations.

2 Preliminary Lemmas

In order to prove our Theorem, we first recall the Nevanlinna Characteristic in an angle, see [1]. We denote by $\overline{\Omega}(\alpha,\beta)$ the closure of $\Omega(\alpha,\beta)$, and set

$$\Omega(\alpha,\beta,r)=\{z\,|\,z\in\Omega(\alpha,\beta),|z|<r\},$$
$$\Omega^*(r,\alpha,\beta)=\{z\,|\,z\in\Omega(\alpha,\beta),|z|>r\}.$$

Let $g(z)$ be meromorphic on the angle $\Omega(\alpha,\beta)$ where $\beta-\alpha\in(0,2\pi]$. Following [1], we define

$$\begin{aligned}A_{\alpha,\beta}(r,g)&=\frac{\omega}{\pi}\int_1^r\Big(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\Big)\{\log^+|g(te^{i\alpha})|+\log^+|g(te^{i\beta})|\}\frac{dt}{t},\\ B_{\alpha,\beta}(r,g)&=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,\\ C_{\alpha,\beta}(r,g)&=2\sum_{1<|b_n|<r}\Big(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}}\Big)\sin\omega(\beta_n-\alpha),\end{aligned}\tag{2.1}$$

where $\omega=\pi/(\beta-\alpha)$, and $b_n=|b_n|e^{i\beta_n}$ are the poles of $g(z)$ in $\overline{\Omega}(\alpha,\beta)$ appeared according to their multiplicities. The Nevanlinna angular characteristic is defined as follows:

$$S_{\alpha,\beta}(r,g)=A_{\alpha,\beta}(r,g)+B_{\alpha,\beta}(r,g)+C_{\alpha,\beta}(r,g).\tag{2.2}$$

Especially, we use

$$\sigma_{\alpha,\beta}(g)=\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,g)}{\log r}$$

to denote the order of $S_{\alpha,\beta}(r,g)$.

We say an open set hyperbolic if it has at least three boundary points in $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Let W be a hyperbolic open set in $\mathbb{C}$. For an $a\in C\setminus W$, define

$$C_W(a)=\inf\{\lambda_W(z)|z-a|:\forall z\in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. It is well known that if every component of W is simply connected, then $C_W(a)\geq 1/2$.

Lemma 2.1 (see [7, 10]) Let $f(z)$ be analytic in $\Omega^*(r_0,\theta_1,\theta_2)$, U be a hyperbolic domain and $f:\Omega^*(r_0,\theta_1,\theta_2)\to U$. If there exists a point $a\in\partial U\setminus\{\infty\}$, such that $C_U(a)>0$, then there exists a constant $d>0$ such that for sufficiently small $\epsilon>0$, we have

$$|f(z)|=O(|z|^d),\ z\to\infty,\ z\in\Omega^*(r_0,\theta_1+\epsilon,\theta_2-\epsilon).$$

Before we show Lemma 2.2, which gives some estimates for the logarithmic derivative of functions being analytic in an angle, we will introduce the definition of R-set (see [3]). Set $B(z_n,r_n)=\{z:|z-z_n|<r_n\}$, if $\sum\limits_{n=1}^{\infty}r_n<\infty$ and $z_n\to\infty$, then $\bigcup\limits_{n=1}^{\infty}B(z_n,r_n)$ is called an R-set. Clearly, the set $\{\,|z|:z\in\bigcup\limits_{n=1}^{\infty}B(z_n,r_n)\}$ is of finite linear measure.

Lemma 2.2 (see [10]) Let $z = r\exp(i\psi), r_0 + 1 < r$ and $\alpha \le \psi \le \beta$ where $0 < \beta - \alpha \le 2\pi$. Suppose that $n(\ge 2)$ is an integer, and $g(z)$ is analytic in $\Omega(r_0, \alpha, \beta)$ with $\sigma_{\alpha,\beta}(g) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then for every $\varepsilon_j \in (0, \frac{\beta_j - \alpha_j}{2})$ $(j = 1, 2, \cdots, n-1)$ outside a set of linear measure zero with

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s \quad \text{and} \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, j = 2, 3, \cdots, n-1,$$

there exist $K > 0, M > 0$ only depending on $g, \varepsilon_1, \cdots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, and not depending on z such that

$$|\frac{g'(z)}{g(z)}| \le Kr^M(\sin k(\psi - \alpha))^{-2}$$

and

$$|\frac{g^{(n)}(z)}{g(z)}| \le Kr^M\Big(\sin k(\psi - \alpha)\prod_{j=1}^{n-1}\sin k_{\varepsilon_j}(\psi - \alpha_j)\Big)^{-2} \tag{2.3}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set D, where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)$ $(j = 1, 2, \cdots, n-1)$.

Lemma 2.3 (see [18]) If f is transcendental and entire, then $F(f)$ has no unbounded multi-connected component.

Lemma 2.4 (see [19]) Let $A_0(z), \cdots, A_{n-1}(z)$, $A_0(z) \not\equiv 0$ be entire functions such that for real constants α, β, μ, θ_1, θ_2, where $0 \le \beta < \alpha$, $\mu > 0$ and $\theta_1 < \theta_2$, we have

$$|A_0(z)| \ge e^{\alpha|z|^\mu} \tag{2.4}$$

and

$$|A_k(z)| \le e^{\beta|z|^\mu}, \quad k = 1, \cdots, n-1 \tag{2.5}$$

as $z \to \infty$ with $\theta_1 \le \arg z \le \theta_2$. Then every solution $f \not\equiv 0$ equation (1.2) has infinite order.

3 Proof of Theorem 1.2

Firstly, we assume that $\mathrm{mes}\Delta(f) < \theta_2 - \theta_1$, and set $\zeta = \theta_2 - \theta_1 - \mathrm{mes}\Delta(f)$. Since $\Delta(f)$ is closed, clearly $S = (0, 2\pi) \setminus \Delta(f)$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$ $(i = 1, 2, \cdots, m)$ satisfying $[\alpha_i, \beta_i] \subset S$ and $\mathrm{mes}(S \setminus \bigcup_{i=1}^{m} I_i) < \zeta/4$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see

$$(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset, \quad \Omega^*(r, \alpha_i, \beta_i) \cap J(f) = \emptyset$$

for sufficiently large r. This implies that for each $i = 1, 2, \cdots, m$, there exist the corresponding r_i and unbounded Fatou component U_i of $F(f)$ such that $\Omega^*(r_i, \alpha_i, \beta_i) \subset U_i$. By Lemma 2.3, $F(f)$ has no unbounded multi-connected component, so we can take a unbounded and connected section Γ_i of ∂U_i, then the mapping $f : \Omega^*(r_i, \alpha_i, \beta_i) \to \mathbb{C} \setminus \Gamma_i$ is analytic. Since we have chosen Γ_i such that $\mathbb{C} \setminus \Gamma_i$ is simply connected, so for any $a \in \Gamma_i \setminus \{\infty\}$, we have

$C_{\mathbb{C}\setminus\Gamma_i}(a) \geq 1/2$. Applying Lemma 2.1 to f in every $\Omega^*(r_i, \alpha_i, \beta_i)$, there exists a positive constant d such that for $z \in \bigcup\limits_{i=1}^{m} \Omega^*(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon)$,

$$|f(z)| = O(|z|^d) \quad \text{as} \quad |z| \to \infty, \tag{3.1}$$

where $0 < \varepsilon < \min\{\zeta/(16m), (\beta_i - \alpha_i)/8,\ i = 1, 2, \cdots, m\}$. Thus, recall the definition of $S_{\alpha,\beta}(r, f)$, we immediately see

$$S_{\alpha_i+\varepsilon,\,\beta_i-\varepsilon}(r, f) = O(1) \quad (i = 1, 2, \cdots, m). \tag{3.2}$$

Therefore by Lemma 2.2, there exists two constants $M > 0$ and $K > 0$ such that

$$\left|\frac{f^{(s)}(z)}{f(z)}\right| \leq Kr^M \ \ (s = 1, 2, \cdots, n-1) \tag{3.3}$$

for all $z \in \bigcup\limits_{i=1}^{m} \Omega(\alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside a R-set H.

Set $D = (\theta_1, \theta_2)$. Clearly,

$$\text{mes}(D \cap S) = \text{mes}(D \setminus (\Delta(f) \cap D) \geq \text{mes}D - \text{mes}\Delta(f) > \frac{3\zeta}{4} > 0. \tag{3.4}$$

Therefore

$$\text{mes}\big((\bigcup_{i=1}^{m} I_i) \cap D\big) = \text{mes}(S \cap D) - \text{mes}\big((S \setminus \bigcup_{i=1}^{m} I_i) \cap D\big) > \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2}. \tag{3.5}$$

Thus there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \bigcup\limits_{i=1}^{m} I_i \subset S$ such that for infinitely many j,

$$\text{mes}(D \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0. \tag{3.6}$$

Then, for sufficiently large r,

$$\int_F |A_0(re^{i\theta})|d\theta \quad \geq \quad \big(\text{mes}(D \cap (\alpha, \beta)) - 4\varepsilon\big)e^{br^c} \geq \frac{\zeta}{4m}e^{br^c}, \tag{3.7}$$

where $F = D \cap (\alpha + 2\varepsilon, \beta - 2\varepsilon)$ and ε is sufficiently small such that $\varepsilon < \frac{\beta-\alpha}{10}$.

On the other hand, coupling (1.2) and (3.3) leads

$$\begin{aligned} \int_F |A_0(r_je^{i\theta})|d\theta \quad &\leq \quad \int_F \Big(\sum_{s=1}^{n-1}\Big|\frac{f^{(s)}(re^{i\theta})}{f(re^{i\theta})}\Big||A_i(re^{i\theta})| + \Big|\frac{f^{(n)}(re^{i\theta})}{f(re^{i\theta})}\Big|\Big)d\theta + O(1) \\ &\leq \quad M_1r^Me^{ar^c} + O(1), \end{aligned} \tag{3.8}$$

where M_1 is a positive constant. Combining (3.7) and (3.8) gives out $e^{(b-a)r^c} \leq \frac{4M_1mr^M}{\zeta}$. Clearly, it is a contradiction.

4 Proof of Theorem 1.1

Conversely, assume that $\mathrm{mes}(\Delta(f)\cap\Delta(f^{(k)})) < \theta_2-\theta_1$, and set $\zeta = \theta_2-\theta_1-\mathrm{mes}(\Delta(f)\cap\Delta(f^{(k)}))$. Next, we will find an interval I such that $I \not\subset (\Delta(f)\cap\Delta(f^{(k)})$ satisfying $I \subset \Delta(f^{(k)})^C$ and $I \subset \Delta(f)$, where $\Delta(f^{(k)})^C = [0,2\pi)\setminus\Delta(f^{(k)})$, and obtain the assertion by reduction to a contradiction in this interval. Set $D = (\theta_1, \theta_2)$.

Step 1 In this step, we shall prove that $\mathrm{mes}(D\setminus\Delta(f)) = 0$. Otherwise, since D is open and $\Delta(f)$ is closed, we can find a ray $\arg z = \theta_0$ such that $\theta_0 \notin \Delta(f)$ and there exists some positive constant η such that $(\theta_0-\eta, \theta_0+\eta) \subset D$ and $\Omega^*(r, \theta_0-\eta, \theta_0+\eta)\cap J(f) = \emptyset$ for sufficiently large r. Then following similar discussion as in Theorem 1.1, we have (3.1) holds for $\Omega^*(r, \theta_0-\eta, \theta_0+\eta)$. Therefore it follows that

$$\begin{aligned}\int_{F_1}|A_0(re^{i\theta})|d\theta &\leq \int_{F_1}\Big(\sum_{s=1}^{n-1}\Big|\frac{f^{(s)}(re^{i\theta})}{f(re^{i\theta})}\Big||A_i(re^{i\theta})| + |\frac{f^{(n)}(re^{i\theta})}{f(re^{i\theta})}\Big|\Big)d\theta + O(1) \quad (4.1)\\ &\leq M_2\mathrm{mes}F_1r^Me^{ar^c} + O(1),\end{aligned}$$

where $F_1 = D\cap(\alpha,\beta)$ and M_2 is a constant.

On the other hand,

$$\int_{F_1}|A_0(re^{i\theta})|d\theta \geq \Big(\mathrm{mes}(D\cap(\alpha,\beta)) - 4\varepsilon\Big)e^{br^c} \geq \mathrm{mes}F_1e^{br^c}. \quad (4.2)$$

(4.1) and (4.2) lead to a contradiction.

Step 2 From Theorem 1.2, we already know

$$\mathrm{mes}\Delta(f) \geq \theta_2-\theta_1. \quad (4.3)$$

And from Step 1, we have

$$\mathrm{mes}(\Delta(f)\cap D) \geq \theta_2-\theta_1-\frac{\zeta}{4}. \quad (4.4)$$

Since $\Delta(f^{(k)})$ is closed, clearly $S = (0,2\pi)\setminus\Delta(f^{(k)})$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$ $(i = 1,2,\cdots,m)$ satisfying

$$I_i \subset \Delta(f^{(k)})^C, \mathrm{mes}(\Delta(f^{(k)})^C\setminus\bigcup_{i=1}^m I_i) < \frac{\zeta}{4}. \quad (4.5)$$

Thus for sufficiently large r,

$$\begin{aligned}&\mathrm{mes}(\Delta(f)\cap D\cap(\bigcup_{i=1}^m I_i)) + \mathrm{mes}(\Delta(f)\cap D\cap\Delta(f^{(k)}))\\ =\ &\mathrm{mes}((\Delta(f)\cap D)\cap(\Delta(f^{(k)})\cup\bigcup_{i=1}^m I_i)) \geq \theta_2-\theta_1-\frac{\zeta}{2},\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{mes}(\Delta(f)\cap D\cap(\bigcup_{i=1}^{m}I_i)) &= \theta_2-\theta_1-\frac{\zeta}{2}-\mathrm{mes}(\Delta(f)\cap D\cap\Delta(f^{(k)})) \\ &\geq \theta_2-\theta_1-\frac{\zeta}{2}-\mathrm{mes}(\Delta(f)\cap\Delta(f^{(k)})) \\ &= \frac{\zeta}{2}.\end{aligned} \tag{4.6}$$

Therefore there exist some I_i such that

$$\mathrm{mes}(\Delta(f)\cap D\cap I_i)>\frac{\zeta}{2m}. \tag{4.7}$$

Thus we can choose a ray $\arg z=\theta$ and sufficiently small $\eta>0$ such that $(\theta-\eta,\theta+\eta)\subset I_i$ and

$$\mathrm{mes}(\Delta(f)\cap D\cap(\theta-\eta,\theta+\eta))>\frac{\zeta}{4m}. \tag{4.8}$$

For the angular domain $\Omega(\theta-\eta,\theta+\eta)$, it is easy to see

$$(\theta-\eta,\theta+\eta)\cap\Delta(f^{(k)})=\emptyset,\quad \Omega^*(r,\theta-\eta,\theta+\eta)\cap J(f^{(k)})=\emptyset$$

for sufficiently large r. This implies that there exist the corresponding r_i and unbounded Fatou component U of $F(f^{(k)})$ such that $\Omega^*(r,\theta-\eta,\theta+\eta)\subset U$, see [18]. We take an unbounded and connected section Γ of ∂U, then the mapping $f^{(k)}:\Omega^*(r,\theta-\eta,\theta+\eta)\to\mathbb{C}\backslash\Gamma$ is analytic. Since we have chosen Γ such that $\mathbb{C}\backslash\Gamma$ is simply connected, so for any $a\in\Gamma\backslash\{\infty\}$, we have $C_{\mathbb{C}\backslash\Gamma}(a)\geq 1/2$. Applying Lemma 2.1 to $f^{(k)}$ in every $\Omega^*(r,\theta-\eta,\theta+\eta)$, there exists a positive constant d and R such that for $z\in\Omega^*(R,\theta-\eta,\theta+\eta)$,

$$|f^{(k)}(z)|=O(|z|^d),\quad \text{as}\quad |z|\to\infty. \tag{4.9}$$

For $z=re^{i\theta}\in\Omega^*(R,\theta-\eta,\theta+\eta)$, take a curve γ which connecting $Re^{i\theta}$ to $re^{i\theta}$ along $\arg z=\theta_j$. So we deduce from (4.9) that

$$|f^{(k-1)}(z)|\leq\int_\gamma|f^{(k)}(z)||dz|+c_k\leq O(|z|^dL(\gamma))+c_k\leq O(r^{d+1}),$$

where $L(\gamma)$ denotes the length of γ. Similarly, we have

$$\begin{aligned}|f^{(k-2)}(z)| &\leq \int_\gamma|f^{(k-1)}(z)||dz|+c_{k-1} \\ &\leq O(|z|^{d+1}L(\gamma))+c_{k-1} \\ &\leq O(r^{d+2}), \\ &\vdots \\ |f(z)| &\leq \int_\gamma|f'(z)||dz|+c_1 \\ &\leq O(|z|^{d+k-1}L(\gamma))+c_1 \\ &\leq O(r^{d+k}),\end{aligned} \tag{4.10}$$

where $c_1, c_2, \cdots, c_k$ are constants. Therefore, by the definition of $S_{\alpha,\beta}(r,f)$, we immediately see

$$S_{\theta-\eta,\theta+\eta}(r,f) = O(1). \tag{4.11}$$

Then by Lemma 2.2, there exists two constants $M > 0$ and $K > 0$ such that

$$\Big|\frac{f^{(s)}(z)}{f(z)}\Big| \leq Kr^M \quad (s = 1, 2, \cdots, n-1) \tag{4.12}$$

for all $z \in \Omega(\theta-\eta, \theta+\eta)$, outside a R-set H_1.

It follows from (1.3) that

$$\int_{\theta-\eta}^{\theta+\eta} |A_0(r_j e^{i\theta})| d\theta \geq 2\eta e^{br^c}. \tag{4.13}$$

On the other hand, coupling (1.2) and (4.12) leads

$$\begin{aligned} & \int_{\theta-\eta}^{\theta+\eta} |A_0(r_j e^{i\theta})| d\theta \\ \leq \; & \int_{\theta-\eta}^{\theta+\eta} \Big(\sum_{s=1}^{n-1} \Big|\frac{f^{(n)}(re^{i\theta})}{f(re^{i\theta})}\Big| |A_i(re^{i\theta})| + \Big|\frac{f^{(s)}(re^{i\theta})}{f(re^{i\theta})}\Big|\Big) d\theta + O(1) \\ \leq \; & 2\eta M_3 r^M e^{ar^c} + O(r), \end{aligned} \tag{4.14}$$

where M_3 is a constant. By (4.13) and (4.14), we can obtain a contradiction since $b > a \geq 0$.

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线性微分方程的整函数解及其导数的Julia集的公共极限方向

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摘要: 本文主要研究了线性微分方程解的Julia集的极限方向问题. 利用值分布论的方法, 在一定条件下, 获得了这类方程非平凡解的Julia集的极限方向分布的下界, 推广了相关结果.

关键词: 极限方向; 整函数; Julia集; 复微分方程

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