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A NOTE ON S-SPACES

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Abstract: In this paper, we study some properties of s-spaces. By means of the addition theorem and the theory of remainders, the following properties are established: (1) if an s-space X is the union of a countable family of metrizable subspaces, then X is sequential; (2) if G is a non-locally compact topological group with a compactification bG such that $Y = bG \setminus G$ is hereditarily an s-space, then either G is separable and metrizable, or G is σ-compact, which generalize and improve some results about s-spaces by Arhangel'skii.

Keywords: s-space; Lindelöf Σ-space; remainder; metrizable; topological group

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1 Introduction

Clearly, every subset of a topological space X can be obtained from open (or closed) subsets of X by the operations of intersection and union, and many classes of topological spaces have been defined in such a way. Among them are s-spaces (see [1–3]) and Lindelöf Σ-spaces (see [10]). Fix a space Z and an arbitrary family $\mathcal{S}$ of open (or closed) subsets of Z, and let $\mathcal{S}_\delta$ be the family of all sets that can be represented as the intersection of some subfamily of $\mathcal{S}$. We will say that the family $\mathcal{S}$ is an open (or closed) source of a subspace X in Z if X is the union of some subfamily of $\mathcal{S}_\delta$. A space X is called an s-space if there exists a countable open source for X in some (every) compactification bX of X. A space X is called a Lindelöf Σ-space if there exists a countable closed source for X in some (every) compactification bX of X.

The following statement establishes the relationship between s-spaces and Lindelöf Σ-spaces.

Theorem 1.1 [2] Suppose that X is a nowhere locally compact space with a remainder Y. Then X is a Lindelöf Σ-space if and only if Y is an s-space.

The next result about remainders is due to Henriksen and Isbell [9].

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Theorem 1.2 A Tychonoff space X is of countable type if and only if the remainder in any (or some) Hausdorff compactification of X is Lindelöf.

Recall that a space X is of countable type if every compact subset P of X is contained in a compact subset $F \subset X$ that has a countable base in X. Every p-space, as well as each metrizable space, is of countable type. From Theorem 1.1 and Theorem 1.2 above, we can see that every s-space is of countable type.

Recall that a paratopological group G is a group G with a topology such that the multiplication is jointly continuous. A semitopological group G is a group G with a topology such that the multiplication is separately continuous. A paratopological group with the inversion being continuous is called a topological group. Clearly, every topological group is a paratopological group, and a paratopological group is a semitopological group. It is well known that each semitopological group is homogeneous.

In this paper, we investigate addition theorems for s-spaces, and obtain several sufficient conditions that the union of some family of s-spaces is also an s-space. A sufficient condition for an s-space to be sequential is established. We also study the remainders about s-spaces, and some results about topological groups with a remainder being an s-space are obtained.

Throughout this paper, a space always means a Tychonoff topological space. By a remainder of a Tychonoff space X, we mean the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. $\overline{A}^X$ stands for the closure of A in X.

In general, we follow [6] in terminology and notation.

2 Main Results

In [2], Arhangel'skii studied the addition theorem for s-spaces, and established the following statement.

Theorem 2.1 [2] If a space X is the union of a countable family η of dense subspaces of X such that each $Z \in \eta$ is an s-space, then X is also an s-space.

We complement Arhangel'skii's result above as follows.

Lemma 2.2 The sum space of a countable family of s-spaces is an s-space.

Proof Assume that $X = \bigoplus\limits_{i\in\omega} X_i$ and each X_i is an s-space. Fix a compactification bX of X, and let bX_i be the closure of X_i in bX for $i \in \omega$. Then each bX_i is a compactification of X_i. Since X_i is an s-space, there exists a countable open source $\mathcal{O}_i$ in bX_i. Observing that each X_i is open in X, we can fix an open subset U_i of bX such that $U_i \cap X = X_i$. Clearly, X_i is dense in U_i. Hence $\overline{U_i}^{bX} = \overline{X_i}^{bX} = bX_i$, which follows that U_i is contained in bX_i. Put $\mathcal{S}_i = \{O \cap U_i : O \in \mathcal{O}_i\}$ for each $i \in \omega$. It is easy to see that $\mathcal{S}_i$ is a countable open source of X_i in bX. Therefore, $\bigcup\limits_{i\in\omega} \mathcal{S}_i$ is a countable family of open subsets of bX. It remains to show that $\bigcup\limits_{i\in\omega} \mathcal{S}_i$ is a source of X in bX. Take any distinct points x, y such that $x \in X, y \in bX \setminus X$. There exists X_i such that $x \in X_i$. Since $\mathcal{S}_i$ is a source of X_i in bX, $\bigcap\{S : x \in S \in \mathcal{S}_i\}$ is contained in X_i. Hence, we can take $S \in \mathcal{S}_i \subset \bigcup\limits_{i\in\omega} \mathcal{S}_i$ such that $x \in S \subset bX \setminus \{y\}$. Therefore, X is an s-space.

Theorem 2.3 Let X be the union of a countable family η of closed subspaces such that each $Z \in \eta$ is an s-space. If η is locally finite in X, then X is an s-space.

Proof Let Y be the sum space of η, i.e., $Y = \bigoplus \eta$. By Lemma 2.2, Y is an s-space. Let $f : Y \to X$ be the canonical mapping that restricts to the identity on each $Z \in \eta$. Since η is a family of closed subsets of X and locally finite in X, it follows that f is a perfect mapping. By Theorem 2.13 in [2], the image of an s-space under a perfect mapping is an s-space. Therefore, X is an s-space.

Corollary 2.4 If X be the union of a finite family η of closed subspaces and each $Z \in \eta$ is an s-space, then X is an s-space.

The following example shows that the assumption in Theorem 2.3 that η is locally finite cannot be dropped.

Example 1 The union of a countable family of closed s-subspaces need not be an s-space.

Proof Fix a σ-compact X such that X is not a p-space (for instance, the σ-product of ω_1 copies of two-elements topological group). Since each compact space is an s-space, X is the union of a countable family of closed s-spaces. We claim that X is not an s-space. Assume the contrary. Let bX be a compactification of X. By Theorem 1.1, $bX \setminus X$ is a Lindelöf Σ-space. Notice that X is also a Lindelöf Σ-space. By Corollary 6.3 in [2], a Lindelöf Σ-space Y is a p-space provided that Y is a subspace of a Lindelöf p-space Z and $Z \setminus Y$ is also a Lindelöf Σ-space. Hence, X is a p-space since X is a Lindelöf Σ-space. This is a contradiction. Therefore, X is not an s-space.

For open s-subspaces the circumstances is different, which can be seen from the following result whose proof is similar with Lemma 2.2.

Theorem 2.5 If X is the union of a countable family η of open subspaces such that each $Z \in \eta$ is an s-space, then X is an s-space.

Theorem 2.6 Let X be the union of a countable family η of metrizable subspaces. If X is an s-space, then X is a sequential space.

Proof Since X is an s-space, X is a k-space by Corollary 2.12 in [2]. In fact, it follows from the fact that every space of point-countable type is a k-space. Fix any non-closed subset A of X. Then there is a compact subset K of X such that $A \cap K$ is not closed in K. By [11], every compact space that is the union of a countable family of metrizable subspaces is sequential. Since K is the union of a countable metrizable subspaces, it follows that K is sequential. Hence, there is a sequence $\{x_n : n \in \omega\}$ of $A \cap K$ converging to a point $x \in K \setminus A \subset X \setminus A$. Therefore, X is a sequential space.

Corollary 2.7 Let X be a topological group that is an s-space. If X is the union of a countable family η of metrizable subspaces, then X is metrizable.

Proof By the assumption and Theorem 2.6, X is sequential. Hence, X has countable tightness. Since X is an s-space, it is of countable type. Fix a compact subset K of X such that K has a countable base in X. Since K is a compact space with countable tightness, K has countable π-character by [12]. Then it follows from the fact K having a countable base in X that X has countable π-character at each point of K. Since X is homogeneous, X has countable π-character. Hence, X is first countable since it is a topological group (see Proposition 5.2.6 in [4]). Therefore, X is metrizable (see Theorem 3.3.12 in [4]).

In [2], Arhangel'skii proved that s-spaces are preserved by a perfect mapping in both directions. It is also known that the image of a Lindelöf Σ-space under any continuous mapping is also a Lindelöf Σ-space [10]. However, the image of an s-space under a continuous closed mapping need not be an s-space.

Theorem 2.8 The image of an s-space under a continuous closed mapping need not be an s-space.

Proof Let $X = \bigoplus_{i\in\omega} I_i$ be the sum space of ω copies of closed unit interval I, where each I_i is homeomorphic to I. Let 0_i be the zero of I_i, and identify all 0_is to be one point 0. Then we obtain a quotient space Y of X with respect to the canonical mapping $f : X \to Y$ defined by $f(0_i) = 0$ for each $i \in \omega$, and $f(x) = x$ for each $x \in X \setminus \{0_i : i \in \omega\}$. Clearly, f is a continuous closed mapping. Since X is separable and metrizable, X is an s-space.

Claim Y is not an s-space.

Assume the contrary. By Theorem 7.1 in [2], $w(Z) = nw(Z)$ provided that Z is an s-space, where $w(Z)$ and $nw(Z)$ denotes the weight and network weight of Z, respectively. Since Y has a countable network, it follows that Y has a countable base, which contradicts with the fact that Y is not first countable.

The following results complement Theorem 1.1.

Theorem 2.9 If B is a compact space and a subspace X of B is a Lindelöf Σ-space, then the subspace $B \setminus X$ of B is an s-space.

Proof Let Y be the closure of X in B. Then Y is a compactification of X, and there exists a countable closed source $\mathcal{F}$ of X in Y. Clearly, $\mathcal{O} = \{B \setminus F : F \in \mathcal{F}\}$ is a countable family of open subsets of B. Let Z be the closure of $B \setminus X$ in B, and $\mathcal{S} = \{O \cap Z : O \in \mathcal{O}\}$. Obviously, Z is a compactification of $B \setminus X$ and $\mathcal{S}$ is a countable open source of $B \setminus X$ in Z. Therefore, $B \setminus X$ is an s-space.

Example 2 There exists a compact space B and its subspace X which is an s-space, the subspace $B \setminus X$ of B need not be a Lindelöf Σ-space.

Proof Let $B = C_1 \cup C_2$ be the Alexandroff double of the circle, where $C_i = \{(x, y) : x^2 + y^2 = i\}$, $i = 1, 2$ (see Example 3.1.26 in [6]). It is known that B is a compact space, C_1 is a compact subspace of B, and hence C_1 is an s-space. Since C_2 is an open discrete subspace of B with cardinality 2^ω, it follows that C_2 is not a Lindelöf Σ-space.

Theorem 2.10 Suppose that B is a compact space, and X is a subspace of B such that X is dense in some open subspace U of B. If X is an s-space, then $B \setminus X$ is a Lindelöf Σ-space.

Proof Let Y be the closure of X in B. Then Y is a compactification of X, and there exists a countable open source $\mathcal{O}$ of X in Y. Notice that U is an open subspace of Y. It is easy to see that $\mathcal{V} = \{O \cap U : O \in \mathcal{O}\}$ is a family of open subsets of B and a source of X in Y. Let Z be the closure of $B \setminus X$ in B. Then the family $\mathcal{F} = \{(B \setminus V) \cap Z : V \in \mathcal{V}\}$ is a countable closed source of $B \setminus X$ in Z. Therefore, $B \setminus X$ is a Lindelöf Σ-space.

In the end, we study some spaces with a compactification such that the remainder is (locally) an s-space.

Theorem 2.11 Let X be a non-locally compact homogeneous space with a compactification bX such that the remainder $Y = bX \setminus X$ is locally an s-space. Then X is a Lindelöf Σ-space and Y is an s-space.

Proof Since Y is locally an s-space and every closed subspace of an s-space is also an s-space, we can fix an open subspace U of Y such that the closure of U in Y, denoted by F, is an s-space. Since X be a non-locally compact homogeneous space, it is nowhere locally compact. Therefore, Y is dense in bX. Let Z be the closure of F in bX. Then $Z \setminus F$ is a Lindelöf Σ-space and contained in X. Clearly, $Z \setminus F$ is a closed subspace of X and has non-empty interior in X. It follows that X is locally a Lindelöf Σ-space. Since every s-space is of countable type, Y is of locally countable type. By [13], every space of locally countable type is of countable type. Hence, Y is of countable type. By Theorem 1.2, X is a Lindelöf space. It follows that X is covered by a countable family of its Lindelöf Σ-subspaces. Therefore, X is a Lindelöf Σ-space (see Proposition 5.3.8 in [4]). Hence, Y is an s-space by Theorem 1.1.

Corollary 2.12 If a first-countable paratopological group G has a compactification bG such that the remainder $Y = bG \setminus G$ is locally an s-space, then G is metrizable.

Proof If G is locally compact, then G is a topological group by [5]. A first-countable topological group is metrizable by Theorem 3.3.12 in [4]. Therefore, G is metrizable.

If G is non-locally compact, then G is a Lindelöf Σ-space by Theorem 2.11. Since a semitopological group with countable π-character has a G_δ-diagonal (see Corollary 5.7.5 in [4]), G has a G_δ-diagonal. Hence, G has a countable network, since every Σ-space with a G_δ-diagonal is a σ-space [8] and every Lindelöf σ-space has a countable network. By Proposition 5.7.14 in [4], a first-countable paratopological group with a countable network has a countable base, so has G. Therefore, G is metrizable.

Theorem 2.13 Let G be a non-locally compact topological group with a compactification bG such that the remainder $Y = bG \setminus G$ is an s-space and is the union of a countable family η of metrizable subspaces. Then G is separable and metrizable.

Proof Since Y is an s-space, it is of countable type. Take an arbitrary point $y \in Y$ and a compact subset $K \subset Y$ such that $y \in K$ and K has a countable base in Y. From the proof of Corollary 2.7 we can see that Y has countable π-character at y. It follows that Y has countable π-character. Since Y is dense in bG, it follows that bG has countable π-character at each point of Y.

By [7], every countably compact space that is the union of a countable family of D-spaces is compact. Since every metrizable space is a D-space and Y is non-compact, it follows that Y is not countably compact. Then there is a countable closed subset $A \subset Y$ which is discrete in Y. Since bG is compact, there exists a point $c \in G$ such that c is a accumulation point of A.

For every $a \in A$, we take a countable π-base η_a of bG at a. Then the family $\bigcup\limits_{a \in A} \eta_a$

is a countable π-base of bG at c. Put $\mathcal{O} = \{O \cap G : O \in \bigcup_{a\in A} \eta_a\}$. Since G is dense in bG, it follows that $\mathcal{O}$ is a countable π-base of G at c. Hence, G has countable π-character since it is homogeneous. Therefore, it follows from G being a topological group that G is metrizable. Clearly, G is Lindelöf, since it is a Lindelöf Σ-space. Therefore, G is separable and metrizable.

Theorem 2.14 Let G be a non-locally compact semitopological group with a compactification bG such that the remainder $Y = bG \setminus G$ is an s-space and is the union of a countable family η of metrizable subspaces. Then G has a countable network.

Proof From the proof of Corollary 2.7 we can see that G has countable π-character. Since G is a semitopological group, it follows that G has a G_δ-diagonal. Clearly, G is a Lindelöf Σ-space. Therefore, G has a countable network.

Theorem 2.15 Let G be a non-locally compact topological group with a compactification bG such that the remainder $Y = bG \setminus G$ is hereditarily an s-space. Then either G is separable and metrizable, or G is σ-compact.

Proof Clearly, both G and Y are dense in bG. Then Y is nowhere locally compact, which implies that Y is dense-in-itself. By Theorem 7.11 in [2], if a dense-in-itself space is hereditarily an s-space, then it is first-countable. Hence, Y is first-countable.

If Y is not countably compact, then from the proof of Theorem 2.13 we can see that G is separable and metrizable. If Y is countably compact, then Y is Čech-complete by Theorem 3.6 in [2]. It follows that G is σ-compact.

Corollary 2.16 Let G be a non-locally compact topological group with a compactification bG such that the remainder $Y = bG \setminus G$ is hereditarily an s-space. If G has the Baire property, then G is separable and metrizable.

Proof Suppose to the contrary that G is not separable and metrizable. Then G is σ-compact by Theorem 2.15. Hence, it follows from G having the Baire property that G is locally compact. This is a contradiction.

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s-空间的一个注记

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摘 要: 本文研究了s-空间的性质. 利用加法定理及剩余性质, 得到以下结论: (1) 如果s-空间X是可数多个度量子空间的并, 则X是序列空间; (2) 如果非局部紧拓扑群G 在某个紧化bG中的剩余是遗传s-空间, 则G是可分度量空间或σ-紧空间. 以上性质推广了Arhangel'skii关于s-空间的一些已有结论.

关键词: s-空间; Lindelöf Σ-空间; 剩余; 可度量的; 拓扑群

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