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THE COEFFICIENT INEQUALITY OF RELATED TO CLOSE-TO-CONVEX FUNCTIONS

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Abstract: In this paper, we introduce some new subclasses of close-to-convex functions. By using subordination relationship, we give coefficient inequalities of these subclasses, which generalizes some known results and interesting new results are obtained.

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1 Introduction

Let H denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Let

$$T = \{q \in H : q(z) = z - \sum_{n=2}^{\infty} |d_n| z^n\}.$$

It is obvious that $T \subset H$. Let Ω denote the class of functions $w(z)$ regular in U and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in U$.

Let f, g be analytic in U. Then g is said to be subordinate to f, written $g \prec f$, if there exists a Schwarz function $\omega(z) \in \Omega$, such that $g(z) = f(\omega(z))$ $(z \in U)$. In particular, if the function $f(z)$ is univalent in U, then

$$g(z) \prec f(z) \ \ (z \in U) \Longleftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

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Biography: Niu Xiaomeng (1982–), female, mongolian, born at Chifeng, Inner Mongolia, lecturer, major in complex analysis and application.

Let $P(A,B)$ $(-1 \le B < A \le 1)$ denote the class of functions of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$, which are analytic in U and satisfying the condition $p(z) \prec \frac{1+Az}{1+Bz}$. It is clear that $P(1,-1) = P$, the well-known class of positive real functions (see [1]). The classes of all starlike functions, convex functions and close-to-convex functions are respectively denoted by S^*, K and C.

Li and Tang [2] defined the following two subclasses of the function class H,

$$S^*_\beta(A,B) = \left\{ g \in H : \frac{zg'(z)}{g(z)} - \beta \left| \frac{zg'(z)}{g(z)} - 1 \right| \prec \frac{1+Az}{1+Bz}, \beta \ge 0, -1 \le B < A \le 1 \right\},$$
$$K_\beta(A,B) = \left\{ g \in H : \frac{(zg'(z))'}{g'(z)} - \beta \left| \frac{(zg'(z))'}{g'(z)} - 1 \right| \prec \frac{1+Az}{1+Bz}, \beta \ge 0, -1 \le B < A \le 1 \right\}.$$

It is obvious that

$$g(z) \in K_\beta(A,B) \Longleftrightarrow zg'(z) \in S^*_\beta(A,B). \tag{1.2}$$

$K_1(1,-1) = UCV$ is the class of uniformly convex functions (see [3–5]), $S^*_1(1,-1) = S_p$ is the class of parabolic starlike functions (see [4]).

In this paper, we generalize the class of $S^*_\beta(A,B)$ and $K_\beta(A,B)$ obtained by Li and Tang [2], and give the following two subclasses of the function class H.

Definition 1.1 Let the function $f(z) \in H$, if there exists a function $g(z) \in S^*_\beta(A,B)$ such that

$$\frac{zf'(z)}{g(z)} - \alpha \left| \frac{zf'(z)}{g(z)} - 1 \right| \prec \frac{1+Cz}{1+Dz} \quad (\alpha \ge 0, -1 \le D < C \le 1, z \in U),$$

then $f(z) \in C_{\alpha,\beta}(A,B;C,D)$.

Definition 1.2 Let the function $f(z) \in H$, if there exists a function $g(z) \in K_\beta(A,B)$ such that

$$\frac{(zf'(z))'}{g'(z)} - \alpha \left| \frac{(zf'(z))'}{g'(z)} - 1 \right| \prec \frac{1+Cz}{1+Dz} \quad (\alpha \ge 0, -1 \le D < C \le 1, z \in U),$$

then $f(z) \in Q_{\alpha,\beta}(A,B;C,D)$.

Obviously, we have

$$f(z) \in Q_{\alpha,\beta}(A,B;C,D) \Longleftrightarrow zf'(z) \in C_{\alpha,\beta}(A,B;C,D). \tag{1.3}$$

Remark 1.3 For suitable choices of parameters A, B, C, D involved in Definitions 1.1 and 1.2, we also obtain the following subclasses which were studied in many earlier works.

(i) $C_{\beta,\beta}(A,B;A,B) = S^*_\beta(A,B)$ $(f(z) = g(z))$ and $Q_{\beta,\beta}(A,B;A,B) = K_\beta(A,B)$ $(f(z) = g(z))$ (see Li et al. [2]);

(ii) $C_{0,0}(1,-1;1,-1) = C$ (see Reade [6]);

(iii) $C_{0,0}(A,B;C,D) = K(A,B;C,D)$ (see Harjinder et al. [7]);

(iv) $C_{0,0}(1,-1;C,D) = K(C,D)$ (see Harjinder et al. [7] and Mehrok [8]).

Let

$$\overline{S}^*_\beta(A,B)=S^*_\beta(A,B)\cap T,\ \ \overline{K}_\beta(A,B)=K_\beta(A,B)\cap T,\ \ \overline{S}_p=S_p\cap T,$$
$$\overline{C}_{\alpha,\beta}(A,B;C,D)=C_{\alpha,\beta}(A,B;C,D)\cap T,\ \ \overline{Q}_{\alpha,\beta}(A,B;C,D)=Q_{\alpha,\beta}(A,B;C,D)\cap T.$$

In this paper, we obtain unsolved coefficients inequalities of $S^*_\beta(A,B)$ and $K_\beta(A,B)$ defined in [2], and discuss coefficients inequalities of $C_{\alpha,\beta}(A,B;C,D)$ and $Q_{\alpha,\beta}(A,B;C,D)$. Some of our results generalize previously known results obtained by [6–11].

2 Coefficients Inequalities of $S^*_\beta(A,B)$ and $K_\beta(A,B)$

In order to obtain unsolved coefficients inequalities of $S^*_\beta(A,B)$ and $K_\beta(A,B)$ defined in [2], we need the following lemmas.

Lemma 2.1 (see [12]) Let $\alpha\ge 0$, $a,b\in R$, $a\ne b$ and $|b|\le 1$. If $p(z)$ is an analytic function with $p(0)=1$, then

$$p(z)-\alpha|p(z)-1|\prec\frac{1+az}{1+bz}\Longleftrightarrow p(z)(1-\alpha e^{-i\varphi})+\alpha e^{-i\varphi}\prec\frac{1+az}{1+bz}\quad(\varphi\in R).$$

Lemma 2.2 (see [13]) Let $P(z)=\frac{1+Cw(z)}{1+Dw(z)}=1+\sum\limits_{n=1}^{\infty}p_nz^n$, then

$$|p_n|\le C-D.$$

Lemma 2.3 (see [9]) Let $g(z)=\sum\limits_{k=q}^{\infty}b_kz^k$ and $G(z)=\sum\limits_{k=q}^{\infty}D_kz^k$, $q\ge 0$. If $g(z)=w(z)G(z)$, where $w(z)\in\Omega$, then $b_q=0$ and

$$\sum_{k=q+1}^{n}|b_k|^2\le\sum_{k=q}^{n-1}|D_k|^2\quad(n=q+1,q+2,\cdots).$$

We now prove coefficient inequalities of $\overline{S}^*_1(A,B)$ and $\overline{K}_1(A,B)$.

Theorem 2.4 Let $q(z)=z-\sum\limits_{n=2}^{\infty}|d_n|z^n\in T$, then $q(z)\in\overline{S_1}^*(A,B)$ if and only if

$$\sum_{n=2}^{\infty}[(2-B)(n-1)-(nB-A)]|d_n|\le A-B,\tag{2.1}$$

where $-1\le B<0$.

Proof Suppose that inequality (2.1) is true. We define the function $p(z)$ by

$$p(z)=\frac{zq'(z)}{q(z)}-\left|\frac{zq'(z)}{q(z)}-1\right|.$$

To prove $q(z)\in\overline{S}^*_1(A,B)$, it suffices to show that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|<1\quad(z\in U).$$

Since $B<0, -1\le B<A\le 1$, we have $nB-A=(n-1)B+B-A<0$, therefore

$$\begin{aligned}\left|\frac{p(z)-1}{A-Bp(z)}\right| &= \left|\frac{\frac{zq'(z)}{q(z)}-\left|\frac{zq'(z)}{q(z)}-1\right|-1}{A-B\left(\frac{zq'(z)}{q(z)}-\left|\frac{zq'(z)}{q(z)}-1\right|\right)}\right| \\ &= \left|\frac{zq'(z)-q(z)-e^{i\theta}|zq'(z)-q(z)|}{Aq(z)-Bzq'(z)+Be^{i\theta}|zq'(z)-q(z)|}\right| \quad \left(\frac{q(z)}{|q(z)|}=e^{i\theta}\right) \\ &= \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n-e^{i\theta}\left|-\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n\right|}{(A-B)z+\sum\limits_{n=2}^{\infty}(nB-A)|d_n|z^n+Be^{i\theta}\left|-\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n\right|}\right| \\ &\le \frac{2\sum\limits_{n=2}^{\infty}(n-1)|d_n||z|^n}{(A-B)|z|-\sum\limits_{n=2}^{\infty}|nB-A||d_n||z|^n-|B|\sum\limits_{n=2}^{\infty}(n-1)|d_n||z|^n} \\ &= \frac{2\sum\limits_{n=2}^{\infty}(n-1)|d_n||z|^{n-1}}{(A-B)+\sum\limits_{n=2}^{\infty}(nB-A)|d_n||z|^{n-1}+B\sum\limits_{n=2}^{\infty}(n-1)|d_n||z|^{n-1}} \\ &\le \frac{2\sum\limits_{n=2}^{\infty}(n-1)|d_n|}{(A-B)+\sum\limits_{n=2}^{\infty}(nB-A)|d_n|+B\sum\limits_{n=2}^{\infty}(n-1)|d_n|}.\end{aligned}$$

From (2.1), we obtain

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|\le 1\ (z\in U)\quad (|z|=1).$$

By using Maximum modulus theorem, we have $q(z)\in \overline{S}_1^*(A,B)$.

Conversely, suppose that $q(z)\in \overline{S}_1^*(A,B)$, then

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|<1\quad (z\in U)$$

or

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n+e^{i\theta}\left|\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n\right|}{(A-B)z+\sum\limits_{n=2}^{\infty}(nB-A)|d_n|z^n+Be^{i\theta}\left|\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n\right|}\right|<1.$$

Using the fact that $|\mathrm{Re}z|\le|z|$ for all z, it follows that

$$\mathrm{Re}\left\{\frac{\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n+e^{i\theta}\left|\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n\right|}{(A-B)z+\sum\limits_{n=2}^{\infty}(nB-A)|d_n|z^n+Be^{i\theta}\left|\sum\limits_{n=2}^{\infty}(n-1)|d_n|z^n\right|}\right\}<1. \tag{2.2}$$

Now choose $z = r \in [0,1) \subset U$ on the real axis, it is easy to show that $\theta = 0$ or π. Taking $\theta = 0$ in (2.2), we obtain

$$\frac{2\sum\limits_{n=2}^{\infty}(n-1)|d_n|r^{n-1}}{(A-B)+\sum\limits_{n=2}^{\infty}(nB-A)|d_n|r^{n-1}+B\sum\limits_{n=2}^{\infty}(n-1)|d_n|r^{n-1}} < 1.$$

Let $r \to 1^-$, we obtain (2.1) .

Remark 2.5 Since $\overline{S}_1^*(1,-1) = \overline{S}_p$, taking $A = 1$, $B = -1$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.6 (see [10]) Let $f(z) = z - \sum\limits_{n=2}^{\infty}|a_n|z^n$, then $f \in \overline{S}_p$ if and only if

$$\sum_{n=2}^{\infty}(2n-1)|a_n| \le 1.$$

Corollary 2.7 Let $q(z) = z - \sum\limits_{n=2}^{\infty}|d_n|z^n \in \overline{S}_1^*(A,B)$, then for $-1 \le B < 0$,

$$|d_n| \le \frac{A-B}{(2-B)(n-1)-(nB-A)}.$$

Proof From Theorem 2.4, we obtain

$$\sum_{n=2}^{\infty}[(2-B)(n-1)-(nB-A)]|d_n| \le A-B.$$

Since $B < 0$, $-1 \le B < A \le 1$, we have

$$(2-B)(n-1)-(nB-A) = (2-B)(n-1)-[(n-1)B+(B-A)] > 0.$$

Hence

$$|d_n| \le \frac{A-B}{(2-B)(n-1)-(nB-A)}.$$

From (1.2) and Theorem 2.4, we get the following Theorem 2.8

Theorem 2.8 Let $q(z) = z - \sum\limits_{n=2}^{\infty}|d_n|z^n \in T$, then $q(z) \in \overline{K}_1(A,B)$ if and only if

$$\sum_{n=2}^{\infty}n[(2-B)(n-1)-(nB-A)]|d_n| \le A-B,$$

where $-1 \le B < 0$.

Remark 2.9 Since $\overline{K_1}(1,-1) = UCV$, taking $A = 1, B = -1$ in Theorem 2.8, we obtain the following corollary.

Corollary 2.10 (see [10]) Let $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| z^n$, then $f \in UCV$ if and only if

$$\sum_{n=2}^{\infty} n(2n-1)|a_n| \leq 1.$$

From Theorem 2.8, we get the following Corollary 2.11.

Corollary 2.11 Let $q(z) = z - \sum\limits_{n=2}^{\infty} |d_n| z^n \in \overline{K}_1(A, B)$, then for $-1 \leq B < 0$,

$$|d_n| \leq \frac{A - B}{n[(2-B)(n-1) - (nB - A)]}.$$

Theorem 2.12 Let $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in S_\beta^*(A, B)$, then

$$|b_n| \leq \frac{1}{(n-1)!} \prod_{k=2}^{n} \left| \frac{A - \beta B e^{-i\varphi}}{1 - \beta e^{-i\varphi}} - (k-1)B \right|, \tag{2.3}$$

where

$$\left| \frac{A - \beta B e^{-i\varphi}}{1 - \beta e^{-i\varphi}} - (n-1)B \right| \geq (n-2) \quad (n \geq 3) \tag{2.4}$$

and

$$e^{-i\varphi} = \left| \frac{zg'(z)}{g(z)} - 1 \right| \cdot \left(\frac{zg'(z)}{g(z)} - 1 \right)^{-1}.$$

Proof Suppose that $g \in S_\beta^*(A, B)$. Then, from Lemma 2.1, we obtain

$$\frac{zg'(z)}{g(z)}(1 - \beta e^{-i\varphi}) + \beta e^{-i\varphi} \prec \frac{1 + Az}{1 + Bz}.$$

It follows from the definition of subordination that

$$\frac{zg'(z)}{g(z)}(1 - \beta e^{-i\varphi}) + \beta e^{-i\varphi} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \in \Omega,$$

which is equivalent to

$$\frac{zg'(z)}{g(z)} = \frac{1 + \frac{A - \beta B e^{-i\varphi}}{1 - \beta e^{-i\varphi}} w(z)}{1 + Bw(z)}.$$

After some computation, we obtain

$$\sum_{k=1}^{\infty} (k-1) b_k z^k = \sum_{k=2}^{\infty} (k-1) b_k z^k = w(z) \sum_{k=1}^{\infty} \left(\frac{A - \beta B e^{-i\varphi}}{1 - \beta e^{-i\varphi}} - Bk \right) b_k z^k \quad (b_1 = 1).$$

From Lemma 2.3, we obtain

$$\sum_{k=2}^{n} (k-1)^2 |b_k|^2 \leq \sum_{k=1}^{n-1} \left| \frac{A - \beta B e^{-i\varphi}}{1 - \beta e^{-i\varphi}} - Bk \right|^2 |b_k|^2, \quad n = 2, 3, \cdots,$$

which evidently yields

$$|b_n|^2 \leq \frac{1}{(n-1)^2} \times \left[\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - B \right|^2 \right.$$
$$\left. + \sum_{k=2}^{n-1} \left(\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - Bk \right|^2 - (k-1)^2 \right) |b_k|^2 \right], \ n = 2, 3, \cdots . \tag{2.5}$$

By using (2.4), all the terms under the summation sign in (2.5) are positive. It is obvious from (2.5) that $|b_2| \leq |\frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - B|$ satisfies (2.3). Assume that (2.1) is true for all $k = 3, 4, \cdots, n-1$.

We now prove

$$|b_n| \leq \frac{1}{(n-1)!} \prod_{k=2}^{n} \left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (k-1)B \right|.$$

In order to complete the proof, it is sufficient to show that

$$\frac{1}{(m-1)^2} \left[\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - B \right|^2 + \sum_{k=2}^{m-1} \left(\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - Bk \right|^2 - (k-1)^2 \right) |b_k|^2 \right]$$
$$\leq \frac{1}{((m-1)!)^2} \prod_{j=2}^{m} \left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (j-1)B \right|^2, \ m = 3, 4, \cdots, \tag{2.6}$$

where $\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (m-1)B \right| \geq (m-2)$.

We now use mathematical induction to prove (2.6).

It readily follows from $|b_2| \leq \left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - B \right|$ and $\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - 2B \right| \geq 1$ that (2.6) is true for $m = 3$. Assume that (2.6) is true for all m, $3 < m \leq n-1$. Then from (2.5), we have

$$\begin{aligned}
|b_n|^2 \leq & \frac{1}{(n-1)^2} \left[\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - B \right|^2 + \sum_{k=2}^{n-1} \left(\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - Bk \right|^2 - (k-1)^2 \right) |b_k|^2 \right] \\
= & \frac{(n-2)^2}{(n-1)^2} \frac{1}{(n-2)^2} \left[\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - B \right|^2 + \sum_{k=2}^{n-2} \left(\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - Bk \right|^2 - (k-1)^2 \right) |b_k|^2 \right] \\
& + \frac{1}{(n-1)^2} \left[\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (n-1)B \right|^2 - (n-2)^2 \right] |b_{n-1}|^2 \\
\leq & \frac{(n-2)^2}{(n-1)^2} \frac{1}{((n-2)!)^2} \prod_{j=2}^{n-1} \left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (j-1)B \right|^2 \\
& + \frac{1}{((n-1)!)^2} \left[\left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (n-1)B \right|^2 - (n-2)^2 \right] \prod_{j=2}^{n-1} \left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (j-1)B \right|^2 \\
= & \frac{1}{((n-1)!)^2} \prod_{j=2}^{n} \left| \frac{A-\beta B e^{-i\varphi}}{1-\beta e^{-i\varphi}} - (j-1)B \right|^2 .
\end{aligned}$$

Hence

$$|b_n| \leq \frac{1}{(n-1)!}\prod_{j=2}^{n}\left|\frac{A-\beta Be^{-i\varphi}}{1-\beta e^{-i\varphi}}-(j-1)B\right|.$$

Remark 2.13 (i) If $f(z) \in S_0^*(1,-1) = S^*$, then $|a_n| \leq n$, we get Theorem 4.10 [11].
(ii) Taking $\beta = 0$ in Theorem 2.3, we get Theorem 1 [9].

Corollary 2.14 Let $g \in S_\beta^*(A,B)$, then

$$|b_n| \leq \frac{1}{(n-1)!}\prod_{j=2}^{n}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right], \ \ \beta \neq 1, \tag{2.7}$$

where

$$\frac{||A|-\beta|B||}{1+\beta}-(n-1)|B| \geq n-2 \ \ (n \geq 3). \tag{2.8}$$

Proof From (2.8), we obtain

$$\begin{aligned}\left|\frac{A-\beta Be^{-i\varphi}}{1-\beta e^{-i\varphi}}-(n-1)B\right| &\geq \left|\left|\frac{A-\beta Be^{-i\varphi}}{1-\beta e^{-i\varphi}}\right|-(n-1)|B|\right| \\ &\geq \frac{||A|-\beta|B||}{1+\beta}-(n-1)|B| \geq n-2 \ \ (n \geq 3).\end{aligned}$$

By using Theorem 2.4, we obtain

$$|b_n| \leq \frac{1}{(n-1)!}\prod_{j=2}^{n}\left|\frac{A-\beta Be^{-i\varphi}}{1-\beta e^{-i\varphi}}-(j-1)B\right|.$$

Therefore

$$|b_n| \leq \frac{1}{(n-1)!}\prod_{j=2}^{n}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right], \ \ \beta \neq 1.$$

From (1.2) and Theorem 2.3, we get the following Theorem 2.15.

Theorem 2.15 Let $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in K_\beta(A,B)$, then

$$|b_n| \leq \frac{1}{n!}\prod_{k=2}^{n}\left|\frac{A-\beta Be^{-i\varphi}}{1-\beta e^{-i\varphi}}-(k-1)B\right|,$$

where

$$\left|\frac{A-\beta Be^{-i\varphi}}{1-\beta e^{-i\varphi}}-(n-1)B\right| \geq n-2 \ \ \ (n \geq 3)$$

and

$$e^{-i\varphi} = \left|\frac{zg'(z)}{g(z)}-1\right| \cdot \left(\frac{zg'(z)}{g(z)}-1\right)^{-1}.$$

Remark 2.16 If $f(z) \in K_0(1,-1) = K$, then $|a_n| \leq 1$, we get Theorem 4.13 [11].
By using Theorem 2.15, it is easy to obtain the following Corollary 2.17.

Corollary 2.17 Let $g \in K_\beta(A,B)$, then

$$|b_n| \le \frac{1}{n!}\prod_{j=2}^{n}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right],\ \ \beta \ne 1,$$

where

$$\frac{||A|-\beta|B||}{1+\beta}-(n-1)|B| \ge n-2 \quad (n \ge 3).$$

3 Coefficients Inequalities of $C_{\alpha,\beta}(A,B;C,D)$ and $Q_{\alpha,\beta}(A,B;C,D)$

In this section, by using Corollary 2.7, Corollary 2.11, Corollary 2.14 and Corollary 2.16, we obtain coefficient inequalities of $C_{\alpha,\beta}(A,B;C,D)$ and $Q_{\alpha,\beta}(A,B;C,D)$.

Theorem 3.1 Let $f(z) \in C_{\alpha,\beta}(A,B;C,D)$, then for $\frac{||A|-\beta|B||}{1+\beta}-(n-1)|B| \ge n-2$ $(n \ge 3)$,

$$\begin{aligned}|a_n| \le &\frac{1}{n!}\prod_{j=2}^{n}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right]\\ &+\frac{C-D}{|1-\alpha|n}\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right]\right),\end{aligned} \tag{3.1}$$

where $0 \le \alpha, \beta \ne 1$.

Proof Suppose that $f(z) \in C_{\alpha,\beta}(A,B;C,D)$. Then, there exits $g \in S^*_\beta(A,B)$ such that

$$\frac{zf'(z)}{g(z)}-\alpha\left|\frac{zf'(z)}{g(z)}-1\right| \prec \frac{1+Cz}{1+Dz}.$$

From Lemma 2.1, we obtain

$$\frac{zf'(z)}{g(z)}(1-\alpha e^{-i\varphi})+\alpha e^{-i\varphi} \prec \frac{1+Cz}{1+Dz} \quad (\varphi \in R).$$

It follows from the definition of subordination that

$$\frac{zf'(z)}{g(z)}(1-\alpha e^{-i\varphi})+\alpha e^{-i\varphi} = \frac{1+Cw(z)}{1+Dw(z)} = p(z),\ \ p(z) \in P,$$

or, equivalently,

$$zf'(z)(1-\alpha e^{-i\varphi}) = (p(z)-\alpha e^{-i\varphi})g(z).$$

After some computation, we have

$$\begin{aligned}&(z+2a_2z^2+3a_3z^3+\cdots)(1-\alpha e^{-i\varphi})\\ =&(1-\alpha e^{-i\varphi}+p_1z+\cdots+p_nz^n+\cdots)(z+b_2z^2+\cdots+b_nz^n+\cdots).\end{aligned} \tag{3.2}$$

Equating the coefficients of z^n in (3.2), we get

$$na_n = b_n + \frac{1}{1-\alpha e^{-i\varphi}}(p_1b_{n-1}+p_2b_{n-2}+\cdots+p_{n-2}b_2+p_{n-1}).$$

From Lemma 2.2, we obtain

$$n|a_n| \le |b_n| + \frac{C-D}{|1-\alpha|}(1+\sum_{k=2}^{n-1}|b_k|).$$

By using Corollary 2.14, we get

$$\begin{aligned} n|a_n| \le & \frac{1}{(n-1)!}\prod_{j=2}^{n}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right] \\ & + \frac{C-D}{|1-\alpha|}\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right]\right), \end{aligned}$$

which proves (3.1).

Similarly, by using Corollary 2.16, we can prove the following result.

Theorem 3.2 Let $f(z) \in Q_{\alpha,\beta}(A,B;C,D)$, then for $\frac{||A|-\beta|B||}{1+\beta} - (n-1)|B| \ge n-2$ $(n \ge 3)$,

$$\begin{aligned} |a_n| \le & \frac{1}{n \cdot n!}\prod_{j=2}^{n}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right] \\ & + \frac{C-D}{|1-\alpha|n^2}\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\left[\frac{|A|+\beta|B|}{|1-\beta|}+(j-1)|B|\right]\right), \end{aligned} \tag{3.3}$$

where $0 \le \alpha, \beta \ne 1$.

By using Corollary 2.7, we get

Theorem 3.3 Let $f(z) \in \overline{C}_{\alpha,1}(A,B;C,D)$, then

$$|a_n| \le \frac{1}{n}\frac{A-B}{(2-B)(n-1)-(nB-A)} + \frac{C-D}{|1-\alpha|n}\left(1+\sum_{k=2}^{n-1}\frac{A-B}{(2-B)(k-1)-(kB-A)}\right),$$

where $0 \le \alpha \ne 1$ and $-1 \le B < 0$.

By using Corollary 2.11, we have

Theorem 3.4 Let $f(z) \in \overline{Q}_{\alpha,1}(A,B;C,D)$, then

$$|a_n| \le \frac{1}{n^2}\frac{A-B}{(2-B)(n-1)-(nB-A)} + \frac{C-D}{|1-\alpha|n^2}\left(1+\sum_{k=2}^{n-1}\frac{A-B}{(2-B)(k-1)-(kB-A)}\right),$$

where $0 \le \alpha \ne 1$ and $-1 \le B < 0$.

Remark 3.5 (i) Taking $A = 1, B = -1, C = 1, D = -1, \alpha = 0, \beta = 0$ in Theorem 3.1, we obtain $|a_n| \le n$ which is the result due to Reade [6].

(ii) Putting $\alpha = 0, \beta = 0, -1 \le B < 0 < A \le 1$ in Theorem 3.1, we get the results obtained by Harjinder and Mehrok [7, Theorem 3.1].

(iii) If $f(z) \in C_{0,0}(1,-1;C,D) = K(C,D)$, then $|a_n| \le 1 + \frac{(n-1)(C-D)}{2}$, we have the results obtained by Mehrok [8].

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近于凸函数的系数不等式

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摘要: 本文研究了近于凸函数的新子类. 利用从属关系的方法, 获得了这些子类的系数不等式, 推广了一些已知结果并获得了新结果.

关键词: 解析函数; 近于凸; 一致凸; 从属; 系数不等式

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