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ON THE UNIQUENESS OF ELLIPSOID SOLUTIONS TO THE L_p-MINKOWSKI PROBLEM

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Abstract: In this paper, we study the L_p-Minkowski problem (under the assumption that the solutions are ellipsoids centered at the origin). Through the relation between support function and Gauss curvature, we obtain the uniqueness of ellipsoid solutions for $p < 1$, and generalize the uniqueness result for L_p-Minkowski problem and Christoffel-Minkowski problem of L_p-sum.

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1 Introduction

The Minkowski problem was popular for more than one hundred years. It had a significant impact on 20th century mathematics. L_p-Minkowski problem introduced by Lutwak [1] was intensively studied in recent decades. There existed many good references on the L_p-Minkowski problem [1–24]. However, very little is known on the uniqueness of the L_p-Minkowski problem for $p < 1$, even in $\mathbb{R}^3$.

In this paper, we discuss the uniqueness of L_p-Minkowski problem for all $p \in \mathbb{R}$ in general dimensions under the assumption that the solutions are ellipsoids centered at the origin, which needs to study the following Monge-Ampère equation

$$\det\left(h_{ij} + h\delta_{ij}\right) = h^{p-1} \quad \text{on} \quad \mathbb{S}^n, \tag{1.1}$$

where h is the support function (see Definition 2.1) of convex bodies, h_{ij} are the second-order covariant derivations of h with respect to any orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ on the unit sphere $\mathbb{S}^n$, δ_{ij} is the Kronecker delta and $p \in \mathbb{R}$. Similarly, we obtain the uniqueness for the Christoffel-Minkowski problem of L_p-sum under the same assumption, which needs to study a k-Hessian equation as follows

$$\sigma_k\left(h_{ij} + h\delta_{ij}\right) = C_n^k h^{p-1} \quad \text{on} \quad \mathbb{S}^n, \tag{1.2}$$

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where $k \in \{1, 2, \cdots, n\}$, $C_n^k = \frac{n!}{k!(n-k)!}$ and σ_k is the k-th elementary symmetric function whose definition is the following: for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

The definition can be extended to a symmetric matrix $W \in \mathbb{R}^{n \times n}$ by $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W)$ is the eigenvalue vector of W.

(1.1) comes from the geometry of convex bodies. A compact convex subset of Euclidean space $\mathbb{R}^{n+1}$ with a nonempty interior is a convex body. Minkowski developed a few basic concepts on convex bodies: support function, Minkowski sum and mixed volumes (see, e.g. [9, 25]).

The classical Minkowski problem asks the existence of a convex body whose surface area measure is prescribed. It was studied by [26–34] (or see [9] for history) and many others. The uniqueness of solutions to the classical Minkowski problem was solved by the Brunn-Minkowski inequality (Gardner gave some equivalent inequalities in [35]): let $Q_1, Q_2 \subset \mathbb{R}^{n+1}$ be two convex bodies and $0 < \lambda < 1$, then

$$\mathrm{Vol}\,((1-\lambda)Q_1 + \lambda Q_2) \geqslant \mathrm{Vol}\,(Q_1)^{1-\lambda}\mathrm{Vol}\,(Q_2)^{\lambda}, \tag{1.3}$$

where $\mathrm{Vol}(\cdot)$ denotes the volume of a convex body and '+' denotes the Minkowski sum (see Definition 2.2). The equality in (1.3) holds if and only if Q_1 and Q_2 are translates.

Firey [36] extended the Minkowski sum to the general cases for $p \geqslant 1$, which is called L_p-sum (see Definition 2.3). Later, in [1], Lutwak generalized the classical surface area measure (see Definition 2.5) to the L_p surface area measure (see Definition 2.6) for $p \geqslant 1$ and studied the generalised Minkowski problem, which was called L_p-Minkowski problem thereafter. Given a finite Borel measure m on $\mathbb{S}^n$, the L_p-Minkowski problem concerns whether there exists a unique convex body $Q \subset \mathbb{R}^{n+1}$ such that m is the L_p surface area measure of Q. Let μ denote the surface area measure of Q, then the L_p-Minkowski problem is equivalent to solving the equation

$$d\mu = h^{p-1} dm, \tag{1.4}$$

where h denotes the support function of Q. Obviously, the classical Minkowski problem is a special case of L_p-Minkowski problem for $p = 1$. In the smooth category, (1.4) is equivalent to considering the following Monge-Ampère equation

$$\det(h_{ij} + h\delta_{ij}) = f h^{p-1} \quad \text{on} \quad \mathbb{S}^n, \tag{1.5}$$

where f is a positive continuous function on $\mathbb{S}^n$.

Lutwak [1] proved the existence of solutions to (1.4) when $p > 1$, except for $p = n + 1$, under an evenness assumption. Then in [2], Lutwak and Oliker obtained a C^∞ solution to the even L_p-Minkowski problem for $p > 1$. Lutwak, Yang and Zhang [6] obtained the existence of solutions to discrete and non-discrete L_p-Minkowski problems with a normalized volume for all $p > 1$, still under the evenness assumption. Without the evenness assumption, Chou and Wang [8] solved (1.5) for general measures for $p > 1$. In addition, Hug, Lutwak, Yang and Zhang [7] obtained a different proof of the existence of solutions to the L_p-Minkowski problem for $p > n+1$ and to the discrete measure for $p > 1$. A $C^{2,\alpha}$ solution to (1.5) for $p \geqslant n+1$ was given by Chou and Wang [8] and Guan and Lin [14] independently. When $1 < p < n+1$, the solution convex body may have the origin on the boundary (see, e.g. [8, 7]), thus it is not necessary to discuss the $C^{2,\alpha}$ regularity. However, for the discrete case, Hug, Lutwak, Yang and Zhang [7] obtained that the solution polytope always has the origin in its interior for $p > 1$ with $p \neq n+1$.

The cases $p < 1$ are difficult to settle. Chou and Wang [8] got the weak solution to (1.5) when $-n-1 < p < n+1$. Also, some special cases were studied. In [10], Böröczky, Lutwak, Yang and Zhang gave the existence of solutions to the even L_0-Minkowski problem. Zhu [11] studied the discrete L_0-Minkowski problem without the evenness assumption. In $\mathbb{R}^2$, Stancu [4] studied the discrete L_0-Minkowski problem. And in $\mathbb{R}^3$, Firey [37] built a mathematical model to describe the ultimate shape of worn stones. This is a parabolic problem related to the L_0-Minkowski problem when f is a constant. Chou and Wang studied the critical case $p = -n-1$ in [8]. In [38], Lu and Wang established the existence of rotationally symmetric solutions of (1.5) for $p = -n-1$ (see [39, 22]). When $p = -n-1$ and $f \equiv 1$, all solutions to (1.5) are ellipsoids centered at the origin, see [40–42].

The uniqueness of L_p-Minkowski problem for $p > 1$ and $p \neq n+1$ (the uniqueness upto a dilation when $p = n+1$) was solved by the Brunn-Minkowski-Firey inequality [1]: let $Q_1, Q_2 \subset \mathbb{R}^{n+1}$ are two convex bodies that contain the origin in their interiors, $p > 1$ and $0 < \lambda < 1$, then

$$\mathrm{Vol}\,((1-\lambda)\circ Q_1 +_p \lambda \circ Q_2) \geqslant \mathrm{Vol}\,(Q_1)^{1-\lambda}\mathrm{Vol}\,(Q_2)^{\lambda}, \tag{1.6}$$

where '$+_p$' is the L_p-sum and '$\circ$' is the Firey multiplication. The equality in (1.6) holds if and only if $Q_1 = Q_2$. However, the uniqueness for $p < 1$ is difficult and still open because the Brunn-Minkowski inequality for $p < 1$ is still open. In [12], Jian, Lu and Wang obtained that for any $-n-1 < p < 0$, there exists a positive function $f \in C^\infty(\mathbb{S}^n)$ to guarantee that (1.5) has two different solutions, which means that we need more conditions to consider the uniqueness. The uniqueness results of polygonal L_0-Minkowski problem in $\mathbb{R}^2$ were given by Stancu in [5]. And in $\mathbb{R}^3$, Huang, Liu and Xu [13] obtained the uniqueness of L_p-Minkowski problem for $-1 \leqslant p < 1$ when $f \equiv 1$ for the C^4 smooth convex bodies. Chen and Zhou obtained the generalised dual Minkowski inequalities in [43].

Christoffel-Minkowski problem arises in the study of surface area functions and it asks the existence of a convex body Q whose k-th elementary symmetric function of all principal radii of the boundary is prescribed [9]. It needs to solve a k-Hessian equation

$$\sigma_k(h_{ij} + h\delta_{ij}) = f \quad \text{on} \quad \mathbb{S}^n, \tag{1.7}$$

where σ_k is the k-th elementary symmetric function defined in the beginning. When $k=n$, (1.7) is the classical Minkowski problem. A necessary condition for (1.7) to have a solution [33] is

$$\int_{\mathbb{S}^n} x_i f(x)dx = 0, \quad \forall i = 1, 2, \cdots, n+1. \tag{1.8}$$

Guan and Ma [46] gave a sufficient condition for the existence of a unique convex solution to (1.7), and Guan, Ma and Zhou [3] proved (1.8) is sufficient for (1.7) to have an admissible solution. The convex solution and admissible solution are defined in Definition 2.7.

Similarly, we can consider the L_p analog of Christoffel-Minkowski problem, which we call the Christoffel-Minkowski problem of L_p-sum, or equivalently

$$\sigma_k(h_{ij} + h\delta_{ij}) = fh^{p-1} \quad \text{on} \quad \mathbb{S}^n. \tag{1.9}$$

When $p=1$, (1.9) is reduced to (1.7), and when $k=n$, (1.9) is reduced to (1.5). When $p \geqslant k+1$ and $1 \leqslant k < n$, under the condition that the function $0 < f \in C^m(\mathbb{S}^n)$ $(m \geqslant 2)$ satisfies $\left(\left(f^{\frac{-1}{p+k+1}}\right)_{ij} + \delta_{ij} f^{\frac{-1}{p+k+1}}\right) \geqslant 0$ on $\mathbb{S}^n$, Hu, Ma and Shen [45] obtained the Christoffel-Minkowski problem of L_p-sum has a unique convex body that has the origin in its interior with a $C^{m+1,\alpha}$ $(0<\alpha<1)$ boundary (the uniqueness upto a dilation when $p=k+1$). The uniqueness of (1.9) for $1<p<k+1$ can be obtained via the Alexandrov-Fenchel inequality [44]. However, the uniqueness of (1.9) for $p<1$ is still open.

In this paper, we consider the uniqueness of L_p-Minkowski problem and Christoffel-Minkowski problem of L_p-sum for $p<1$ when the solutions to (1.1) and (1.2) are ellipsoids centered at the origin.

Our main result is

Theorem 1.1 If the solution to (1.1) is an ellipsoid centered at the origin, then the uniqueness holds for any $p \in \mathbb{R}\backslash\{-n-1\}$ (the uniqueness holds upto a dilation when $p=n+1$). And when $p=-n-1$, the solutions to (1.1) are all ellipsoids centered at the origin with a volume ω_{n+1}, where ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$.

Theorem 1.2 If the solution to (1.2) is an ellipsoid centered at the origin, then the uniqueness holds for any $p \in \mathbb{R}$ (the uniqueness holds upto a dilation when $p=k+1$).

The organization of this paper is as follows: after the preliminary Section 2, we discuss the ellipsoid solutions of L_p-Minkowski problem (Theorem 1.1) in Section 3. Then in Section 4, we prove Theorem 1.2.

2 Preliminary

Associated with a convex body is its support function.

Definition 2.1 Let $Q \subset \mathbb{R}^{n+1}$ be a convex body and M be its boundary. The support function of Q (or M) is defined by

$$h(x) = \max\{\langle x, y\rangle : y \in Q\}, \quad \forall x \in \mathbb{R}^{n+1}, \tag{2.1}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in $\mathbb{R}^{n+1}$.

If M is smooth and strictly convex, then it can be represented by its inverse Gauss map $\nu:\mathbb{S}^n\to M$. Then the support function of M can be represented by

$$h(x)=\langle x,\nu(x)\rangle,\quad \forall x\in\mathbb{S}^n, \tag{2.2}$$

and be positively homogeneous degree 1 after being extended to $\mathbb{R}^{n+1}$ by $h(x)=|x|\,h(x/\,|x|)$ for all $x\in\mathbb{R}^{n+1}$.

Clearly, the support function of a convex body is convex and positively homogeneous degree 1, thus it is determined by its value on $\mathbb{S}^n$ completely. Conversely, any continuous function h on $\mathbb{S}^n$, which can be convex after being extended to be positively homogeneous degree 1 on $\mathbb{R}^{n+1}$, can determine a convex body by

$$Q=\bigcap_{x\in\mathbb{S}^n}\left\{y\in\mathbb{R}^{n+1}:\langle x,y\rangle\leqslant h(x)\right\}.$$

Definition 2.2 Given two convex bodies $Q_1,Q_2\in\mathcal{K}$ with respective support function h_1 and h_2, and $\lambda,\mu\geqslant 0$ $(\lambda^2+\mu^2>0)$, the Minkowski sum $\lambda Q_1+\mu Q_2\in\mathcal{K}$ is defined by the convex body whose support function is $\lambda h_1+\mu h_2$, which means

$$\lambda Q_1+\mu Q_2=\bigcap_{x\in\mathbb{S}^n}\left\{y\in\mathbb{R}^{n+1}:\langle x,y\rangle\leqslant \lambda h_1(x)+\mu h_2(x)\right\}. \tag{2.3}$$

Let $\mathcal{K}_0$ collect convex bodies in $\mathcal{K}$ that contain the origin in their interiors. In 1962, Firey [36] generalized the concept of Minkowski sum from $p=1$ to L_p-sum for $p\geqslant 1$ as follows.

Definition 2.3 For $p\geqslant 1$, given two convex bodies $Q_1,Q_2\in\mathcal{K}_0$ with respective support function h_1 and h_2, and $\lambda,\mu\geqslant 0\quad(\lambda^2+\mu^2>0)$, the L_p-sum $\lambda\circ Q_1+_p\mu\circ Q_2\in\mathcal{K}_0$ is the convex body with support function $(\lambda h_1^p+\mu h_2^p)^{\frac{1}{p}}$, which means

$$\lambda\circ Q_1+_p\mu\circ Q_2=\bigcap_{x\in\mathbb{S}^n}\left\{y\in\mathbb{R}^{n+1}:\langle x,y\rangle^p\leqslant \lambda h_1^p(x)+\mu h_2^p(x)\right\}, \tag{2.4}$$

where '$+_p$' means the L_p summation and '$\circ$' means Firey multiplication.

It is clear that $\lambda\circ Q=\lambda^{\frac{1}{p}}Q$. And if $p=1$, they are equal.

Furthermore, we consider the set of positive support functions in $\mathcal{S}$, denoted by $\mathcal{S}_0$, i.e., $\mathcal{S}_0=\mathcal{S}\cap\{h>0\}$. Then we can further extend the L_p-sum (2.4) to any $p\in\mathbb{R}$. For $0<\lambda<1$ and $a,b>0$, define

$$M_p(a,b,\lambda)=\begin{cases}\min\{a,b\}, & \text{if}\quad p=-\infty,\\ ((1-\lambda)a^p+\lambda b^p)^{\frac{1}{p}}, & \text{if}\quad p\in(-\infty,0)\cup(0,\infty),\\ a^{1-\lambda}b^{\lambda}, & \text{if}\quad p=0,\\ \max\{a,b\}, & \text{if}\quad p=\infty.\end{cases} \tag{2.5}$$

$M_p(a,b,\lambda)$ is increasing with respect to p, namely, if $-\infty\leqslant p<q\leqslant\infty$, then

$$M_p(a,b,\lambda)\leqslant M_q(a,b,\lambda), \tag{2.6}$$

where $M_p(a,b,\lambda)=M_q(a,b,\lambda)$ if and only if $a=b>0$.

Definition 2.4 For $Q_1,Q_2\in\mathcal{K}_0$ with respective support function $h_1,h_2\in\mathcal{S}_0$, $\lambda\in(0,1)$ and $p\in\mathbb{R}$, the generalised L_p-sum is

$$(1-\lambda)\circ Q_1+_p\lambda\circ Q_2=\bigcap_{x\in\mathbb{S}^n}\left\{y\in\mathbb{R}^{n+1}:\langle x,y\rangle\leqslant M_p(h_1(x),h_2(x),\lambda)\right\}. \tag{2.7}$$

It is obvious that when $p\geqslant 1$, the convex body defined by (2.7) is the L_p-sum (2.4).

Definition 2.5 Suppose $Q\in\mathcal{K}$, the surface area measure $S(Q,\cdot)$ of Q is a Borel measure defined on $\mathbb{S}^n$, such that

$$\int_{\mathbb{S}^n}h_{Q'}(\omega)S(Q,d\omega)=\lim_{\varepsilon\to0^+}\frac{\mathrm{Vol}\,(Q+\varepsilon Q')-\mathrm{Vol}\,(Q)}{\varepsilon} \tag{2.8}$$

for any convex body $Q'\in\mathcal{K}$, where $h_{Q'}$ is the support function of Q' and $Q+\varepsilon Q'$ is the Minkowski sum defined in Definition 2.2.

Definition 2.6 For $p\geqslant 1$ and $Q\in\mathcal{K}_0$, the L_p surface area measure $S_p(Q,\cdot)$ of Q is a Borel measure defined on $\mathbb{S}^n$ satisfying

$$\frac{1}{p}\int_{\mathbb{S}^n}h^p_{Q'}(\omega)S_p(Q,d\omega)=\lim_{\varepsilon\to0^+}\frac{\mathrm{Vol}\,(Q+_p\varepsilon\circ Q')-\mathrm{Vol}\,(Q)}{\varepsilon} \tag{2.9}$$

for any convex body $Q'\in\mathcal{K}_0$, where $Q+_p\varepsilon\circ Q'$ is the L_p-sum defined in Definition 2.3.

The relationship between the classical and L_p surface area measure is

$$S_p(Q,\cdot)=h_Q^{1-p}S(Q,\cdot). \tag{2.10}$$

Definition 2.7 A function $u\in C^2(\mathbb{S}^n)$ is called convex if $(u_{ij}+u\delta_{ij})>0$ on $\mathbb{S}^n$. For $1\leqslant k\leqslant n$, let Γ_k be the convex cone in $\mathbb{R}^n$ determined as

$$\Gamma_k=\{\sigma_1(\lambda)>0,\sigma_2(\lambda)>0,\cdots,\sigma_k(\lambda)>0\}.$$

Suppose $u\in C^2(\mathbb{S}^n)$, we say u is k-convex if for any $x\in\mathbb{S}^n$, $W(x)=\{u_{ij}(x)+u(x)\delta_{ij}\}\in\Gamma_k$. Furthermore, u is called an admissible solution to (1.7) if u is k-convex and satisfies (1.7).

Now we represent the Gauss curvature of a convex body by its support function.

Assume that $M\subset\mathbb{R}^{n+1}$ is a smooth, closed and uniformly strictly convex hypersurface enclosing the origin and parameterised by its inverse Gauss map $\nu:\mathbb{S}^n\to M$. Let h be the support function of M, $\{e_1,e_2,\cdots,e_n\}$ be the local orthonormal frame on $\mathbb{S}^n$ and ∇_i be the covariant differentiation on M along the direction e_i. Differentiate (2.2) twice along e_i and e_j, then we have

$$G_{ij}=\nabla_{ij}h+h\delta_{ij}, \tag{2.11}$$

where G_{ij} is the second fundamental form of M. The details can be found in [47]. Let g_{ij} be the metric of M, then according to the relation $\nabla_i x=G_{ik}g^{km}\nabla_m\nu$, we have

$$\delta_{ij}=\langle\nabla_i x,\nabla_j x\rangle=G_{ik}g^{km}G_{js}g^{sl}\langle\nabla_m\nu,\nabla_l\nu\rangle=G_{ik}G_{jm}g^{km},$$

thus

$$G^{jk} = G_{jm} g^{km}. \tag{2.12}$$

Due to the uniformly convexity of M, the Gauss curvature K of M can be represented by its support function as follows

$$K = \det\left(G_{jm} g^{km}\right) = \det\left(G^{jk}\right),$$

by (2.11), we have

$$\frac{1}{K} = \det(G_{jk}) = \det(\nabla_{jk} h + h\delta_{jk}). \tag{2.13}$$

Remark 2.8 The principal radii of M are eigenvalues of matrix $\{h_{ij} + h\delta_{ij}\}$.

3 Proof of Theorem 1.1

We can see that $h = 1$ is a solution to (1.1). Denote $\mathbb{M}$ by the boundary of the ellipsoid centered at the origin in Theorem 1.1. To prove the uniqueness of solutions to (1.1), we need to prove that $\mathbb{M}$ is a unit sphere when $p \in \mathbb{R}\backslash\{-n-1, n+1\}$. Choose a suitable orthonormal frame on $\mathbb{R}^{n+1}$ such that $\mathbb{M}$ is in the following form

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} + \frac{x_{n+1}^2}{a_{n+1}^2} = 1 \quad (a_1, a_2, \cdots, a_n, a_{n+1} > 0). \tag{3.1}$$

Setting $p_0 = p - 1$, according to (2.13), (1.1) is equivalent to

$$\frac{1}{\mathbb{K}} = h^{p_0}, \tag{3.2}$$

where $\mathbb{K}$ is the Gauss curvature of $\mathbb{M}$ and h is the support function of $\mathbb{M}$.

3.1 The Gauss Curvature $\mathbb{K}$ of $\mathbb{M}$

The lower semi-surface of $\mathbb{M}$ is

$$\begin{aligned} &\mathbb{M}^- : \mathbb{R}^n \supset \Omega \to \mathbb{R}^{n+1}, \\ &(x_1, x_2, \cdots, x_n) \mapsto (x_1, x_2, \cdots, x_n, u^-(x_1, x_2, \cdots, x_n)) \\ &(x_i \in [-a_i, a_i], a_i > 0, i = 1, 2, \cdots, n). \end{aligned} \tag{3.3}$$

Set $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, then

$$\begin{array}{rccc} \mathbb{M}^- : & \mathbb{R}^n \supset \Omega & \to & \mathbb{R}^{n+1}, \\ & x & \mapsto & (x, u^-(x)), \end{array}$$

where

$$u^-(x) = -a_{n+1}\sqrt{1 - \sum_{i=1}^n \frac{x_i^2}{a_i^2}}, \quad x_i \in [-a_i, a_i], \tag{3.4}$$

then $\mathbb{M}^-$ is the graph of u^-. Set

$$u^+(x) = a_{n+1}\sqrt{1-\sum_{i=1}^{n}\frac{x_i^2}{a_i^2}}, \tag{3.5}$$

then $u^- = -u^+$. When $u^+ \neq 0$, we have

$$u^-_{x_i}(x) = \frac{a_{n+1}^2}{a_i^2}\frac{x_i}{u^+(x)} \quad (i=1,2,\cdots,n), \tag{3.6}$$

$$u^-_{x_ix_j}(x) = \frac{a_{n+1}^2}{a_i^2}\left(\frac{\delta_{ij}}{u^+(x)} + \frac{a_{n+1}^2}{a_j^2}\frac{x_ix_j}{(u^+(x))^3}\right) \quad (i,j=1,2,\cdots,n). \tag{3.7}$$

Then

$$\begin{aligned}
\det(D^2u^-) &= \det\left(\frac{a_{n+1}^2}{a_i^2}\left(\frac{\delta_{ij}}{u^+} + \frac{a_{n+1}^2}{a_j^2}\frac{x_ix_j}{(u^+)^3}\right)\right)\\
&= \frac{a_{n+1}^{2n}}{(u^+)^n}\det\left(E_n + \begin{pmatrix}\frac{a_{n+1}^2x_1}{(u^+)^2}\\ \frac{a_{n+1}^2x_2}{(u^+)^2}\\ \vdots\\ \frac{a_{n+1}^2x_n}{(u^+)^2}\end{pmatrix}\left(\frac{x_1}{a_1^2},\frac{x_2}{a_2^2},\cdots,\frac{x_n}{a_n^2}\right)\right)\prod_{i=1}^{n}\frac{1}{a_i^2}\\
&= \frac{a_{n+1}^{2n}}{(u^+)^n}\det\left(1 + \left(\frac{x_1}{a_1^2},\frac{x_2}{a_2^2},\cdots,\frac{x_n}{a_n^2}\right)\begin{pmatrix}\frac{a_{n+1}^2x_1}{(u^+)^2}\\ \frac{a_{n+1}^2x_2}{(u^+)^2}\\ \vdots\\ \frac{a_{n+1}^2x_n}{(u^+)^2}\end{pmatrix}\right)\prod_{i=1}^{n}\frac{1}{a_i^2}\\
&= \frac{a_{n+1}^{2n}}{(u^+)^n}\left(1+\frac{a_{n+1}^2}{(u^+)^2}\sum_{i=1}^{n}\frac{x_i^2}{a_i^2}\right)\prod_{i=1}^{n}\frac{1}{a_i^2}\\
&= \frac{a_{n+1}^{2n+2}}{(u^+)^{n+2}}\prod_{i=1}^{n}\frac{1}{a_i^2}.
\end{aligned}$$

In the third equality above, we have used

$$\det(\lambda E_m + A_{mn}B_{nm}) = \lambda^{m-n}\det(\lambda E_n + A_{nm}B_{mn}),$$

where $\lambda \in \mathbb{R}$ is a constant, A_{mn} is a real $m\times n$ matrix and E_m is a m-order identity matrix.

Also, we have

$$(1+\left|Du^-\right|^2)^{\frac{n+2}{2}} = \left(1+\frac{a_{n+1}^4}{(u^+)^2}\sum_{i=1}^{n}\frac{x_i^2}{a_i^4}\right)^{\frac{n+2}{2}}.$$

Then the Gauss curvature $\mathbb{K}^-$ of $\mathbb{M}^-$ is

$$\begin{aligned}
\mathbb{K}^- &= \frac{\det(D^2u^-)}{(1+|Du^-|^2)^{\frac{n+2}{2}}}\\
&= a_{n+1}^{2n+2}\left(\prod_{i=1}^{n}\frac{1}{a_i^2}\right)\left(\left(u^+\right)^2 + a_{n+1}^4\sum_{i=1}^{n}\frac{x_i^2}{a_i^4}\right)^{-\frac{n+2}{2}}.
\end{aligned}$$

By (3.5), we have

$$\mathbb{K}^- = (\prod_{i=1}^{n+1} \frac{1}{a_i^2}) \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{-\frac{n+2}{2}}.$$

According to the symmetry of ellipsoids, the Gauss curvature $\mathbb{K}$ of $\mathbb{M}$ is

$$\mathbb{K} = (\prod_{i=1}^{n+1} \frac{1}{a_i^2}) \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{-\frac{n+2}{2}}. \tag{3.8}$$

Remark 3.1 Although u^+ is present in the denominator of $\det(D^2u^-)$ and $1+|Du^-|^2$, the quotient $\mathbb{K}^-$ of them avoids the case. Therefore we can also use (3.8) to obtain the Gauss curvature of $\mathbb{M}$ when $u^+ = 0$ because of the continuity of Gauss curvature of ellipsoids.

3.2 The Support Function h of $\mathbb{M}$

The unit outer normal at an arbitrary point $P = (x_1, x_2, \cdots, x_n, x_{n+1})$ on $\mathbb{M}$ is

$$N(P) = \frac{(\frac{x_1}{a_1^2}, \frac{x_2}{a_2^2}, \cdots, \frac{x_n}{a_n^2}, \frac{x_{n+1}}{a_{n+1}^2})}{\sqrt{\sum\limits_{i=1}^{n+1} \frac{x_i^2}{a_i^4}}},$$

then the support function h at P is

$$h(P) = \langle P, N(P) \rangle = \frac{\sum\limits_{i=1}^{n+1} \frac{x_i^2}{a_i^2}}{\sqrt{\sum\limits_{i=1}^{n+1} \frac{x_i^2}{a_i^4}}} = \frac{1}{\sqrt{\sum\limits_{i=1}^{n+1} \frac{x_i^2}{a_i^4}}}.$$

Thus the support function h of $\mathbb{M}$ is

$$h = \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{-\frac{1}{2}}. \tag{3.9}$$

3.3 Proof of Theorem 1.1

Proof Inserting (3.8) and (3.9) into (3.2), we have

$$\left(\prod_{i=1}^{n+1} a_i^2 \right) \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{\frac{n+2}{2}} = \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{-\frac{p_0}{2}},$$

thus

$$\left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{\frac{n+2+p_0}{2}} = \prod_{i=1}^{n+1} a_i^{-2}. \tag{3.10}$$

In order that (3.10) is true for all $P \in \mathbb{M}$, for any $j \in \{1,2,\cdots,n+1\}$, pick $P_j = (0,\cdots,0,a_j,0,\cdots,0)$ in (3.10), then we have

$$a_j^{n+2+p_0} = \prod_{i=1}^{n+1} a_i^2, \quad \forall j \in \{1,2,\cdots,n+1\}. \tag{3.11}$$

Then

$$\begin{cases} p_0 = -n-2, \\ \prod\limits_{i=1}^{n+1} a_i = 1, \end{cases} \quad \text{or} \quad \begin{cases} p_0 \neq -n-2, \\ a_1 = a_j, \quad \forall j. \end{cases}$$

Case 1 When $\begin{cases} p_0 = -n-2, \\ \prod\limits_{i=1}^{n+1} a_i = 1, \end{cases}$ the volume of the ellipsoid surrounded by $\mathbb{M}$ is a constant ω_{n+1}, where ω_{n+1} is the volume of the $(n+1)$-dimensional unit ball in $\mathbb{R}^{n+1}$.

Case 2 When $\begin{cases} p_0 \neq -n-2, \\ a_1 = a_j, \quad \forall j, \end{cases}$ by (3.11), we have

$$a_1^{2n} = a_1^{n+p_0},$$

then

$$p_0 = n \quad \text{or} \quad \begin{cases} p_0 \neq n, \\ a_1 = 1. \end{cases}$$

Hence, for all $p_0 \in \mathbb{R} \setminus \{n, -n-2\}$, $\mathbb{M}$ is a unit sphere; when $p_0 = n$, $\mathbb{M}$ is an arbitrary sphere; when $p_0 = -n-2$, the product of all the half-axis of $\mathbb{M}$ is 1.

4 Proof of Theorem 1.2

Let M be a uniformly convex hypersurface that can be represented by the graph of a C^2 function u, then the first and second fundamental form of M are

$$\mathbf{I} = (\delta_{ij} + u_i u_j), \quad \mathbf{II} = \frac{1}{\sqrt{1+|Du|^2}}(u_{ij}),$$

respectively, where Du, (u_{ij}) are the gradient and Hessian matrix of u, respectively, and (u_{ij}) is invertible because of the uniformly convexity of M.

4.1 Proof of Theorem 1.2 for $k = 1$

When $k = 1$, (1.2) is reduced to

$$\sigma_1(h_{ij} + h\delta_{ij}) = nh^{p_0} \quad \text{on} \quad \mathbb{S}^n. \tag{4.1}$$

We can see that $h = 1$ is a solution to (1.2). Similarly, let $\mathbb{M}$ represented by (3.1) be the boundary of the ellipsoid centered at the origin in Theorem 1.2. To prove the uniqueness of solutions to (1.2), we need to prove that $\mathbb{M}$ is a unit sphere when $p_0 \in \mathbb{R}\setminus\{1\}$. $\mathbb{M}^-$ is represented by the graph of u^-, then denote the inverse matrix of the Hessian matrix of u^- by

$$(u^-_{ij})^{-1} = ((u^-)^{ij}),$$

we have

$$\sigma_1(h_{ij} + h\delta_{ij}) = \sqrt{1 + |\nabla u^-|^2}\left(\sum_{i=1}^{n}(u^-)^{ii} + \sum_{i,j=1}^{n} u_i^- u_j^- (u^-)^{ij}\right). \tag{4.2}$$

Proof For the lower semi-surface $\mathbb{M}^-$, we have

$$\sigma_1(h_{ij} + h\delta_{ij}) = \sqrt{1 + \sum_{i=1}^{n}\left(\frac{a_{n+1}^4}{a_i^4}\frac{x_i^2}{x_{n+1}^2}\right)}\left(\sum_{i=1}^{n}(u^-)^{ii} + \sum_{i,j=1}^{n} u_i^- u_j^- (u^-)^{ij}\right). \tag{4.3}$$

Inserting (4.3) and (3.9) into (4.1), we have

$$\sum_{i=1}^{n}(u^-)^{ii} + \sum_{i,j=1}^{n} u_i^- u_j^- (u^-)^{ij} = n\frac{|x_{n+1}|}{a_{n+1}^2}\left(\sum_{i=1}^{n+1}\frac{x_i^2}{a_i^4}\right)^{-\frac{p_0+1}{2}}. \tag{4.4}$$

Since

$$\delta_{ij} = \sum_{m=1}^{n} u_{im}^-(u^-)^{mj} = \sum_{m=1}^{n}\frac{a_{n+1}^2}{a_i^2}\frac{\delta_{im}}{u^+}(u^-)^{mj} + \sum_{m=1}^{n}\frac{u_i^- u_m^-(u^-)^{mj}}{u^+} \quad (i,j = 1,2,\cdots,n),$$

then

$$u^+ = \frac{a_{n+1}^2}{a_i^2}(u^-)^{ii} + \sum_{m=1}^{n} u_i^- u_m^-(u^-)^{mi} \quad (i = 1,2,\cdots,n)$$

and

$$nu^+ = a_{n+1}^2\sum_{i=1}^{n}\frac{(u^-)^{ii}}{a_i^2} + \sum_{i,m=1}^{n} u_i^- u_m^-(u^-)^{mi},$$

thus (4.4) is equivalent to

$$na_{n+1}\sqrt{1 - \sum_{i=1}^{n}\frac{x_i^2}{a_i^2}} - a_{n+1}^2\sum_{i=1}^{n}\frac{(u^-)^{ii}}{a_i^2} + \sum_{i=1}^{n}(u^-)^{ii} = n\frac{|x_{n+1}|}{a_{n+1}^2}\left(\sum_{i=1}^{n+1}\frac{x_i^2}{a_i^4}\right)^{-\frac{p_0+1}{2}}. \tag{4.5}$$

In order that (4.5) is true for any P on $\mathbb{M}^-$, taking $P_{n+1} = (0,0,\cdots,0,-a_{n+1})$, we have

$$\left[na_{n+1} - a_{n+1}^2\sum_{i=1}^{n}\frac{(u^-)^{ii}}{a_i^2} + \sum_{i=1}^{n}(u^-)^{ii}\right]\Bigg|_{P_{n+1}} = na_{n+1}^{p_0}. \tag{4.6}$$

For all $j \in \{1,2,\cdots,n\}$, fixed, at $P_j = (0,\cdots,0,a_j,0,\cdots,0)$, we have

$$\left[-a_{n+1}^2\sum_{i=1}^{n}\frac{(u^-)^{ii}}{a_i^2} + \sum_{i=1}^{n}(u^-)^{ii}\right]\Bigg|_{P_j} = 0. \tag{4.7}$$

Without loss of generality, assume that

$$a_{n+1} = \min\{a_1, a_2, \cdots, a_n, a_{n+1}\}.$$

Since (u^-_{ij}) is positive definite on $\mathbb{M}^-$, we have $(u^-)^{ii} > 0$ $(i = 1, 2, \cdots, n)$, then (4.7) shows that

$$\left[\sum_{i=1}^{n}(1 - \frac{a_{n+1}^2}{a_i^2})(u^-)^{ii}\right]\Bigg|_{P_j} = 0,$$

then

$$1 - \frac{a_{n+1}^2}{a_i^2} = 0 \Rightarrow a_i = a_{n+1} \ (i = 1, 2, \cdots, n). \tag{4.8}$$

Using (4.8) in (4.6), we obtain

$$a_{n+1} = a_{n+1}^{p_0}.$$

Thus

$$\begin{cases} p_0 = 1, \\ a_i = a_{n+1}, \end{cases} \quad \text{or} \quad \begin{cases} p_0 \neq 1, \\ a_i = a_{n+1} = 1 \end{cases} \quad (i = 1, 2, \cdots, n).$$

Hence we obtain the following results: $\mathbb{M}$ is a unit sphere for all $p_0 \in \mathbb{R} \setminus \{1\}$, and $\mathbb{M}$ is an arbitrary sphere when $p_0 = 1$.

4.2 Proof of Theorem 1.2 for $1 < k < n$

It is complicated to compute the Hessian matrix $((u^-)^{ij})$ for the intermediate cases

$$\sigma_k(h_{ij} + h\delta_{ij}) = C_n^k h^{p_0} \quad \text{on} \quad \mathbb{S}^n, \quad k \in \{2, 3, \cdots, n-1\}. \tag{4.9}$$

According to the above discussion, we can pick some special points on the boundary first, then calculate the Hessian matrix at these special points, and use equation (4.9) to obtain the conclusion finally. In this part, we need to prove that $\mathbb{M}$ is a unit sphere for any $p_0 \in \mathbb{R}\setminus\{k\}$.

Proof For the lower semi-surface $\mathbb{M}^-$, at point $P_{n+1} = (0, 0, \cdots, 0, -a_{n+1})$, according to (3.6), (3.7), we have

$$Du^- = 0, \quad u^-_{ij} = \frac{a_{n+1}}{a_i^2}\delta_{ij} \quad (i, j = 1, 2, \cdots, n),$$

then

$$\big((u^-)^{ij}\big) = \mathrm{diag}\,(\frac{a_1^2}{a_{n+1}}, \frac{a_2^2}{a_{n+1}}, \cdots, \frac{a_n^2}{a_{n+1}}).$$

Thus

$$\begin{aligned} \sigma_k(h_{ij} + h\delta_{ij})|_{P_{n+1}} &= \sigma_k\left(\mathrm{diag}\left(\frac{a_1^2}{a_{n+1}}, \frac{a_2^2}{a_{n+1}}, \cdots, \frac{a_n^2}{a_{n+1}}\right)\right) \\ &= a_{n+1}^{-k}\sigma_k(a_1^2, a_2^2, \cdots, a_n^2). \end{aligned}$$

Using (4.9), we have

$$\sigma_k(a_1^2, a_2^2, \cdots, a_n^2) = C_n^k a_{n+1}^{p_0+k}. \tag{4.10}$$

For $i=1,2,\cdots,n$, fixed, at point $P_i=(0,\cdots,0,\frac{\sqrt{2}}{2}a_i,0,\cdots,0,-\frac{\sqrt{2}}{2}a_{n+1})$, we have

$$\begin{aligned}&Du^-=(0,\cdots,0,\frac{a_{n+1}}{a_i},0,\cdots,0),\\&u^-_{ii}=2\sqrt{2}\frac{a_{n+1}}{a_i^2},\ \ u^-_{mm}=\sqrt{2}\frac{a_{n+1}}{a_m^2}\ \ (m=1,\cdots,i-1,i+1,\cdots,n),\\&u^-_{mj}=0\ \ (m,j=1,2,\cdots,n,\ m\neq j).\end{aligned}$$

Then

$$\begin{aligned}&\big((u^-)^{ij}\big)=\frac{1}{\sqrt{2}}\mathrm{diag}\left(\frac{a_1^2}{a_{n+1}},\cdots,\frac{a_{i-1}^2}{a_{n+1}},\frac{a_i^2}{2a_{n+1}},\frac{a_{i+1}^2}{a_{n+1}},\cdots,\frac{a_n^2}{a_{n+1}}\right),\\&(\delta_{ij}+u^-_iu^-_j)=\mathrm{diag}\,(1,\cdots,1,1+\frac{a_{n+1}^2}{a_i^2},1,\cdots,1).\end{aligned}$$

Hence

$$\sigma_k(h_{ij}+h\delta_{ij})|_{P_i}=\frac{(a_i^2+a_{n+1}^2)^{\frac{k}{2}}}{2^{\frac{k}{2}}a_i^ka_{n+1}^k}\sigma_k(a_1^2,\cdots,a_{i-1}^2,\frac{a_i^2+a_{n+1}^2}{2},a_{i+1}^2,\cdots,a_n^2).$$

Using (4.9), we have

$$\sigma_k(a_1^2,\cdots,a_{i-1}^2,\frac{a_i^2+a_{n+1}^2}{2},a_{i+1}^2,\cdots,a_n^2)=C_n^k\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}a_{n+1}^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}. \tag{4.11}$$

For $k\in\{2,3,\cdots,n-1\}$, denote

$$\begin{aligned}&\Sigma_1=\sigma_{k-1}(a_1^2,\cdots,a_{i-1}^2,a_{i+1}^2,\cdots,a_n^2)>0,\\&\Sigma_2=\sigma_k(a_1^2,\cdots,a_{i-1}^2,a_{i+1}^2,\cdots,a_n^2)>0,\end{aligned}$$

then

$$a_i^2\Sigma_1+\Sigma_2=C_n^ka_{n+1}^{p_0+k}, \tag{4.12}$$

$$\frac{a_i^2+a_{n+1}^2}{2}\Sigma_1+\Sigma_2=C_n^k\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}a_{n+1}^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}. \tag{4.13}$$

Next we prove $a_i=a_{n+1}$.

Case 1 $p_0+k\geqslant 0$. Divided (4.13) by (4.12), we have

$$\frac{\frac{a_i^2+a_{n+1}^2}{2}+\frac{\Sigma_2}{\Sigma_1}}{a_i^2+\frac{\Sigma_2}{\Sigma_1}}=\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}. \tag{4.14}$$

If $a_i\geqslant a_{n+1}$, then right hand side of (4.14) is

$$\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}\geqslant\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(2a_i^2)^{\frac{p_0+k}{2}}}=1,$$

while left hand side of (4.14) is

$$\frac{\frac{a_i^2+a_{n+1}^2}{2}+\frac{\Sigma_2}{\Sigma_1}}{a_i^2+\frac{\Sigma_2}{\Sigma_1}} \leqslant \frac{a_i^2+\frac{\Sigma_2}{\Sigma_1}}{a_i^2+\frac{\Sigma_2}{\Sigma_1}}=1,$$

then

$$\frac{\frac{a_i^2+a_{n+1}^2}{2}+\frac{\Sigma_2}{\Sigma_1}}{a_i^2+\frac{\Sigma_2}{\Sigma_1}}=\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}=1 \Rightarrow a_i=a_{n+1}.$$

Similarly, if $a_i \leqslant a_{n+1}$, we also have $a_i=a_{n+1}$.

Case 2 $p_0+k<0$. Subtracting (4.13) from (4.12), we have

$$\frac{a_i^2-a_{n+1}^2}{2}\Sigma_1=C_n^k a_{n+1}^{p_0+k}\left(1-\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}\right). \tag{4.15}$$

Then we prove $a_i=a_{n+1}$ by contradiction. If $a_i \neq a_{n+1}$, then Σ_1 can be represented as

$$\Sigma_1=\frac{2C_n^k a_{n+1}^{p_0+k}}{a_i^2-a_{n+1}^2}\left(1-\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}\right). \tag{4.16}$$

Note that $\Sigma_1=\sigma_{k-1}(a_1^2,\cdots,a_{i-1}^2,a_{i+1}^2,\cdots,a_n^2)$ is independent of a_i^2, thus

$$\begin{aligned}
0=&\frac{\partial\Sigma_1}{\partial(a_i^2)}\\
=&-\frac{2C_n^k a_{n+1}^{p_0+k}}{(a_i^2-a_{n+1}^2)^2}\left(1-\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}\right)\\
&-\frac{2C_n^k a_{n+1}^{p_0+k}}{a_i^2-a_{n+1}^2}\frac{p_0+k}{2}\left(\frac{2a_i^2}{a_i^2+a_{n+1}^2}\right)^{\frac{p_0+k-2}{2}}\frac{2a_{n+1}^2}{(a_i^2+a_{n+1}^2)^2}\\
=&-\frac{2C_n^k a_{n+1}^{p_0+k}}{(a_i^2-a_{n+1}^2)^2}\left(1-\frac{2^{\frac{p_0+k}{2}}a_i^{p_0+k}}{(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}}\right)-\frac{(p_0+k)2^{\frac{p_0+k}{2}}C_n^k a_{n+1}^{p_0+k+2}a_i^{p_0+k-2}}{(a_i^2-a_{n+1}^2)(a_i^2+a_{n+1}^2)^{\frac{p_0+k+2}{2}}}.
\end{aligned}$$

The above equality is equivalent to

$$(a_i^2-a_{n+1}^2)\left(2^{\frac{p_0+k}{2}}a_i^{p_0+k}-(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}\right)=\frac{(p_0+k)2^{\frac{p_0+k-2}{2}}(a_i^2-a_{n+1}^2)^2a_{n+1}^2a_i^{p_0+k-2}}{a_i^2+a_{n+1}^2}. \tag{4.17}$$

The right hand side of (4.17) is positive. If $a_i>a_{n+1}$, then

$$2a_i^2>a_i^2+a_{n+1}^2 \Rightarrow (2a_i^2)^{\frac{p_0+k}{2}}<(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}}.$$

If $a_i<a_{n+1}$,

$$(2a_i^2)^{\frac{p_0+k}{2}}>(a_i^2+a_{n+1}^2)^{\frac{p_0+k}{2}},$$

then the left hand side of (4.17) is negative. This is a contradiction.

Hence $a_i=a_{n+1}$ $(i=1,2,\cdots,n)$.

Using (4.10), we have

$$C_n^k a_{n+1}^{2k} = C_n^k a_{n+1}^{p_0+k} \quad \Rightarrow \quad p_0 = k \quad \text{or} \quad \begin{cases} p_0 \neq k, \\ a_{n+1} = 1. \end{cases}$$

Thus we have

$$\begin{cases} p_0 = k, \\ a_i = a_{n+1} \end{cases} \quad \text{or} \quad \begin{cases} p_0 \neq k, \\ a_i = a_{n+1} = 1 \end{cases} \quad (i = 1, 2, \cdots, n).$$

Now we have the following results: for any $p_0 \in \mathbb{R} \setminus \{k\}$, $\mathbb{M}$ is a unit sphere, and if $p_0 = k$, $\mathbb{M}$ is an arbitrary sphere.

Now we complete the proof of Theorem 1.2.

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L_p-Minkowski 问题椭球解的唯一性

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摘要: 本文研究了 L_p-Minkowski 问题 (解是中心在原点的椭球的假定下). 利用支撑函数与高斯曲率的关系, 获得了当 $p < 1$ 时椭球解的唯一性, 推广了 L_p-Minkowski 问题以及 L_p-和的 Christoffel-Minkowski 问题的唯一性结果.

关键词: 唯一性; Minkowski问题; Monge-Ampère方程; k-Hessian方程

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