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RIGIDITY THEOREMS OF THE SPACE-LIKE λ-HYPERSURFACES IN THE LORENTZIAN SPACE $\mathbb{R}_1^{n+1}$

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Abstract: In this paper, we study complete space-like λ-hypersurfaces in the Lorentzian space $\mathbb{R}_1^{n+1}$. By using the property of generalized $\mathcal{L}$-operator and some integral inequalities, we obtain some rigidity theorems for these hypersurfaces including the complete space-like self-shrinkers with weight in $\mathbb{R}_1^{n+1}$, which generalize some related results in the Euclidean space.

Keywords: Lorentzian space; rigidity theorems; space-like λ-hypersurfaces; self-shrinkers

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1 Introduction

For $\varepsilon = \pm 1$, let $\mathbb{E}_\varepsilon^{n+1}$ be the Euclidean space $\mathbb{R}^{n+1}$ (when $\varepsilon = 1$) or the Lorentzian space $\mathbb{R}_1^{n+1}$ (when $\varepsilon = -1$). The standard inner product on $\mathbb{E}_\varepsilon^{n+1}$ is given by

$$\langle X, Y\rangle = X_1Y_1 + X_2Y_2 + \cdots + \varepsilon X_{n+1}Y_{n+1}.$$

Let M be an immersed space-like hypersurface in $\mathbb{E}_\varepsilon^{n+1}$ of which the induced metric g is positive definite and $x : M^n \to \mathbb{E}_\varepsilon^{n+1}$ be the corresponding immersion of M. In this paper, we also use x to denote the position vector of M. Thus x and the unit normal vector N are taken as smooth $\mathbb{R}^{n+1}$-valued functions on M^n. For a suitably chosen function s on M^n, if the position vector x and the mean curvature H of M satisfy

$$H + \varepsilon s\langle x, N\rangle := \lambda = \text{const}, \tag{1.1}$$

then M is called a λ-hypersurface with the weight function s.

When $s \equiv 0$, the corresponding λ-hypersurfaces reduce to hypersurfaces with constant mean curvature which have been studied extensively. For example, Calabi considered in [1] the maximal space-like hypersurfaces M^n in the Lorentzian space $\mathbb{R}_1^{n+1}$ and proposed some

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Bernstein-type problems for a nonlinear equation; for a given complete space-like hypersurface M^n in $\mathbb{R}^{n+1}_1$ with constant mean curvature, it was proved by Xin [24] that if the Gauss map image of M^n is inside a bounded subdomain of the hyperbolic n-space $\mathbb{H}^n$, then M^n must be a hyperplane. A similar result was also proved earlier in [22] with extra assumptions. In [3], Cao, Shen and Zhu further extended the result by showing that if the image of the Gauss map lies inside a horoball of $\mathbb{H}^n$, M^n is necessarily a hyperplane. Later, Wu [23] generalized the above mentioned results and proved a more general Bernstein theorem for complete space-like hypersurfaces in Lorentzian space with constant mean curvature.

If $\lambda = 0$, then M^n is called a self-shrinker with weight s. In particular, if moreover s is chosen to be a constant, then M^n is nothing but the usual self-shrinker which play an important role in the study of the mean curvature flow because they describe all possible blow-ups at a given singularity of the mean curvature flow (see [5] or [8] with $\lambda = 0$). Very recently, there have appeared some interesting results for space-like self-shrinkers. For example, after the submission of this paper, we were kindly informed the following results: Chen-Qiu [4] proved a rigidity (uniqueness) theorem that any complete m-dimensional space-like self-shrinker in a pseudo-Euclidean space $\mathbb{R}^{m+n}_n$ must be an affine plane, which is clearly a very important Bernstein-type result; Liu-Xin [19] obtained two rigidity theorems for closed (w.r.t the Euclidean topology) or complete space-like self-shrinkers M^m in $\mathbb{R}^{m+n}_n$ by using restrictions on the growth of either $|H|^2$ or the log of the w-function; and Ding-Wang [11] proved a Bernstein-type theorem for space-like graph self-shrinkers of higher codimension by assuming the sub-exponential decay of the metric determinant $\det g$. As for the geometries of self-shrinkers in Euclidean space, a lot of interesting results were obtained in recent years, including some gap theorems and rigidity theorems. Details of this can be found in, for example, [2, 7, 10, 12–14, 16, 18] etc.

According to [15], λ-hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ were firstly studied by Mcgonagle and Ross in [20] with $s = \frac{1}{2}$; Guang [15] also studied the λ-hypersurfaces in $\mathbb{R}^{n+1}$ with $s = \frac{1}{2}$ and proved a Bernstein-type theorem showing that smooth λ-hypersuafaces which are entire graphs and with a polynomial volume growth are necessarily hyperplanes in $\mathbb{R}^{n+1}$.

If one takes $\varepsilon = s = 1$ in (1.1), the corresponding λ-hypersurfaces are exactly what Cheng and Wei defined and studied in [8], where the authors successfully introduced a weighted volume functional and proved that the λ-hypersuafaces in the Euclidean space $\mathbb{R}^{n+1}$ are nothing but the critical points of the above functional. Later, Cheng, Ogaza and Wei (see [6, 9]) obtained some rigidity and Bernstein-type theorems for these complete λ-hypersurfaces. In particular, the following result is proved.

Theorem 1.1 [6] Let $x : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with weight $s = -1$ and a polynomial area growth. Then, either x is isometric to one of the following embedded hypersurfaces

1. the sphere $S^n(r) \subset \mathbb{R}^{n+1}$ with radius $r > 0$;
2. the hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$;

3. the cylinder $S^1(r) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$;
4. the cylinder $S^{n-1}(r) \times \mathbb{R} \subset \mathbb{R}^{n+1}$,

or, there exists some $p \in M^n$ such that the squared norm S of the second fundamental form of x satisfies

$$\left(\sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H(p) - \lambda)^2 > 1 + \frac{n\lambda^2}{4(n-1)}.$$

In this paper, we consider space-like λ-hypersurfaces $x : M^n \to \mathbb{R}^{n+1}_1$ in the Lorentzian space $\mathbb{R}^{n+1}_1$ (so that $\varepsilon = -1$). After extending the definitions of λ-hypersurfaces, we generalize the $\mathcal{L}$-operator that has been effectively used by many authors (see the operators $\tilde{\mathcal{L}}$ and $\mathcal{L}$ defined, respectively in (2.4) and (2.14)). We shall use these generalized operators to extend Theorem 1.1 to the complete space-like λ-hypersurfaces in $\mathbb{R}^{n+1}_1$.

Let a be a nonzero constant and denote $\epsilon = \mathrm{Sgn}\,(a\langle x, x\rangle)$, where $\langle \cdot, \cdot \rangle$ is the Lorentzian product. We shall study λ-hypersurfaces in $\mathbb{R}^{n+1}_1$ either with weight $s = \epsilon a$ when $\langle x, x\rangle \neq 0$, or with weight $s = \langle x, x\rangle$. Not that, in the first case, we can always choose an a such that $a\langle x, x\rangle > 0$ implying $\epsilon = 1$.

Now for a given hypersurface M^n, let S denote the squared norm of the second fundamental form, and A, I denote the shape operator and the identity map, respectively. Then the rigidity theorems we prove in this paper are stated as follows.

Theorem 1.2 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like λ-hypersurface with weight $s = \epsilon a$ and $\langle x, x\rangle \neq 0$, where a is a constant, such that

$$\int_{M^n} \left(\left| \nabla \left(S - \frac{H^2}{n} \right) \right| + \left| \mathcal{L} \left(S - \frac{H^2}{n} \right) \right| \right) e^{-\frac{\epsilon a \langle x, x\rangle}{2}} dV_{M^n} < +\infty, \tag{1.2}$$

in which the differential operator $\mathcal{L}$ is defined by (2.4). Then, either x is totally umbilical and thus isometric to one of the following two hypersurfaces:

1. the hyperbolic space $\mathbb{H}^n(c) \subset \mathbb{R}^{n+1}_1$ with the sectional curvature $c < 0$;
2. the Euclidean space $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$,

or, there exists some $p \in M^n$ such that, at p,

$$\left(\sqrt{S - \frac{H^2}{n}} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} + \epsilon a < 0. \tag{1.3}$$

Remark 1.1 Theorem 1.2 consists of two conclusions according to the two assumptions $\langle x, x\rangle > 0$ and $\langle x, x\rangle < 0$. In the first case, we have $\epsilon a > 0$. It follows that (1.3) always fails to true when $\lambda = 0$. So we can obtain a Bernstein-type theorem for the usual space-like self-shrinkers which is unfortunately much weaker compared with that by Chen and Qiu in [4]. This, on the other hand, motivates us that Theorem 1.2 can be further improved in general. For example, one may prove with great possibility an improvement of Theorem 1.2 by making applications of the idea and method that are used in [4].

Theorem 1.3 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like λ-hypersurface with weight $s = \langle x, x\rangle$. Suppose that

$$\int_{M^n} \left(\left|\nabla \left(S - \frac{H^2}{n}\right)\right| + \left|\tilde{\mathcal{L}} \left(S - \frac{H^2}{n}\right)\right| \right) e^{-\frac{\langle x,x\rangle^2}{4}} dV_{M^n} < +\infty, \tag{1.4}$$

$$4A^2 - \frac{4HA}{n} + \left(S - \frac{H^2}{n}\right) I \geq 0, \tag{1.5}$$

where the differential operator $\tilde{\mathcal{L}}$ is defined by (2.14). Then, either x is totally umbilical and thus isometric to one of the following two embedded hypersurfaces:

1. the hyperbolic space $\mathbb{H}^n(c) \subset \mathbb{R}^{n+1}_1$ with an arbitrary $c < 0$;
2. the Euclidean space $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$,

or, there exists some $p \in M^n$ such that

$$\left(\sqrt{S(p) - \frac{H^2(p)}{n}} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H(p) - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} < 0. \tag{1.6}$$

Theorems 1.2 and 1.3 will be proved in Section 3; Some necessary lemmas are given in Section 2.

As a direct corollary of Theorem 1.3, we obtain

Theorem 1.4 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like λ-hypersurface with weight $s = \langle x, x\rangle$. Suppose that (1.4) and (1.5) are satisfied. If

$$\left(\sqrt{S - \frac{1}{n}H^2} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} \geq 0, \tag{1.7}$$

then one of the following two conclusions must hold:

1. $\lambda \leq \left(\frac{n}{2}\right)^{\frac{3}{4}}$, and x is isometric to the hyperbolic space $\mathbb{H}^n(-r^{-2}) \subset \mathbb{R}^{n+1}_1$ with $r \geq \left(\frac{n}{2}\right)^{\frac{1}{4}}$;
2. $\lambda = 0$ and x is isometric to the Euclidean space $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$.

Proof If condition (1.7) is satisfied for a hypersurface $\mathbb{H}^n(-r^{-2})$ with $r > 0$, then by the fact that $x = rN$, we have $\lambda = H - \langle x, x\rangle\langle x, N\rangle = \frac{n}{r} - r^3$. If follows that

$$\begin{aligned} 0 \leq & \left(\sqrt{S - \frac{1}{n}H^2} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} \\ = & \frac{1}{n}(H^2 - 2\lambda H) = \frac{1}{r^2}(2r^4 - n). \end{aligned} \tag{1.8}$$

Therefore $r \geq \left(\frac{n}{2}\right)^{\frac{1}{4}}$ which implies directly that $\lambda \leq \left(\frac{n}{2}\right)^{\frac{3}{4}}$. As for the Euclidean space $\mathbb{R}^n$, $\lambda = 0$ is direct by the definition of λ-hypersurfaces.

A similar corollary of Theorem 1.2 can also be derived, which is omitted here.

Corollary 1.5 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like λ-hypersurface with weight $s = \langle x, x\rangle$. Suppose $S - \frac{H^2}{n}$ is constant. If (1.5) and (1.7) are satisfied, then x is isometric to either the hyperbolic space $\mathbb{H}^n(-r^{-2}) \subset \mathbb{R}^{n+1}_1$ with $r \geq \left(\frac{n}{2}\right)^{\frac{1}{4}}$ or the hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$.

Proof Since $S - \frac{H^2}{n}$ is constant, condition (1.4) in Theorem 1.3 is trivially satisfied. Then Corollary 1.5 follows direct from Theorem 1.4.

Remark 1.2 For the special case that $\lambda = 0$, that is, for the case of "self-shrinker" with weight s, the following two conclusions can be easily seen from Theorem 1.4.

Theorem 1.6 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like self-shrinker with weight $s = \langle x, x\rangle$. Suppose that (1.4) and (1.5) are satisfied, then x is isometric to one of the following two embedded hypersurfaces:

1. the hyperbolic space $\mathbb{H}^n\left(-\frac{1}{\sqrt{n}}\right) \subset \mathbb{R}^{n+1}_1$;
2. the Euclidean space $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$.

Proof When $\lambda = 0$, it is clear that (1.7) is trivially satisfied. Furthermore, for a hyperbolic space $\mathbb{H}^n(-r^{-2}) \subset \mathbb{R}^{n+1}_1$, $\lambda = 0$ also implies that $r^2 = \sqrt{n}$.

Corollary 1.7 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like self-shrinker with weight $s = \langle x, x\rangle$. If $S - \frac{H^2}{n}$ is constant and (1.5) is satisfied, then x is isometric to the either the hyperbolic space $\mathbb{H}^n\left(-\frac{1}{\sqrt{n}}\right) \subset \mathbb{R}^{n+1}_1$ or the hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$.

Proof The assumption that $S - \frac{H^2}{n}$ is constant directly means that (1.4) is trivially satisfied.

2 Preliminaries and Lemmas

First, we fix the following convention for the ranges of indices

$$1 \le i, j, k, \cdots \le n, \quad 1 \le A, B, C \cdots \le n+1.$$

Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a connected space-like hypersurface of the $(n+1)$-dimensional Lorentzian space $\mathbb{R}^{n+1}_1$ and $\{e_A\}_{A=1}^{n+1}$ be a local orthonormal frame field of $\mathbb{R}^{n+1}_1$ along x with dual coframe field $\{\omega^A\}_{A=1}^{n+1}$ such that, when restricted to x, $e_1, \ldots, e_n$ are tangent to x and thus $N := e_{n+1}$ is the unit normal vector of x. Then with the connection forms ω_A^B, we have

$$dx = \sum_i \omega^i e_i, \quad de_i = \sum_j \omega_i^j e_j + \omega_i^{n+1} e_{n+1}, \quad de_{n+1} = \sum_i \omega_{n+1}^i e_i.$$

By restricting these forms to M^n and using Cartan's lemma, we have

$$\omega^{n+1} = 0, \quad \omega_i^{n+1} = \omega_{n+1}^i = \sum_{j=1}^n h_{ij}\omega^j, \quad h_{ij} = h_{ji},$$

where h_{ij} are nothing but the components of the second fundamental form h of x, that is, $h = \sum h_{ij}\omega^i\omega^j$. Then the mean curvature H of x is given by $H = \sum\limits_{j=1}^n h_{jj}$. Denote

$$h_{ijk} = (\nabla h)_{ijk} = (\nabla_k h)_{ij}, \quad h_{ijkl} = (\nabla^2 h)_{ijkl} = (\nabla_l(\nabla h))_{ijk}, \tag{2.1}$$

where ∇ is the Levi-Civita connection of the induced metric and $\nabla_i := \nabla_{e_i}$. Then the Gauss equations, Codazzi equations and Ricci identities are given respectively by

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \quad h_{ijk} = h_{ikj}, \tag{2.2}$$

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im}R_{jmkl} + \sum_{m=1}^{n} h_{mj}R_{imkl}, \tag{2.3}$$

where R_{ijkl} are the components of the Riemannian curvature tensor. For a function F defined on M^n, the covariant derivatives of F are denoted by

$$F_{,i} = (\nabla F)_i = \nabla_i F, \quad F_{,ij} = (\nabla^2 F)_{ij} = (\nabla_j(\nabla F))_i, \quad \cdots .$$

Let Δ be the Laplacian operator of the induced metric on M^n. In case that $\langle x, x\rangle$ does not change its sign, we can define

$$\mathcal{L}v = \triangle v - \epsilon a\langle x, \nabla v\rangle, \quad \forall v \in C^2(M^n), \tag{2.4}$$

where, for any constant a, $\epsilon = \mathrm{Sgn}\,(a\langle x, x\rangle)$. Then $\mathcal{L}$ is an elliptic operator and

$$\mathcal{L}v = e^{\frac{\epsilon a\langle x,x\rangle}{2}} \mathrm{div}\left(e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right), \quad \forall v \in C^2(M^n). \tag{2.5}$$

In fact, for $v \in C^2(M^n)$, we find

$$\begin{aligned} & e^{\frac{\epsilon a\langle x,x\rangle}{2}} \mathrm{div}\left(e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right) \\ =& e^{\frac{\epsilon a\langle x,x\rangle}{2}} \left(e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \mathrm{div}\,(\nabla v) + \left\langle \nabla e^{-\frac{\epsilon a\langle x,x\rangle}{2}}, \nabla v\right\rangle\right) \\ =& e^{\frac{\epsilon a\langle x,x\rangle}{2}} \left(e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \triangle v + e^{-\frac{\epsilon a\langle x,x\rangle}{2}} (-\epsilon a\langle x, x_i\rangle)\langle e_i, \nabla v\rangle\right) \\ =& \triangle v - \epsilon a\langle x, \nabla v\rangle = \mathcal{L}v. \end{aligned}$$

Lemma 2.1 [5] Let $x : M^n \to \mathbb{R}_1^{n+1}$ be a complete space-like hypersurface for which $\langle x, x\rangle$ does not change its sign. Then, for any C^1-function u on M^n with compact support, it holds that

$$\int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n} = -\int_{M^n} \langle \nabla v, \nabla u\rangle e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n}, \quad \forall v \in C^2(M^n). \tag{2.6}$$

Proof By (2.5), we find

$$\begin{aligned} \int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n} &= \int_{M^n} u\left(e^{\frac{\epsilon a\langle x,x\rangle}{2}} \mathrm{div}\left(e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right)\right) e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n} \\ &= \int_{M^n} u\,\mathrm{div}\left(e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right) dV_{M^n} \\ &= \int_{M^n} \left(\mathrm{div}\left(ue^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right) - \left\langle \nabla u, e^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right\rangle\right) dV_{M^n} \\ &= \int_{M^n} \mathrm{div}\left(ue^{-\frac{\epsilon a\langle x,x\rangle}{2}} \nabla v\right) dV_{M^n} - \int_{M^n} \langle \nabla u, \nabla v\rangle e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n}. \end{aligned}$$

Hence there are two cases to be considered:

Case (1) M^n is compact without boundary. In this case, we can directly use the divergence theorem to get

$$\int_{M^n} \operatorname{div}\left(ue^{-\frac{\epsilon a\langle x,x\rangle}{2}}\nabla v\right) dV_{M^n} = 0.$$

Case (2) M^n is complete and noncompact. In this case, we can find a geodesic ball $B_r(o)$ big enough such that $\operatorname{Supp} u \subset B_r(o)$. It follows that

$$\begin{aligned}\int_{M^n} \operatorname{div}\left(ue^{-\frac{\epsilon a\langle x,x\rangle}{2}}\nabla v\right) dV_{M^n} &= \int_{B_r(o)} \operatorname{div}\left(ue^{-\frac{\epsilon a\langle x,x\rangle}{2}}\nabla v\right) dV_{B_r(o)}\\ &= -\int_{\partial B_r(o)} \left\langle N, ue^{-\frac{\epsilon a\langle x,x\rangle}{2}}\nabla v\right\rangle dV_{\partial B_r(o)} = 0.\end{aligned}$$

It follows that

$$\int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n} = -\int_{M^n} \langle\nabla v, \nabla u\rangle e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n}.$$

Corollary 2.2 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like hypersurface. If u, v are C^2-functions satisfying

$$\int_{M^n} (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n} < +\infty, \tag{2.7}$$

then we have

$$\int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n} = -\int_{M^n} \langle\nabla u, \nabla v\rangle e^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n}. \tag{2.8}$$

Proof We will use square brackets $[\cdot]$ to denote weighted integrals

$$[f] = \int_{M^n} fe^{-\frac{\epsilon a\langle x,x\rangle}{2}} dV_{M^n}. \tag{2.9}$$

Given any ϕ that is C^1-with compact support, we can apply Lemma 2.1 to ϕu and v to get

$$[\phi u\mathcal{L}v] = -[\phi\langle\nabla v, \nabla u\rangle] - [u\langle\nabla v, \nabla\phi\rangle]. \tag{2.10}$$

Now we fix one point $o \in M$ and, for each $j = 1, 2, \cdots$, let B_j be the intrinsic ball of radius j in M^n centered at o. Define ϕ_j to be one smooth cutting-off function on M^n that cuts off linearly from one to zero between B_j and B_{j+1}. Since $|\phi_j|$ and $|\nabla\phi_j|$ are bounded by one, $\phi_j \to 1$ and $|\nabla\phi_j| \to 0$, as $j \to +\infty$. Then the dominated convergence theorem (which applies because of (2.7)) shows that, as $j \to +\infty$, we have the following limits

$$[\phi_j u\mathcal{L}v] \to [u\mathcal{L}v], \tag{2.11}$$

$$[\phi_j\langle\nabla v, \nabla u\rangle] \to [\langle\nabla v, \nabla u\rangle], \tag{2.12}$$

$$[u\langle\nabla v, \nabla\phi\rangle] \to 0. \tag{2.13}$$

Replacing ϕ in (2.10) with ϕ_j, we obtain the corollary.

Next we consider the case that $s=\langle x,x\rangle$ and define

$$\tilde{\mathcal{L}}v=\triangle v-\langle x,x\rangle\langle x,\nabla v\rangle,\quad \forall v\in C^2(M^n). \tag{2.14}$$

Then similar to (2.5), we have for all $v\in C^2(M^n)$,

$$\begin{aligned}
&e^{\frac{\langle x,x\rangle^2}{4}}\operatorname{div}\left(e^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)\\
=&e^{\frac{\langle x,x\rangle^2}{4}}\left(e^{-\frac{\langle x,x\rangle^2}{4}}\operatorname{div}(\nabla v)+\left\langle\nabla e^{-\frac{\langle x,x\rangle^2}{4}},\nabla v\right\rangle\right)\\
=&e^{\frac{\langle x,x\rangle^2}{4}}\left(e^{-\frac{\langle x,x\rangle^2}{4}}\triangle v+e^{-\frac{\langle x,x\rangle^2}{4}}\left(-\frac{2\langle x,x\rangle}{4}2\langle x,x_i\rangle\right)\langle e_i,\nabla v\rangle\right)\\
=&\triangle v-\langle x,x\rangle\langle x,\nabla v\rangle=\tilde{\mathcal{L}}v.
\end{aligned}\tag{2.15}$$

Lemma 2.3 If $x:M^n\to\mathbb{R}^{n+1}_1$ is a complete space-like hypersurface, u is a C^1-function with compact support, and v is a C^2-function, then

$$\int_{M^n}u(\tilde{\mathcal{L}}v)e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}=-\int_{M^n}\langle\nabla v,\nabla u\rangle e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}. \tag{2.16}$$

Proof Using (2.15), we have

$$\begin{aligned}
\int_{M^n}u(\tilde{\mathcal{L}}v)e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}&=\int_{M^n}u\left(e^{\frac{\langle x,x\rangle^2}{4}}\operatorname{div}\left(e^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)\right)e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}\\
&=\int_{M^n}u\operatorname{div}\left(e^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)dV_{M^n}\\
&=\int_{M^n}\left(\operatorname{div}\left(ue^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)-\left\langle\nabla u,e^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right\rangle\right)dV_{M^n}\\
&=\int_{M^n}\operatorname{div}\left(ue^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)dV_{M^n}-\int_{M^n}\langle\nabla u,\nabla v\rangle e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}.
\end{aligned}$$

(1) If M^n is compact without boundary, then by the divergence theorem,

$$\int_{M^n}\operatorname{div}\left(ue^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)dV_{M^n}=0.$$

(2) If M^n is complete and noncompact, then there exists some geodesic ball $B_r(o)$ big enough such that $\operatorname{Supp}u\subset B_r(o)$. It follows that

$$\begin{aligned}
&\int_{M^n}\operatorname{div}\left(ue^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)dV_{M^n}=\int_{B_r(o)}\operatorname{div}\left(ue^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right)dV_{B_r(o)}\\
=&-\int_{\partial B_r(o)}\left\langle N,ue^{-\frac{\langle x,x\rangle^2}{4}}\nabla v\right\rangle dV_{\partial B_r(o)}=0.
\end{aligned}$$

Therefore

$$\int_{M^n}u(\tilde{\mathcal{L}}v)e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}=-\int_{M^n}\langle\nabla v,\nabla u\rangle e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}.$$

Corollary 2.4 Let $x : M^n \to \mathbb{R}^{n+1}_1$ be a complete space-like hypersurface. If u, v are C^2-functions satisfying

$$\int_{M^n} (|u\nabla v| + |\nabla u||\nabla v| + |u\tilde{\mathcal{L}}v|)e^{-\frac{\langle x,x\rangle^2}{4}} dV_{M^n} < +\infty, \tag{2.17}$$

then we have

$$\int_{M^n} u(\tilde{\mathcal{L}}v)e^{-\frac{\langle x,x\rangle^2}{4}} dV_{M^n} = -\int_{M^n} \langle \nabla u, \nabla v\rangle e^{-\frac{\langle x,x\rangle^2}{4}} dV_{M^n}. \tag{2.18}$$

Proof The proof is the same as that of Corollary 2.2 and is omitted.

The following lemma is also needed in this paper.

Lemma 2.5 [21] Let $\mu_1, \cdots, \mu_n$ be real numbers satisfying

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = \beta^2$$

with β a nonnegative constant. Then

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \le \sum_i \mu_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3$$

with either equality holds if and only if $(n-1)$ of μ_i are equal to each other.

3 Proof of Main Theorems

3.1 Proof of Theorem 1.2

Since $H - \epsilon a\langle x, N\rangle = \lambda$, we have

$$\begin{aligned}
H_{,i} =& (\lambda + \epsilon a\langle x, N\rangle)_{,i} = \epsilon a\langle x, N\rangle_{,i} = \sum_k \epsilon a h_{ik}\langle x, e_k\rangle,\\
H_{ij} =& \sum_k \epsilon a h_{ikj}\langle x, e_k\rangle + \sum_k \epsilon a h_{ik}\langle X_j, e_k\rangle + \sum_k \epsilon a h_{ik}\langle x, e_{k,j}\rangle\\
=& \sum_k \epsilon a h_{ikj}\langle x, e_k\rangle + \epsilon a h_{ij} + \sum_k \epsilon a h_{ik}h_{kj}\langle x, N\rangle\\
=& \sum_k \epsilon a h_{ikj}\langle x, e_k\rangle + \epsilon a h_{ij} + \sum_k h_{ik}h_{kj}(H - \lambda).
\end{aligned}$$

Using the Codazzi equation in (2.2), we infer

$$\triangle H = \sum_i H_{,ii} = \epsilon a\langle x, \nabla H\rangle + \epsilon a H + S(H - \lambda),$$

where $S = \sum_{i,k} h_{ik}^2$. It then follows that

$$\mathcal{L}H = \triangle H - \epsilon a\langle x, \nabla H\rangle = a\epsilon H + S(H - \lambda),$$

implying that

$$\begin{aligned}\frac{1}{2}\mathcal{L}H^2 =& \frac{1}{2}(\triangle H^2 - \epsilon a\langle x, \nabla H^2\rangle) = \frac{1}{2}\left(\sum_i (H^2)_{,ii} - \epsilon a\langle x, \nabla H^2\rangle\right)\\ =& \frac{1}{2}(2|\nabla H|^2 + 2H\triangle H - 2\epsilon a H\langle x, \nabla H\rangle) = |\nabla H|^2 + H(\triangle H - \epsilon a\langle x, \nabla H\rangle)\\ =& |\nabla H|^2 + \epsilon a H^2 + SH(H-\lambda). \end{aligned} \tag{3.1}$$

By making use of the Ricci identities and the Gauss-Codazzi equations, we have

$$\begin{aligned}\mathcal{L}h_{ij} =& \triangle h_{ij} - \epsilon a\langle x, \nabla h_{ij}\rangle = \sum_k h_{ki,jk} - \epsilon a\langle x, \nabla h_{ij}\rangle\\ =& \sum_k h_{ki,kj} + \sum_{m,k} h_{mi}R^m_{kjk} + \sum_{k,m} h_{km}R^m_{ijk} - \epsilon a\langle x, \nabla h_{ij}\rangle\\ =& \sum_k h_{kk,j} + \sum_{m,k} h_{mi}R_{kmjk} + \sum_{k,m} h_{km}R_{imjk} - \epsilon a\langle x, \nabla h_{ij}\rangle\\ =& H_{,ij} - H\sum_m h_{im}h_{mj} + Sh_{ij} - \epsilon a\langle x, \nabla h_{ij}\rangle\\ =& (\epsilon a + S)h_{ij} - \lambda\sum_k h_{ik}h_{kj}.\end{aligned}$$

Therefore it holds that

$$\begin{aligned}\frac{1}{2}\mathcal{L}S =& \frac{1}{2}\left(\triangle\sum_{i,j}(h_{ij})^2 - \sum_k \epsilon a\langle x, e_k\rangle\left(\sum_{i,j}(h_{ij})^2\right)_{,k}\right)\\ =& \sum_{i,j,k} h^2_{ijk} + (\epsilon a + S)S - \lambda\sum_{i,j,k} h_{ik}h_{kj}h_{ij}\\ =& \sum_{i,j,k} h^2_{ijk} + (\epsilon a + S)S - \lambda f_3,\end{aligned} \tag{3.2}$$

where $f_3 = \sum\limits_{i,j,k} h_{ij}h_{jk}h_{ki}$. Let λ_i be the principal curvatures of x and denote

$$\mu_i = \lambda_i - \frac{H}{n}, \quad 1 \le i \le n.$$

For any point $p \in M^n$, suitably choosing $\{e_1, e_2, \cdots, e_n\}$ around p such that $h_{ij}(p) = \lambda_i(p)\delta_{ij}$. Then at the given point p,

$$f_3 = \sum_i \lambda_i^3 = \sum_i\left(\mu_i + \frac{H}{n}\right)^3 = B_3 + \frac{3}{n}HB + \frac{1}{n^2}H^3,$$

where

$$B = \sum_i \mu_i^2 = S - \frac{H^2}{n}, \quad B_3 = \sum_i \mu_i^3.$$

By a direct computation with (3.1) and (3.2), we have

$$\begin{aligned}\frac{1}{2}\mathcal{L}B =&\frac{1}{2}\mathcal{L}S - \frac{1}{n}\left(\frac{1}{2}\mathcal{L}H^2\right)\\ =&\sum_{i,j,k} h_{ijk}^2 + (\epsilon a + S)S - \lambda f_3 - \frac{1}{n}(|\nabla H|^2 + \epsilon a H^2 + SH(H-\lambda))\\ =&\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (\epsilon a + S)S - \lambda f_3 - \frac{1}{n}\epsilon a H^2 - S(H-\lambda)\frac{H}{n}\\ =&\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (\epsilon a + B)B + \frac{H^2 B}{n} - \lambda B_3 - \frac{2}{n}\lambda H B.\end{aligned}$$

Since $\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = B$, we have by Lemma 2.5

$$|B_3| \leq \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}},$$

where the equality holds if and only if at least $n-1$ of μ_is are equal. Consequently,

$$\begin{aligned}\frac{1}{2}\mathcal{L}B \geq& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + (\epsilon a + B)B + \frac{1}{n}H^2 B - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{3}{2}} - \frac{2}{n}\lambda H B\\ =& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + B\left((B+\epsilon a) + \frac{1}{n}H^2 - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{1}{2}} - \frac{2}{n}\lambda H\right)\\ =& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + B\left(\left(\sqrt{B} - |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 + \epsilon a - \frac{n\lambda^2}{4(n-1)}\right).\end{aligned}$$

Because of (1.2), we can apply Corollary 2.2 to functions 1 and $B = S - \frac{H^2}{n}$ to obtain

$$\begin{aligned}0 \geq& \int_{M^n}\left(\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2\right) e^{-\epsilon\frac{a\langle x,x\rangle}{2}} dV_{M^n}\\ &+ \int_{M^n} B\left(\left(\sqrt{S-\frac{H^2}{n}} - |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 - \frac{n\lambda^2}{4(n-1)} + \epsilon a\right) e^{-\epsilon\frac{a\langle x,x\rangle}{2}} dV_{M^n}.\end{aligned} \tag{3.3}$$

On the other hand, by use of the Codazzi equations and the Schwarz inequality, we find

$$\sum_{i,j,k} h_{ijk}^2 = 3\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i\neq j\neq k\neq i} h_{ijk}^2, \quad \frac{1}{n}|\nabla H|^2 \leq \sum_{i,k} h_{iik}^2.$$

So that

$$\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \geq 2\sum_{i\neq k} h_{iik}^2 + \sum_{i\neq j\neq k\neq i} h_{ijk}^2 \geq 0, \tag{3.4}$$

in which the equalities hold if and only if $h_{ijk} = 0$ for any i, j, k.

If $B \not\equiv 0$ and, for all $p \in M^n$, (1.3) does not hold, that is

$$\left(\sqrt{S-\frac{H^2}{n}}-|\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2+\frac{1}{n}(H-\lambda)^2-\frac{n\lambda^2}{4(n-1)}+\epsilon a\geq 0$$

everywhere on M^n, then the right hand side of (3.3) is nonnegative. It then follows that

$$\sum_{i,j,k} h_{ijk}^2-\frac{1}{n}|\nabla H|^2\equiv 0 \tag{3.5}$$

and

$$\left(\sqrt{S-\frac{H^2}{n}}-|\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2+\frac{1}{n}(H-\lambda)^2-\frac{n\lambda^2}{4(n-1)}+\epsilon a\equiv 0 \tag{3.6}$$

on where $B \neq 0$. By (3.4) and (3.5), the second fundamental form h of x is parallel. In particular, x is isoparametric and thus both B and H are constant. Since $B \neq 0$, the equality (3.6) shows that x is a complete isoparametric space-like hypersurface in $\mathbb{R}_1^{n+1}$ of exactly two distinct principal curvatures one of which is simple. It then follows by [17] and $B \neq 0$ that x is isometric to one of the product spaces $\mathbb{H}^{n-1}(c)\times\mathbb{R}^1\subset\mathbb{R}_1^{n+1}$ and $\mathbb{H}^1(c)\times\mathbb{R}^{n-1}\subset\mathbb{R}_1^{n+1}$. But it is clear that, for both of these two product spaces, the function $\langle x,x\rangle$ does change its sign, contradicting the assumption. This contradiction proves that either $B\equiv 0$, namely, x is totally umbilical and isometric to either of the hyperbolic n-space $\mathbb{H}^n(c)\subset\mathbb{R}_1^{n+1}$ and the Euclidean n-space $\mathbb{R}^n\subset\mathbb{R}_1^{n+1}$, or there exists some $p\in M^n$ such that (1.3) holds.

The proof of Theorem 1.2 is thus finished.

3.2 Proof of Theorem 1.3

Since the idea and method here are the same as those in the proof of Theorem 1.2, we omit the computation detail.

First, by $H-\langle x,x\rangle\langle x,N\rangle=\lambda$, we have

$$\begin{aligned}H_{,i}=&2\langle x,e_i\rangle\langle x,N\rangle+\langle x,x\rangle\sum_k h_{ik}\langle x,e_k\rangle,\\ H_{,ij}=&2\delta_{ij}\langle x,N\rangle+2h_{ij}\langle x,N\rangle^2+2\sum_k h_{jk}\langle x,e_i\rangle\langle x,e_k\rangle\\ &+2\sum_k h_{ik}\langle x,e_k\rangle\langle x,e_j\rangle+\sum_k h_{ikj}\langle x,x\rangle\langle x,e_k\rangle\\ &+\langle x,x\rangle h_{ij}+\sum_k h_{ik}h_{kj}(H-\lambda).\end{aligned}$$

Then by using the Codazzi equation in (2.2), we find

$$\begin{aligned}\triangle H=&2n\langle x,N\rangle+2H\langle x,N\rangle^2+4\sum_{i,k}h_{ik}\langle x,e_i\rangle\langle x,e_k\rangle\\ &+\sum_i H_{,i}\langle x,x\rangle\langle x,e_i\rangle+H\langle x,x\rangle+S(H-\lambda).\end{aligned}$$

Second, by the definition of $\tilde{\mathcal{L}}$, we find

$$\begin{aligned}\tilde{\mathcal{L}}H =&\triangle H - \langle x, x\rangle\langle x, \nabla H\rangle\\ =&2n\langle x, N\rangle + 2H\langle x, N\rangle^2 + 4\sum_{i,k} h_{ik}\langle\ X, e_i\rangle\langle x, e_k\rangle + H\langle x, x\rangle + S(H-\lambda),\end{aligned}$$

implying

$$\begin{aligned}\frac{1}{2}\tilde{\mathcal{L}}H^2 =&\frac{1}{2}(\triangle H^2 - \langle x, x\rangle\langle x, \nabla H^2\rangle)\\ =&|\nabla H|^2 + 2nH\langle x, N\rangle + 2H^2\langle x, N\rangle^2\\ &+ 4H\sum_{i,k} h_{ik}\langle x, e_i\rangle\langle x, e_k\rangle + H^2\langle x, x\rangle + SH(H-\lambda).\end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}\tilde{\mathcal{L}}h_{ij} =&\triangle h_{ij} - \langle x, x\rangle\langle x, \nabla h_{ij}\rangle\\ =&H_{,ij} + \sum_{k,m} h_{mi}R_{kmjk} + \sum_{k,m} h_{km}R_{imjk} - \langle x, x\rangle\langle x, \nabla h_{ij}\rangle\\ =&H_{,ij} + Sh_{ij} - H\sum_m h_{mi}h_{mj} - \langle x, x\rangle\langle x, \nabla h_{ij}\rangle\\ =&2\delta_{ij}\langle x, N\rangle + 2h_{ij}\langle x, N\rangle^2 + 2\sum_k h_{jk}\langle x, e_i\rangle\langle x, e_k\rangle\\ &+ 2\sum_k h_{ik}\langle x, e_k\rangle\langle x, e_j\rangle + \sum_k h_{ikj}\langle x, x\rangle\langle x, e_k\rangle + \langle x, x\rangle h_{ij}\\ &+ \sum_k h_{ik}h_{kj}(H-\lambda) + Sh_{ij} - H\sum_m h_{mi}h_{mj} - \langle x, x\rangle\langle x, \nabla h_{ij}\rangle\\ =&2\delta_{ij}\langle x, N\rangle + 2h_{ij}\langle x, N\rangle^2 + 2\sum_k h_{jk}\langle x, e_i\rangle\langle x, e_k\rangle\\ &+ 2\sum_k h_{ik}\langle x, e_k\rangle\langle x, e_j\rangle + \langle x, x\rangle h_{ij} + Sh_{ij} - \lambda\sum_k h_{ik}h_{kj}.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\tilde{\mathcal{L}}S =&\frac{1}{2}\left(\triangle\sum_{i,j}(h_{ij})^2 - \langle x, x\rangle\left\langle x, \nabla\left(\sum_{i,j}(h_{ij})^2\right)\right\rangle\right)\\ =&\sum_k h_{ijk}^2 + 2H\langle x, N\rangle + 2S\langle x, N\rangle^2 + 4\sum_{i,j,k} h_{ij}h_{jk}\langle x, e_i\rangle\langle x, e_k\rangle\\ &+ \langle x, x\rangle S + S^2 - \lambda f_3,\end{aligned} \tag{3.8}$$

where again $f_3 = \sum\limits_{i,j,k} h_{ij}h_{jk}h_{ki}$.

Denote by $x^\top = \langle x, e_i\rangle e_i$ be the tangential part of the position vector x. Then, as in the proof of Theorem 1.2, we can choose a suitable frame field $\{e_1, e_2, \cdots, e_n\}$ making diagonal the second fundamental form h_{ij} around each point $p \in M^n$, and perform a direct computation using (1.5), (3.7) and (3.8) to obtain

$$\begin{aligned}\frac{1}{2}\tilde{\mathcal{L}}B =& \frac{1}{2}\tilde{\mathcal{L}}S - \frac{1}{n}\left(\frac{1}{2}\tilde{\mathcal{L}}H^2\right)\\ =& \sum_{i,j,k} h_{ijk}^2 + B\langle x, N\rangle^2 - \frac{1}{n}|\nabla H|^2 + x^\top\left(4A^2 + BI - \frac{4HA}{n}\right)(x^\top)^t\\ &+ B^2 + \frac{H^2 B}{n} - \lambda B_3 - \frac{2\lambda HB}{n}\\ \geq& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + B^2 + \frac{H^2 B}{n} - \lambda B_3 - \frac{2\lambda HB}{n},\end{aligned}$$

where assumption (1.5) has been used. Once again we use Lemma 2.5 to get

$$|B_3| \leq \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}},$$

where the equality holds if and only if at least $n-1$ of μ_i are equal. It then follows that

$$\begin{aligned}\frac{1}{2}\tilde{\mathcal{L}}B \geq& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + B^2 + \frac{H^2 B}{n} - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{3}{2}} - \frac{2}{n}\lambda HB\\ =& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + B\left(B + \frac{H^2}{n} - |\lambda|\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{1}{2}} - \frac{2}{n}\lambda H\right)\\ =& \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 + B\left(\left(\sqrt{B} - |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 - \frac{n\lambda^2}{4(n-1)}\right).\end{aligned} \tag{3.9}$$

Because of (1.4), we can apply Corollary 2.4 to functions 1 and $B = S - \frac{H^2}{n}$ to find

$$0 \geq \int_{M^n}\left(\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2\right)e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n} \tag{3.10}$$

$$+\int_{M^n} B\left(\left(\sqrt{B} - |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 - \frac{n\lambda^2}{4(n-1)}\right)e^{-\frac{\langle x,x\rangle^2}{4}}dV_{M^n}. \tag{3.11}$$

If $B \not\equiv 0$ and, for all $p \in M^n$, (1.6) does not hold, that is

$$\left(\sqrt{S(p) - \frac{H^2(p)}{n}} - |\lambda|\frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H(p)-\lambda)^2 - \frac{n\lambda^2}{4(n-1)} \geq 0$$

everywhere on M^n, then the right hand side of (3.10) is nonnegative. It then follows that

$$\sum_{i,j,k} h_{ijk}^2 - \frac{1}{n}|\nabla H|^2 \equiv 0, \tag{3.12}$$

and at points where $B \neq 0$,

$$\left(\sqrt{S(p) - \frac{H^2(p)}{n}} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H(p) - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} = 0. \tag{3.13}$$

By (3.12) and (3.13), the second fundamental form h of x is parallel. In particular, x is isoparametric and thus both B and H are constant. Since $B \neq 0$, equality (3.6) shows that x is a complete isoparametric space-like hypersurface in $\mathbb{R}^{n+1}_1$ of exactly two distinct principal curvatures one of which is simple. It then follows by [17] and $B \neq 0$ that x is isometric to one of the product spaces $\mathbb{H}^{n-1}(c) \times \mathbb{R}^1 \subset \mathbb{R}^{n+1}_1$ and $\mathbb{H}^1(c) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}_1$. But it is clear that, for both of these two product spaces, the function $\langle x, x\rangle$ is not a constant so that both $\mathbb{H}^{n-1}(c) \times \mathbb{R}^1 \subset \mathbb{R}^{n+1}_1$ and $\mathbb{H}^1(c) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}_1$ could not be λ-hypersurfaces with $s = \langle x, x\rangle$. This contradiction proves that either $B \equiv 0$, namely, x is totally umbilical and isometric to either of the hyperbolic n-space $\mathbb{H}^n(c) \subset \mathbb{R}^{n+1}_1$ and the Euclidean n-space $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$, or there exists some $p \in M^n$ such that (1.6) holds.

The proof of Theorem 1.3 is thus finished.

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Lorentz 空间$\mathbb{R}_1^{n+1}$ 中类空λ-超曲面的刚性定理

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摘 要: 本文研究了Lorentz 空间$\mathbb{R}_1^{n+1}$中完备的类空λ-超曲面的刚性问题. 利用推广了的$\mathcal{L}$-算子的性质和一些积分不等式, 最终得到了关于这类超曲面的若干刚性定理, 其中包括$\mathbb{R}_1^{n+1}$中加权的完备类空自收缩子的刚性, 推广了此前欧氏空间完备λ-超曲面的相关结果.

关 键 词: Lorentz 空间; 刚性定理; 类空λ-超曲面; 自收缩子

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