

PERIODIC SOLUTIONS AND PERMANENCE FOR A DELAYED PREDATOR-PREY MODEL WITH MODIFIED LESLIE-GOWER AND HOLLING-TYPE III SCHEMES

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Abstract: In this paper, we study the delayed modified Leslie-Gower predator-prey model with Holling-type III schemes. By applying the coincidence degree theorem and the comparison theorem, sufficient conditions for the existence of positive periodic solutions and permanence are obtained, which extend and complement the previously known result. Furthermore, examples show that the obtained criteria are easily verifiable.

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1 Introduction

Leslie [1] introduced the famous Leslie predator-prey system

$$\begin{aligned}\dot{x}(t) &= x(t)[a - bx(t)] - p(x)y(t), \\ \dot{y}(t) &= y(t)[e - f\frac{y(t)}{x(t)}],\end{aligned}$$

where $x(t), y(t)$ stand for the population (the density) of the prey and the predator at time t , respectively, and $p(x)$ is the so-called predator functional response to prey. The term $f\frac{y}{x}$ of the above equation is called Leslie-Gower term, which measures the loss in the predator population due to rarity (per capita y/x) of its favorite food. In case of severe scarcity, y can switch over to other populations but its growth will be limited by the fact that its most favorite food x is not available in abundance. This situation can be taken care of by adding a positive constant k to the denominator, see [2–7] and references cited therein.

It is well known that time delays play important roles in many biological dynamical systems. In general, delay differential equations exhibit much more complicated dynamics

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than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate (see [5–7]). Furthermore, the existence of periodic solutions may be changed. Naturally, more realistic and interesting models of population interactions should take into account both the seasonality of changing environment and the effects of time delay.

In recent years, Leslie-Gower model with Holling-type II was extensively studied by many scholars, many excellent results were obtained concerned with the persistent property and positive periodic solution of the system (see [18–23] and the reference therein). Because Holling-type III can describe the relationship between the predator and prey clearly. So Zhang et al. [7] studied the following system

$$\begin{cases} x'(t) = x(t) \left[r_1 - b_1 x(t - \tau_1) - \frac{a_1 x(t)y(t)}{x^2(t) + k_1} \right], \\ y'(t) = y(t) \left[r_2 - \frac{a_2 y(t - \tau_2)}{x(t - \tau_2) + k_2} \right], \end{cases} \quad (1.1)$$

where $x(t)$ and $y(t)$ represent the densities of the prey and predator population, respectively; $\tau_i \geq 0$; $r_1, b_1, a_1, k_1, r_2, a_2$, and k_2 are positive values. Some sufficient conditions for the local stability of the positive equilibrium and the existence of periodic solutions via Hopf bifurcation with respect to the two delays are obtained; however, Zhang did not give sufficient conditions for the existence of positive periodic solutions and permanence. Moreover, We know that coincidence degree theory is an important method to investigate the positive periodic solutions, and some excellent results were obtained concerned with the existence of periodic solutions of the predator-prey system (see [8–14] and the references therein).

Stimulated by the above reasons, in this paper, we consider the following system:

$$\begin{cases} x'(t) = x(t) \left[r_1(t) - b(t)x(t - \tau_1(t)) - \frac{a_1(t)x(t)y(t - \sigma(t))}{x^2(t) + k_1^2(t)} \right], \\ y'(t) = y(t) \left[r_2(t) - \frac{a_2(t)y(t - \tau_2(t))}{x(t - \tau_2(t)) + k_2(t)} \right], \end{cases} \quad (1.2)$$

where $x(t)$ and $y(t)$ represent the densities of the prey and predator population, respectively; $b(t), a_1(t), a_2(t), k_1(t), k_2(t), \sigma(t), \tau_i(t), i = 1, 2$ are all positive periodic continuous functions with period $\omega > 0$; $r_i(t) \in C(R, R), i = 1, 2$ are ω -periodic continuous functions. In addition, we request that $\int_0^\omega r_i dt > 0, i = 1, 2$, and the growth functions $r_i(t), i = 1, 2$ are not necessarily positive, because the environment fluctuates randomly. Obviously, where k_1^2, k_2 are positive constants, system (1.1) is the special case of (1.2).

To our knowledge, no such work has been done on the existence of positive periodic solutions and permanence of (1.2). Our aim in this paper is, by using the coincidence degree theory developed by Gaines and Mawhin [15], to derive a set of easily verifiable sufficient conditions for the existence of positive solutions. Then by utilizing the comparison, we obtain sufficient conditions for permanence of system (1.2).

2 Preliminaries

Let X, Z be real Banach spaces, $L : \text{Dom} L \subset X \rightarrow Z$ be a linear mapping, and $N : X \rightarrow Z$ be a continuous mapping. The mapping L is said to be a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim Im } L < +\infty$ and $\text{Im } L$ is closed in Z . If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$. It follows that the restriction L_P of L to $\text{Dom} L \cap \text{Ker } P : (I - P)X \rightarrow \text{Im } L$ is invertible. Denote the inverse of L_P by K_P . The mapping N is said to be L -compact on $\bar{\Omega}$ if Ω is an open bounded subset of X , $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exist an isomorphism $J : \text{Im } L \rightarrow \text{Ker } L$.

Lemma 2.1 (Continuation theorem [15]) Let $\Omega \subset X$ be an open bounded set, L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Suppose that

- (i) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom} L$, $Lx \neq \lambda Nx$;
- (ii) for each $x \in \partial\Omega \cap \text{Ker } L$, $QNx \neq 0$;
- (iii) $\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom} L$.

Lemma 2.2 [17] Suppose that $g \in PC_\omega^1 = \{x : x \in C^1(R, R), x(t + \omega) \equiv x(t)\}$, then

$$0 \leq \max_{s \in [0, \omega]} g(s) - \min_{s \in [0, \omega]} g(s) \leq \frac{1}{2} \int_0^\omega |g'(s)| ds.$$

3 Existence of Periodic Solutions

For convenience, we denote

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt, \quad f^L = \min_{t \in [0, \omega]} f(t), \quad f^U = \max_{t \in [0, \omega]} f(t),$$

where $f(t)$ is a continuous ω -periodic function.

Theorem 3.1 Assume $\bar{r}_1 > \left(\frac{a_1}{2k_1}\right)e^{H_2}$ hold, where H_2 is defined in the proof, then system (1.2) has at least one positive ω -periodic solution.

Proof Let $x(t) = e^{x_1(t)}$, $y(t) = e^{x_2(t)}$, then from (1.2), we have

$$\begin{cases} x_1'(t) = r_1(t) - b(t) \exp\{x_1(t - \tau_1(t))\} - \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)}, \\ x_2'(t) = r_2(t) - \frac{a_2(t) \exp\{x_2(t - \tau_2(t))\}}{\exp\{x_1(t - \tau_2(t))\} + k_2(t)}. \end{cases} \quad (3.1)$$

It is easy to see that if system (3.1) has one ω -periodic solution $(x_1^*(t), x_2^*(t))^T$, then $(x^*(t), y^*(t))^T = (e^{x_1^*(t)}, e^{x_2^*(t)})^T$ is a positive ω -periodic solution of (1.2). Therefore, we only need to prove that (3.1) has at least one ω -periodic solution.

Take $X = Z = \{x(t) = (x_1(t), x_2(t))^T \in C(R, R^2) : x(t + \omega) = x(t)\}$ and denote

$$\|x\| = \max_{t \in [0, \omega]} \{|x_1(t)| + |x_2(t)|\},$$

then X and Z are Banach spaces when they are endowed with the norms $\|\cdot\|$.

We define operators L, P and Q as follows, respectively,

$$L : \text{Dom}L \cap X \rightarrow Z, \quad Lx = x'; \quad P(x) = \frac{1}{\omega} \int_0^\omega x(t)dt, \quad x \in X,$$

$$Q(x) = \frac{1}{\omega} \int_0^\omega z(t)dt, \quad z \in Z,$$

where $\text{Dom}L = \{x \in X : x(t) \in C^1(R, R^2)\}$, and define $N : X \rightarrow Z$ by the form

$$Nx = \begin{bmatrix} r_1(t) - b(t) \exp\{x_1(t - \tau_1(t))\} - \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)} \\ r_2(t) - \frac{a_2(t) \exp\{x_2(t - \tau_2(t))\}}{\exp\{x_1(t - \tau_2(t))\} + k_2(t)} \end{bmatrix}.$$

Evidently, $\text{Ker}L = R^2$, $\text{Im}L = \{z | z \in Z, \int_0^\omega z(t)dt = 0\}$ is closed in Z . $\dim \text{Ker}L = \text{codim Im}L = 2$, and P, Q are continuous projectors such that

$$\text{Im}P = \text{Ker}L, \quad \text{Ker}Q = \text{Im}L = \text{Im}(I - Q),$$

thus L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \text{Im}L \rightarrow \text{Ker}P \cap \text{Dom}L$ has the form

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)dsdt.$$

Thus

$$Q Nx = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \left[r_1(t) - b(t) \exp\{x_1(t - \tau_1(t))\} - \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)} \right] dt \\ \frac{1}{\omega} \int_0^\omega \left[r_2(t) - \frac{a_2(t) \exp\{x_2(t - \tau_2(t))\}}{\exp\{x_1(t - \tau_2(t))\} + k_2(t)} \right] dt \end{bmatrix}$$

and

$$\begin{aligned} & K_P(I - Q)Nx \\ &= \begin{bmatrix} \int_0^t \left[r_1(s) - b(s) \exp\{x_1(s - \tau_1(s))\} - \frac{a_1(s) \exp\{x_1(s) + x_2(s - \sigma(s))\}}{\exp\{2x_1(s)\} + k_1^2(s)} \right] ds \\ \int_0^t \left[r_2(s) - \frac{a_2(s) \exp\{x_2(s - \tau_2(s))\}}{\exp\{x_1(s - \tau_2(s))\} + k_2(s)} \right] ds \end{bmatrix} \\ & - \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \int_0^t \left[r_1(s) - b(s) \exp\{x_1(s - \tau_1(s))\} - \frac{a_1(s) \exp\{x_1(s) + x_2(s - \sigma(s))\}}{\exp\{2x_1(s)\} + k_1^2(s)} \right] dsdt \\ \frac{1}{\omega} \int_0^\omega \int_0^t \left[r_2(s) - \frac{a_2(s) \exp\{x_2(s - \tau_2(s))\}}{\exp\{x_1(s - \tau_2(s))\} + k_2(s)} \right] dsdt \end{bmatrix} \\ & - \begin{bmatrix} \left(\frac{t}{\omega} - \frac{1}{2}\right) \int_0^\omega \left[r_1(s) - b(s) \exp\{x_1(s - \tau_1(s))\} - \frac{a_1(s) \exp\{x_1(s) + x_2(s - \sigma(s))\}}{\exp\{2x_1(s)\} + k_1^2(s)} \right] ds \\ \left(\frac{t}{\omega} - \frac{1}{2}\right) \int_0^\omega \left[r_2(s) - \frac{a_2(s) \exp\{x_2(s - \tau_2(s))\}}{\exp\{x_1(s - \tau_2(s))\} + k_2(s)} \right] ds \end{bmatrix}. \end{aligned}$$

Obviously, QN and $K_P(I - Q)N$ are continuous. Moreover, $QN(\overline{\Omega})$, $K_P(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L -compact on $\overline{\Omega}$, here Ω is any open bounded set in X .

Corresponding to the operator equation $Lx = \lambda Nz$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} x'_1(t) &= \lambda \left[r_1(t) - b(t) \exp\{x_1(t - \tau_1(t))\} - \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)} \right], \\ x'_2(t) &= \lambda \left[r_2(t) - \frac{a_2(t) \exp\{x_2(t - \tau_2(t))\}}{\exp\{x_1(t - \tau_2(t))\} + k_2(t)} \right]. \end{aligned} \quad (3.2)$$

Suppose that $x(t) = (x_1(t), x_2(t))^T \in X$ is an ω -periodic solution of system (3.2) for a certain $\lambda \in (0, 1)$. By integrating (3.2) over the interval $[0, \omega]$, we obtain

$$\int_0^\omega \left[r_1(t) - b(t) \exp\{x_1(t - \tau_1(t))\} - \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)} \right] dt = 0, \quad (3.3)$$

$$\int_0^\omega \left[r_2(t) - \frac{a_2(t) \exp\{x_2(t - \tau_2(t))\}}{\exp\{x_1(t - \tau_2(t))\} + k_2(t)} \right] dt = 0. \quad (3.4)$$

From (3.2)–(3.4), we obtain

$$\begin{aligned} \int_0^\omega |x'_1(t)| dt &= \lambda \int_0^\omega \left| r_1(t) - b(t) \exp\{x_1(t - \tau_1(t))\} - \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)} \right| dt \\ &\leq (\overline{|r_1|} + \bar{r}_1) \omega \end{aligned} \quad (3.5)$$

and

$$\int_0^\omega |x'_2(t)| dt \leq \int_0^\omega \left| r_2(t) - \frac{a_2(t) \exp\{x_2(t - \tau_2(t))\}}{\exp\{x_1(t - \tau_2(t))\} + k_2(t)} \right| dt \leq (\overline{|r_2|} + \bar{r}_2) \omega. \quad (3.6)$$

Noting that $x = (x_1(t), x_2(t))^T \in X$. Then there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$x_i(\xi_i) = \sup_{t \in [0, \omega]} x_i(t), \quad x_i(\eta_i) = \inf_{t \in [0, \omega]} x_i(t), \quad i = 1, 2. \quad (3.7)$$

It follows from (3.3) and (3.7) that

$$\omega \bar{r}_1 \geq \int_0^\omega b(t) \exp\{x_1(t - \tau_1(t))\} dt \geq \omega \bar{b} e^{x_1(\eta_1)},$$

which implies that

$$x_1(\eta_1) \leq \ln \frac{\bar{r}_1}{\bar{b}}. \quad (3.8)$$

It follows from (3.5), (3.8) and Lemma 2.2 that, for any $t \in [0, \omega]$,

$$x_1(t) \leq x_1(\eta_1) + \frac{1}{2} \int_0^\omega |x'_1(t)| dt \leq \ln \frac{\bar{r}_1}{\bar{b}} + \frac{1}{2} \omega (\overline{|r_1|} + \bar{r}_1) \triangleq H_1. \quad (3.9)$$

From (3.7), (3.9) and (3.4) that

$$\frac{\omega \bar{a}_2 e^{x_2(\eta_2)}}{e^{H_1} + k_2^U} \leq \int_0^\omega \frac{a_2 e^{x_2(t)}}{e^{x_1(t)} + k_2(t)} dt = \omega \bar{r}_2,$$

i.e.,

$$x_2(\eta_2) \leq \ln \frac{\bar{r}_2(e^{H_1} + k_2^U)}{\bar{a}_2},$$

which together with (3.6) and Lemma 2.2 imply

$$x_2(t) \leq x_2(\eta_2) + \frac{1}{2} \int_0^\omega |x_2'(t)| dt \leq \ln \frac{\bar{r}_2(e^{H_1} + k_2^U)}{\bar{a}_2} + \frac{1}{2} \omega(|r_2| + \bar{r}_2) \triangleq H_2. \quad (3.10)$$

In addition, from (3.3) and (3.7), we get

$$\begin{aligned} \omega \bar{b} e^{x_1(\xi_1)} &\geq \omega \bar{r}_1 - \int_0^\omega \frac{a_1(t) \exp\{x_1(t) + x_2(t - \sigma(t))\}}{\exp\{2x_1(t)\} + k_1^2(t)} dt \\ &\geq \omega \bar{r}_1 - \omega \left(\frac{\bar{a}_1}{2k_1} \right) e^{H_2}, \end{aligned}$$

which implies that

$$x_1(\xi_1) \geq \ln \frac{\bar{r}_1 - \left(\frac{\bar{a}_1}{2k_1} \right) e^{H_2}}{\bar{b}},$$

then together with (3.5) and Lemma 2.2 imply

$$x_1(t) \geq x_1(\xi_1) - \frac{1}{2} \int_0^\omega |x_1'(t)| dt \geq \ln \frac{\bar{r}_1 - \left(\frac{\bar{a}_1}{2k_1} \right) e^{H_2}}{\bar{b}} - \frac{1}{2} \omega(|r_1| + \bar{r}_1) \triangleq H_3. \quad (3.11)$$

From (3.9), (3.7) and (3.4), we have

$$\frac{\omega \bar{a}_2 e^{x_2(\xi_2)}}{e^{H_3} + k_2^L} \geq \int_0^\omega \frac{a_2(t) \exp\{x_2(t)\}}{\exp\{x_1(t)\} + k_2(t)} dt = \omega \bar{r}_2,$$

i.e.,

$$x_2(\xi_2) \geq \ln \frac{\bar{r}_2(\exp\{H_3\} + k_2^L)}{\bar{a}_2},$$

which, together with (3.6) and Lemma 2.2 imply

$$x_2(t) \geq x_2(\xi_2) - \frac{1}{2} \int_0^\omega |x_2'(t)| dt \geq \ln \frac{\bar{r}_2(\exp\{H_3\} + k_2^L)}{\bar{a}_2} - \frac{1}{2} \omega(|r_2| + \bar{r}_2) \triangleq H_4. \quad (3.12)$$

It follows from (3.9)–(3.12) that

$$\|x\| \leq |H_1| + |H_2| + |H_3| + |H_4| \triangleq H_0. \quad (3.13)$$

Obviously, H_0 is independent of λ .

Considering the following algebraic equations

$$\begin{cases} \bar{r}_1 - \bar{b} \exp\{x_1\} - \frac{1}{\omega} \int_0^\omega \frac{a_1(t) \exp\{x_1 + x_2\}}{\exp\{2x_1\} + k_1^2(t)} dt = 0, \\ \bar{r}_2 - \frac{1}{\omega} \int_0^\omega \frac{a_2(t) \exp\{x_2\}}{\exp\{x_1\} + k_2(t)} dt = 0. \end{cases} \quad (3.14)$$

If system (3.14) has a solution or a number of solutions $x^* = (x_1^*, x_2^*)^T$, then similar arguments as those of (3.9)–(3.12) show that

$$\begin{aligned} x_1^* &\leq \ln \frac{\bar{r}_1}{\bar{b}} \leq H_1, & x_2^* &\leq \ln \frac{\bar{r}_2(e^{H_1} + k_2^U)}{\bar{a}_2} \leq H_2, \\ x_1^* &\geq \ln \frac{\bar{r}_1 - \left(\frac{a_1}{2k_1}\right)e^{H_2}}{\bar{b}} \geq H_3, & x_2^* &\geq \ln \frac{\bar{r}_2(e^{H_3} + k_2^L)}{\bar{a}_2} \geq H_4. \end{aligned}$$

Hence

$$\|x^*\| = \|(x_1^*, x_2^*)^T\| = \max\{|x_1^*| + |x_2^*|\} < H_0. \quad (3.15)$$

Set $\Omega = \{x = (x_1, x_2)^T \in X : \|x\| < H_0\}$. Then, $Lx \neq \lambda Nx$ for $x \in \partial\Omega$ and $\lambda \in (0, 1)$, that is Ω satisfies condition (i) in Lemma 2.1.

Suppose $x \in \partial\Omega \cap \text{Ker}L$ with $\|x\| = H_0$. If (3.14) has at least one solution, we obtain from (3.15) that

$$QNx = \begin{bmatrix} \bar{r}_1 - \bar{b} \exp\{x_1\} - \frac{1}{\omega} \int_0^\omega \frac{a_1(t) \exp\{x_1 + x_2\}}{\exp\{2x_1\} + k_1^2(t)} dt \\ \bar{r}_2 - \frac{1}{\omega} \int_0^\omega \frac{a_2(t) \exp\{x_2\}}{\exp\{x_1\} + k_2(t)} dt \end{bmatrix} \neq 0.$$

If system (3.14) does not have a solution, then

$$QNx = \begin{bmatrix} \bar{r}_1 - \bar{b} \exp\{x_1\} - \frac{1}{\omega} \int_0^\omega \frac{a_1(t) \exp\{x_1 + x_2\}}{\exp\{2x_1\} + k_1^2(t)} dt \\ \bar{r}_2 - \frac{1}{\omega} \int_0^\omega \frac{a_2(t) \exp\{x_2\}}{\exp\{x_1\} + k_2(t)} dt \end{bmatrix} \neq 0.$$

Thus condition (ii) in Lemma 2.1 is satisfied.

Finally in order to prove (iii) in Lemma 2.1 we define homomorphism mapping

$$J : \text{Im}Q \rightarrow \text{Ker}L, \quad x \rightarrow x$$

and

$$H : \text{Dom}X \times [0, 1],$$

$$H(x_1, x_2, \mu) = \begin{bmatrix} \bar{r}_1 - \bar{b}e^{x_1} - \left(\frac{a_1}{2k_1}\right)e^{H_2} \\ \bar{r}_2 - \frac{\bar{a}_2 e^{x_2}}{e^{H_3} + k_2^L} \end{bmatrix} + \mu \begin{bmatrix} \left(\frac{a_1}{2k_1}\right)e^{H_2} - \frac{1}{\omega} \int_0^\omega \frac{a_1 e^{x_1 + x_2}}{e^{2x_1} + k_1^2(t)} dt \\ \frac{\bar{a}_2 e^{x_2}}{e^{H_3} + k_2^L} - \frac{1}{\omega} \int_0^\omega \frac{a_2(t) \exp\{x_2\}}{\exp\{x_1\} + k_2(t)} dt \end{bmatrix},$$

where $\mu \in [0, 1]$ is a parameter. We will show that if $x = (x_1, x_2)^T \in \partial\Omega \cap \text{Ker}L$, $x = (x_1, x_2)^T$ is a constant vector in R^2 with $\max\{|x_1|, |x_2|\} = H_0$, then $H(x_1, x_2, \mu) \neq 0$. Otherwise, suppose that $x = (x_1, x_2)^T \in R^2$ with $\max\{|x_1|, |x_2|\} = H_0$ satisfying $H(x_1, x_2, \mu) = 0$, that is,

$$\begin{aligned} \bar{r}_1 - \bar{b}e^{x_1} - \left(\frac{\bar{a}_1}{2k_1}\right)e^{H_2} + \mu\left[\left(\frac{\bar{a}_1}{2k_1}\right)e^{H_2} - \frac{1}{\omega} \int_0^\omega \frac{a_1 e^{x_1+x_2}}{e^{2x_1} + k_1^2(t)} dt\right] &= 0, \\ \bar{r}_2 - \frac{\bar{a}_2 e^{x_2}}{e^{H_3} + k_2^L} + \mu\left[\frac{\bar{a}_2 e^{x_2}}{e^{H_3} + k_2^L} - \frac{1}{\omega} \int_0^\omega \frac{a_2(t) \exp\{x_2\}}{\exp\{x_1\} + k_2(t)} dt\right] &= 0. \end{aligned}$$

Similar argument as those of (3.14), (3.15) show that

$$\|x\| = \max\{|x_1| + |x_2|\} \leq H_0,$$

which is a contradiction.

Hence by a direct calculation, we have

$$\begin{aligned} \deg\{JQN, \Omega \cap \text{Ker}L, 0\} &= \deg\{H(x_1, x_2, 1), \Omega \cap \text{Ker}L, 0\} \\ &= \deg\{H(x_1, x_2, 0), \Omega \cap \text{Ker}L, 0\} \\ &\neq 0. \end{aligned} \tag{3.16}$$

So (iii) in Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that system (1.2) has at least one positive ω -periodic solution. The proof is completed.

Remark 3.1 It is notable that our result only need $b(t)$, $a_1(t)$, $a_2(t)$, $k_1(t)$, $k_2(t)$, $\tau_i(t)$ $i = 1, 2$, $\sigma(t)$ are all positive ω -periodic continuous functions; but $r_i(t) \in C(R, R)$, $i = 1, 2$ are ω -periodic continuous functions, $\int_0^\omega r_i(t)dt > 0$, $i = 1, 2$, and the growth functions $r_i(t)$, $i = 1, 2$ are not necessarily positive. It is reasonable on the biology. In addition, one can easily find that time delays $\tau_i(t)$, $i = 1, 2$, $\sigma(t)$ do not necessarily remain nonnegative. Moreover, Theorem 3.1 will remain valid for systems (1.2) if the delayed terms are replaced by the term with discrete time delays, state-dependent delays, or deviating argument. Hence, time delays of any type or the deviating argument have no effect on the existence of positive solutions.

If the time delayed term $\sigma(t)$ vanishes, $\tau_1(t) \equiv \tau_1$, $\tau_2 \equiv \tau_2$ and $k_1^2(t) \equiv k_1$, $k_2(t) \equiv k_2$, then system (1.2) is reduced to system (1.1) which was studied by Zhang et al. in [7]. Thus from Theorem 3.1, we have the following result.

Corollary 3.1 Assume $\bar{r}_1 > \frac{\bar{a}_1}{2k_1}e^{H_2^*}$ hold, where

$$H_2^* = \ln \frac{\bar{r}_2^*(e^{H_1^*} + k_2)}{\bar{a}_2} + \frac{1}{2}\omega(\overline{|r_2|} + \bar{r}_2), \quad H_1^* = \ln \frac{\bar{r}_1}{\bar{b}} + \frac{1}{2}\omega(\overline{|r_1|} + \bar{r}_1).$$

Then system (1.1) has at least one positive ω -periodic solution.

Remark 3.2 In [7], Zhang et al. suppose $r_i(t)$, $i = 1, 2$ are positive. From Corollary 3.1, it is easy to known that $r_i(t) \in C(R, R)$, $\int_0^\omega r_i(t)dt > 0$, so $r_i(t)$, $i = 1, 2$ are not necessarily positive. We improve the result of [7].

4 Permanence

Definition 4.1 System (1.2) is said to be permanent if there exist positive constants T , M_i , m_i , $i = 1, 2$, such that any solution $(x(t), y(t))^T$ of (1.2) satisfies $m_1 \leq x(t) \leq M_1$, $m_2 \leq y(t) \leq M_2$ for $t \geq T$.

Lemma 4.1 [16] If $a > 0$, $b > 0$, $\tau(t) \geq 0$, then

- (1) if $y'(t) \leq y(t)[b - ay(t - \tau(t))]$, then there exists a constant $T > 0$ such that $y(t) \leq \frac{b}{a} \exp\{b\tau^U\}$ for $t > T$;
- (2) if $y'(t) \geq y(t)[b - ay(t - \tau(t))]$, then there exists a constant T and M such that $y(t) < M$ for $t > T$, then for any small constant $\varepsilon > 0$, there exists a constant $T^* > T$ such that $y(t) \geq \min\{\frac{b}{a} \exp\{(b - aM)\tau^U\}, \frac{b}{a} - \varepsilon\}$ for $t \geq T^*$.

Lemma 4.2 There exists positive constant T_0 such that the solution $(x(t), y(t))$ of (1.2) satisfies

$$0 < x(t) \leq M_1 \quad \text{and} \quad 0 < y(t) \leq M_2 \quad \text{for } t \geq T_0,$$

where

$$M_1 = \frac{r_1^U}{b_1^L} \exp\{r_1^U \tau_1^U\}, \quad M_2 = \frac{(M_1 + k_2^U)r_2^U}{a_2^L} \exp\{r_2^U \tau_2^U\}.$$

Proof It follows from system (1.2) that

$$x'(t) \leq x(t)[r_1^U - b^L x(t - \tau_1(t))].$$

From Lemma 4.1 yield that there exists a positive constant T_1 such that $x(t) \leq M_1$ for $t \geq T_1$. Then we get

$$y'(t) \leq y(t)[r_2^U - \frac{a_2^L y(t - \tau_2(t))}{M_1 + k_2^U}] \quad \text{for } t \geq T_1.$$

So there exists a positive $T_0 \geq T_1$ such that $y(t) \leq M_2$ for $t \geq T_0$.

Lemma 4.3 If $\Delta_1 > 0$ then there exists a positive constant T^* such that the solution $(x(t), y(t))$ of system (1.2) satisfies

$$x(t) \geq m_1 \quad \text{and} \quad y(t) \geq m_2 \quad \text{for } t \geq T^*,$$

where ε is a small enough positive constant and

$$\Delta_1 = [r_1 - \frac{a_1 M_2}{2k_1}]^L, \quad m_1 = \min \left\{ \frac{\Delta_1}{b^U} \exp\{(\Delta_1 - b^U M_1)\tau_1^U\}, \frac{\Delta_1}{b^U} - \varepsilon \right\},$$

$$m_2 = \min \left\{ \frac{r_2^L k_2^L}{a_2^U} \exp\{(r_2^L - \frac{a_2^U M_2}{k_2^L})\tau_2^U\}, \frac{r_2^L k_2^L}{a_2^U} - \varepsilon \right\}.$$

Proof It follows from Lemma 4.2 and system (1.2) that for $t \geq T_0$,

$$\begin{cases} x'(t) \geq x(t)[\Delta_1 - b^U x(t - \tau_1(t))], \\ y'(t) \geq y(t)[r_2^L - \frac{a_2^U y(t - \tau_2(t))}{k_2^L}], \end{cases} \quad (4.1)$$

which, together with Lemma 4.1 and Lemma 4.2, implies that there exists a positive constant $T^* \geq T_0$ such that $x(t) \geq m_1$ and $y(t) \geq m_2$ for $t \geq T^*$.

From Lemma 4.2 and Lemma 4.3, we can get the following result on the permanence of system (1.1).

Theorem 4.1 If $\Delta_1 > 0$, then system (1.2) is permanent.

Similar to the proofs of Lemma 4.2 and Lemma 4.3, we have

Corollary 4.1 If $\Delta_1 > 0$, then system (1.1) is permanent.

Example 1 Consider the following equation

$$\begin{cases} x'(t) = x(t) \left[r_1(t) - b(t)x(t - \tau_1) - \frac{a_1(t)x(t)y(t)}{x^2(t) + k_1^2} \right], \\ y'(t) = y(t) \left[r_2(t) - \frac{a_2(t)y(t - \tau_2)}{x(t - \tau_2) + k_2} \right], \end{cases} \quad (4.2)$$

where $r_1(t) = 3 + 2\sin(12\pi t)$, $b(t) = 1 - 0.1\sin(12\pi t)$, $a_1(t) = 0.5 + 0.1\sin(12\pi t)$, $k_1^2 = 9$, $r_2(t) = 0.8 + 0.2\sin(12\pi t)$, $\tau_1 = \frac{1}{10}$, $\tau_2 = 0$, $a_2(t) = 0.3 - 0.1\sin(12\pi t)$, and $k_2 = 1$. It is easy to calculation, and all the conditions in Theorems 3.1, 3.2 and 4.1 hold. So we know system (4.3) has at least one positive periodic solution and permanent (see Figures 1, 2, we take $x(0) = 1, y(0) = 5$ and $x(0) = 4, y(0) = 5$).

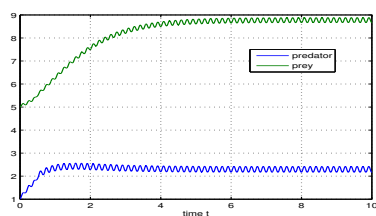


Figure 1

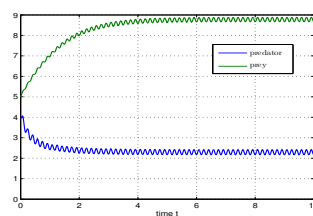


Figure 2

Example 2 If $r_1(t) = 8 + 2\sin(2\pi t)$, $b(t) = 2 - 0.1\sin(2\pi t)$, $a_1(t) = 0.5 + 0.1\sin(2\pi t)$, $k_1^2(t) = 9$, $r_2(t) = 0.8 + 0.2\sin(2\pi t)$, $\tau_1(t) = 1$, $\tau_2(t) = 0.5$, $\sigma(t) = 0$, $a_2(t) = 0.3 - 0.1\sin(2\pi t)$, and $k_2(t) = 1$. It is easy to calculation, and all the conditions in Theorems 3.1, 3.2 and 4.1 hold. So we know system (4.2) has at least one positive periodic solution and permanent (see Figure 3).

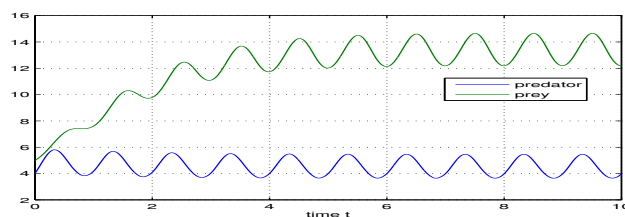


Figure 3

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一类具有修正Leslie-Gower和Holling-type III型的时滞食饵捕食模型的周期解与持久性

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摘要: 本文研究了一类具有修正的Leslie-Gower和Holling-type III型时滞食饵捕食模型. 运用重合度理论和比较定理, 得到系统正周期解和持久性的充分条件. 结论拓展和完善了已有的结论. 最后, 从例子可以看到结论是容易验证的.

关键词: Holling III型反应函数; 时滞; 正周期解; 持久性; 重合度

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