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JULIA SETS AS JORDAN CURVES

ZHUANG Wei
(*Department of Mathematics and Physics, Beijing Institute of Petrochemical Technology, Beijing 102617, China*)

Abstract: In this paper, we consider the Julia sets of NCP maps as Jordan curves. By the way of net and conformal iterated function system, we obtain the general result of the Julia set in which case as the Jordan curves, which generalizes the results of the complex analytic dynamical systems of Julia sets of rational functions as Jordan curves.

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1 Introduction and Main Results

Let $f(z)$ be a rational map of degree $d = \deg f \geq 2$ on the complex sphere $\overline{\mathbb{C}}$. The Julia set $J(f)$ of a rational function f is defined to be the closure of all repelling periodic points of f, and its complement set is called Fatou set $F(f)$. It is known that $J(f)$ is a perfect set (so $J(f)$ is uncountable, and no point of $J(f)$ is isolated), and also that if $J(f)$ is disconnected, then it has infinitely many components.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. We call a compact forward invariant subset $X \subset J(f)$ (i.e. satisfying property $f(X) \subset X$) hyperbolic if there exists $n \geq 1$ such that

$$|(f^n)^{'}(x)| > 1$$

for every $x \in X$ and f^n is topologically conjugate to a subshift of finite type. If only condition $|(f^n)^{'}(x)| > 1$ is satisfied, we call the map $f|_X$ expanding.

We call a rational function $f : J(f) \mapsto J(f)$ hyperbolic if there exists $n \geq 1$ such that

$$\inf\{(f^n)^{'}(z)| : z \in J(f)\} > 1.$$

Denote $CV(f)$ the critical values of a rational function f. Let

$$PCV(f) = \bigcup_{n\geq 1} f^n(CV(f)).$$

It follows from [1, Theorem 2.2] that a rational function $f : J(f) \mapsto J(f)$ is hyperbolic if and only if

$$\overline{PCV(f)} \bigcap J(f) = \emptyset.$$

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Biography: Zhuang Wei (1968–), male, born at Xintai, Shandong, associate professor, major in complex dynamics.

Denote by $J(f)$ the Julia set of a rational function. A rational map f is expansive if the Julia set $J(f)$ contains no critical points of f. It follows from [1] that each hyperbolic rational function is expansive and that a rational function is expansive but not hyperbolic if and only if the Julia set contains no critical points of f but intersect the ω-limit set of critical points.

We call expansive but not hyperbolic rational functions parabolic. It follows from [1] that a rational function $f : J(f) \mapsto J(f)$ is expansive but not hyperbolic if and only if the Julia set $J(f)$ contains no critical points of f but contains at least one parabolic point.

We recall that if $T : X \to X$ is a continuous map of a topological space X, then for every point $x \in X$, the ω-limit set of x denoted by $\omega(x)$ is defined to be the set of all limit points of the sequence $\{T^n(x)\}_{n\geq 0}$. We call a point x recurrent if $x \in \omega(x)$; otherwise x is called non-recurrent.

A rational function $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is called an NCP map if all critical points contained in the Julia set $J(f)$ are non-recurrent.

The class of NCP maps obviously contains all expanding and parabolic maps. It also comprises the important class of so called subexpanding maps which are defined by the requirement that $f|_{\omega(\mathrm{Crit}(f))\cap J(f)}$ is expanding and the class of geometrically finite maps defined by the property that the forward trajectory of each critical point contained in the Julia set is finite and disjoint from ω-limit set.

Let $f(z)$ be a map of degree ≥ 2. A component D of the Fatou set $F(f)$ is said to be completely invariant, if

$$f^{-1}(D) = D = f(D).$$

A Jordan arc γ in $\overline{\mathbb{C}}$ is defined to be the image of the real interval $[0, 1]$ under a homeomorphism φ. If the interval $[0, 1]$ is replaced by the unit circle in the above definition then γ is said to be a Jordan curve.

In this paper, we establish the following main theorem.

Main Theorem Let $f(z)$ be an NCP map of degree ≥ 2, and suppose that $F(f)$ is the union of exactly two completely invariant components. Then $J(f)$ is their common boundary and is a Jordan curve.

2 Preliminaries and the Construction of a Net

Let f be an NCP map. Denote by $\Lambda(f)$ the set of all parabolic periodic points of f (these points belong to the Julia set and have an essential influence on its fractal structure), and $\mathrm{Crit}(f)$ of all critical points of f. We put

$$\mathrm{Crit}(J(f)) = \mathrm{Crit}(f) \cap J(f).$$

Set

$$\mathrm{Sing}(f) = \bigcup_{n\geq 0} f^{-n}(\Lambda(f) \cup \mathrm{Crit}(J(f))).$$

Definition 2.1 We define the conical set $J_c(f)$ of f as follow. First, say x belongs to $J_c(f,r)$ if for any $\epsilon > 0$, there is a neighborhood U of x and $n > 0$ such that $\mathrm{diam}(U) < \varepsilon$ and $f^n : U \to B(f^n(x), r)$ is a homeomorphism. Then set $J_c(f) = \bigcup\limits_{r>0} J_c(f,r)$. We have $x \in J_c(f)$ if and only if arbitrary small neighborhood of x can be blow up univalently by the dynamics to balls of definite size centered at $f^n(x)$.

Lemma 2.1 (see [2]) If $f : J(f) \to J(f)$ is an NCP map, then

$$J_c(f) = J(f) \setminus \mathrm{Sing}(f).$$

Note that Curtis T. McMullen used the term radial Julia set $J_{rad}(f)$ instead of conical set $J_c(f)$ in analogy with Kleinian groups, see ref. [3]. By paper [3], we have the set $\mathrm{Sing}(f)$ is countable.

Let $0 < \lambda < 1$. Then there exist an integer $m \geq 1$, $C > 0$, an open topological disk U containing no critical values of f up to order m and analytic inverse branches $f_i^{-mn} : U \to \overline{\mathbb{C}}$ of $f^{mn} (i = 1, \cdots, k_n \leq d^{nm}, n \geq 0)$, satisfying

(1) $\forall n \geq 0, \forall 1 \leq i \leq k_{n+1}, \exists 1 \leq j \leq k_n, f^m \circ f_i^{-m(n+1)} = f_j^{-mn}$,

(2) $\mathrm{diam}(f_i^{-mn}(U)) \leq c\lambda^n$ for $n = 0, 1, \cdots$ and $i = 1, \cdots, k_n$,

(3) for each fixed $n \geq 1$, for all $i = 1, \cdots, k_n$ the sets $\overline{f_i^{-mn}(U)}$ are pairwise disjoint and $\overline{f_i^{-mn}(U)} \subset U$.

Now we state as a lemma the following consequence of (1)–(3).

Lemma 2.2 For each n, let $\mathcal{N}_n = \bigcup\{f_j^{-n}(U) : j = 1, \cdots, k_n\}$ and let $\mathcal{N} = \bigcup \mathcal{N}_n$. Then $\mathcal{N}$ is a net of $J_c(f)$, i.e., any two sets in $\mathcal{N}$ are either disjoint or one is a subset of the other.

3 Conformal Iterated Function System

In paper [4], Urbanski and Zdunik provided the framework to study infinite conformal iterated function systems. Now we recall this notion and some of its basic properties. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric space X (equipped with a metric ρ) into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x,y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup\limits_{n\geq 1} I^n$, the space of finite words, and for $\tau \in I^*$, $n \geq 1$, let $\phi_\tau = \phi_{\tau_1} \circ \phi_{\tau_2} \circ \cdots \circ \phi_{\tau_n}$. Let $I^\infty = \{\{\tau_n\}_{n=1}^\infty\}$ be the set of all infinite sequences of elements of I. If $\tau \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of τ, we denote by $\tau|_n$ the word $\tau_1\tau_2\cdots\tau_n$. Since given $\tau \in I^\infty$, the diameters of the compact sets $\phi_{\tau|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\tau|_n}(X)$$

is a singleton therefor, denoting its only element by $\pi(\tau)$, defines the coding map

$$\pi : I^{\infty} \to X.$$

The main object in the theory of iterated function systems is the limit set defined as follows.

$$J = \pi(I^{\infty}) = \bigcup_{\tau \in I^{\infty}} \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X) = \bigcap_{n \geq 1} \bigcup_{|\tau|=n} \phi_{\tau}(X).$$

Observe that J satisfied the natural invariance equality, $J = \bigcup\limits_{i \in I} \phi_i(J)$.

Notice (1) If I is finite, then J is compact and this property fails for infinite systems by paper [4].

(2) In Lemma 3.3, we shall build recursively our iterated function system $S_t = \{S_t^1, S_t^2, \cdots, S_t^n\}$, and $n(= I)$ is finite.

Let $X(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X)$, $i \in I^{'}$, where $I^{'}$ ranges over all infinite subsets of I, see ref. [4].

Lemma 3.1 (see [4]) If $\lim\limits_{i \in I} \mathrm{diam}(\phi_i(X)) = 0$, then $\overline{J} = J \cup \bigcup\limits_{\omega \in I^*} \phi_{\omega}(X(\infty))$.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the open set condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$ (we do not exclude $cl\phi_i(U) \cap cl\phi_j(U) \neq \emptyset$).

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to be conformal if $X \subset \mathbb{R}^d$ for some $d \geq 1$ and the following conditions are satisfied.

(a) Open set condition (OSC). $\phi_i(\mathrm{Int}X) \cap \phi_j(\mathrm{Int}X) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.

(b) $\overline{\bigcup\limits_{i \in I} \phi_i(X)} \subset \mathrm{Int}X$.

(c) There exists an open connected set V such that $X \subset V \subset \mathbb{R}^d$ such that all maps ϕ_i, $i \in I$, extend to $\mathcal{C}^1$ conformal diffeomorphisms of V into V (note that for $d = 1$ this just means that all the maps ϕ_i, $i \in I$, are monotone diffeomorphism, for $d = 2$ the words conformal mean holomorphic and antiholomorphic, and for $d = 3$, the maps ϕ_i, $i \in I$ are Möbius transformations).

(d) (Cone condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X$ and there exists an open cone $\mathrm{Con}(x, u, \alpha) \subset \mathrm{Int}(V)$ with vertex x, the symmetry axis determined by vector u of length l and a central angle of Lebesgue measure α, here $\mathrm{Con}(x, u, \alpha) = \{y : 0 < (y - x, u) \leq \cos\alpha ||y - x|| \leq l\}$.

(e) Bounded distortion property (BDP). There exists $K \geq 1$ such that

$$|\phi_{\omega}^{'}(y)| \leq K |\phi_{\omega}^{'}(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi_{\omega}^{'}(x)|$ means the norm of the derivative, see ref. [9, 10].

Definition 3.1 A bounded subset X of a Euclidean space (or Reimann sphere) is said to be porous if there exists a positive constant $c > 0$ such that each open ball B centered at a point of X and of an arbitrary radius $0 < r \leq 1$ contains an open ball of radius cr disjoint from X. If only balls B centered at a fixed point $x \in X$ are discussed above, X is called porous at x, see ref. [5].

Lemma 3.2 (see [5]) The Julia set of each NCP map, if different from $\overline{\mathbb{C}}$, is porous.

Lemma 3.3 If f is an NCP map, then $J_c(f)$ admits a conformal iterated function system satisfying conditions (a)–(e).

Proof Let f be an NCP map. By Lemma 2.2, $J_c(f)$ admits a net such that $B_i \cap B_j = \emptyset$, $i \neq j$. Moreover, we may require the existence of an integer $q \geq 1$ and $\sigma > 0$ such that the following holds:

If $x \in J_c(f)$, say $x \in B_i$, and $f^{qn}(x) \in B_t$, then there exists a unique holomorphic inverse branch $f_x^{-qn} : U(B_t, 2\sigma) \to \overline{\mathbb{C}}$ of f^{qn} sending $f^{qn}(x)$ to x. Moreover $f_x^{-qn}(B_t) \subset B_i$ and taking q sufficiently large, we have

$$f_x^{-qn}(U(B_t, \sigma/2)) \subset \mathrm{Int}(B_i)$$

for sufficiently small σ, then

$$\overline{f_x^{-qn}(B_t)} \subset B_i = \mathrm{Int}(B_i). \tag{3.1}$$

Let $n > 1$ be finite. For every $t = 1, 2, \cdots, n$, we now build recursively our iterated function system S_t as a disjoint union of the families S_t^j, $j \geq 1$, as follows. S_t^1 consists of all the maps f_x^{-q}, where $x, f^q(x) \in J_c(f) \cap B_t$. S_t^2 consists of all the maps f_x^{-2q}, where $x, f^{2q}(x) \in J_c(f) \cap B_t$ and $f^q(x) \notin B_t$. Suppose that the families $S_t^1, S_t^2, \cdots, S_t^{n-1}$ have been already constructed. Then S_t^n is composed of all the maps f_y^{-qn} such that $y, f^{qn}(y) \in J_c(f) \cap B_t$ and $f^{qj}(y) \notin B_t$ for every $1 \leq j \leq n-1$.

Let $V \supset J_c(f)$ be an open set constructed by the net such that it disjoints from the parabolic and critical points and their inverse orbits of f. For any $x \in V$ and finite $n < \infty$, we have

$$0 < |(f^{-n}(x))'| \leq M < \infty,$$

then

$$|(f^{-n}(y))'| \leq K|(f^{-n}(x))'|,$$

where x, $y \in V$ and $1 \leq K < \infty$ is a constant. So condition (e) bounded distortion property (BDP) holds. It is evident that f^n is holomorphic and antiholomorphic of V into V for all $n \geq 1$, then condition (c) holds. Since $J(f)$ is porous, and condition (d) is satisfied. Condition (b) follows immediately from (3.1). In order to prove condition (a), take two distinct maps f_x^{-qm} and f_y^{-qn} belong to S_t. Without loss of generality we may assume that $m \leq n$. Suppose on the contrary that

$$f_x^{-qm}(B_t) \cap f_y^{-qn}(B_t) \neq \emptyset.$$

Then

$$\emptyset \neq f^{qm}(f_x^{-qm}(B_t) \cap f_y^{-qn}(B_t)) \subset B_t \cap f^{qm}(f_y^{-qn}(B_t)) = B_t \cap f_{f^{qm}(y)}^{-q(n-m)}(B_t).$$

Hence $f^{-q(n-m)}_{f^{qm}(y)}(B_t) \subset B_t$, and therefor $f^{qm}(y) \in B_t$. Due to our construction of the system S_t, we have $f^{qn}(y) \in B_t$, and this implies that $m = n$. But then $f_x^{-qn}(B_t) \cap f_y^{-qn}(B_t) = \emptyset$ since f_x^{-qn} and f_y^{-qn} are distinct inverse branches of the same map f^{qn}. This contradiction finishes the proof of Lemma 3.3.

4 Proof of Main Result

Given $x \in \mathbb{C}$, $\theta, r > 0$, we put

$$\mathrm{Con}(x, \theta, r) = \mathrm{Con}(x, \eta, r) \cup \mathrm{Con}(x, -\eta, r),$$

where η is a representative of θ. We recall that a set Y has a tangent in the direction θ at a point $x \in Y$ if for every $r > 0$,

$$\lim_{r \to 0} \frac{\mathcal{H}^1(Y \cap (B(x, r) \setminus \mathrm{Con}(x, \theta, r)))}{r} = 0,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure (see refs. [6, 7]). Following [6], we say that a set Y has a strong tangent in the direction θ at a point x provided for each $0 < \beta \leq 1$, there is a some $r > 0$ such that $Y \cap B(x, r) \subset \mathrm{Con}(x, \theta, \beta)$.

Lemma 4.1 (see [7]) If Y is locally arcwise connected at a point x and Y has a tangent θ at x, then Y has strong tangent θ at x.

We call a point $\tau \in I^\infty$ transitive if $\omega(\tau) = I^\infty$, where $\omega(\tau)$ is the ω-limit set of τ under the shift transformation $\sigma : I^\infty \to I^\infty$. We denote the set of these points by I_t^∞ and put $\Gamma_t = \pi(I_t^\infty)$. We call the Γ_t the set of transitive points of Γ_{S_t} and notice that for every $\tau \in I_t^\infty$, the set $\{\pi(\sigma^n \tau) : n \geq 0\}$ is dense in Γ_{S_t} or $\overline{\Gamma}_{S_t}$.

Lemma 4.2 (see [7]) If $\overline{\Gamma}_{S_t}$ has a strong tangent at a point $x = \pi(\tau)$, $\tau \in I^\infty$, then $\overline{\Gamma}_{S_t}$ has a strong tangent at every point $\overline{\pi(\omega(\tau))}$.

Remark 4.1 If f is an NCP map, by Lemma 3.3, $J_c(f)$ admits a conformal iterated function system S_t. It is obvious that the Julia set $J(f)$ coincides with the limit set $\overline{\Gamma}_{S_t}$ by Lemma 3.1. By Lemma 3.1, 3.3 and 4.2 we have

Lemma 4.3 If f is an NCP map, then $J(f)$ has a strong tangent at every point of $J(f)$.

Proof of Main Theorem Let f be an NCP map and denoted by F_∞ the unbounded component of the Fatou set $F(f)$. As F_∞ is completely invariant, applying Riemann-Hurwitz formula (see §5.4 in [8]) to $f : F_\infty \to F_\infty$, we find that F_∞ has exactly $d - 1$ critical points of f, and all of these lie at ∞. Now take any disk D centered at ∞, which is such that

$$f(\overline{D}) \subset D \subset F_\infty.$$

For each n, let $D_n = f^{-n}(D)$: then D_n is open and connected,

$$D = D_0 \subset D_1 \subset D_2 \subset \cdots ,$$

and as

$$\chi(D_{n+1}) + (d-1) = d\chi(D_n),$$

where $\chi(D_{n+1})$ and $\chi(D_n)$ denote the Euler characteristics of domains D_{n+1} and D_n as above, we see that each D_n is simply connected. Let γ_n be the boundary of D_n; then γ_n is a Jordan curve and f^n is a d^n-fold map of γ_n onto γ_0. Set $\lim\limits_{n\to\infty} \gamma_n = \Gamma$. Roughly speaking, we shall show that γ_n converges to $\partial F_\infty (= J(f))$, i.e. $\Gamma = J(f)$.

If $\xi \in \Gamma$ then there are points ξ_n on γ_n which converge to ξ, so, in particular, ξ is in the closure of F_∞. However, ξ cannot lie in F_∞ else it has a compact neighbourhood K lying in some D_n (for the D_j are an open cover of K), and hence not meeting $\gamma_n, \gamma_{n+1}, \cdots$ for sufficiently large n. We deduce that $\Gamma \subset J(f)$.

$J(f)$ is porous, then $J_c(f)$ admits a conformal iterated function system $S_t = \{f_t^{-i} : t \in s\}$ for finite s satisfying conditions (a)–(e) by Lemma 3.3.

To prove that $J(f) \subset \Gamma$, let $w \in J(f)$ be a repelling fixed point (or an image of a repelling fixed point) and l be the straight line determined by the strongly tangent direction of $J(f)$ at w as in Lemma 4.3. Then w is an attracting fixed point of f^{-1}. Moreover,

$$f^{-1} : U(w) \to U(w)$$

is a conformal map, where $U(w)$ is a disk centered at w. Now suppose that $J(f)$ is not contained in Γ. Consider $x \in J(f) \setminus \Gamma$ such that $x \in U(w)$, then $\lim\limits_{n\to\infty} f^{-n}(x) = w$ and for every $n \geq 0$, we have $f^{-n}(x) \in J(f)$. Since the map $f^{-1} : U(w) \to U(w)$ is conformal, we get

$$\angle(w - f^{-n}(x), l) = \angle(f^{-n}(w) - f^{-n}(x), l) = \angle(f^{-n}(w-x), f^{-n}(l)) = \angle(x - w, l).$$

It follows that w and $f^{-n}(x) (n \geq 0)$ are contained in the same line $\widetilde{l} \neq l$ and this implies that $\widetilde{l}$ is the strongly tangent straight line of $J(f)$ at w. Therefore, we conclude that l is not a strongly tangent straight line of $J(f)$ at w. This contradiction proves that $J(f) \subset \Gamma$.

Remark If Main Theorem only with the hypothesis: the Fatou set $F(f)$ has a completely invariant component, $J(f)$ need not be a Jordan curve; for example, the map $z \mapsto z^2 - 1$ is expanding on its Julia set (certainly NCP map), see Theorem 9.7.5 and Figure 1.5.1 in [8].

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Julia集为Jordan曲线

庄 伟

(北京石油化工学院数理系, 北京 102617)

摘要: 本文研究了NCP映射的Julia集为Jordan曲线的问题. 利用网格和共形迭代函数系统的方法, 获得了Julia集在那种情况下为Jordan曲线的一般结果, 推广了有理函数的Julia集为Jordan曲线的复解析动力系统方面的结果.

关键词: Julia集; 网格; 共形迭代函数系统; Jordan曲线

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