Vol. 38 (2018)
No. 2

数 学 杂 志
J. of Math. (PRC)

A SCHUR'S LEMMA FOR BAKRY-EMERY RICCI CURVATURE ON KÄHLER MANIFOLDS

HUANG Guang-yue

(*Department of Mathematics, Henan Normal University, Xinxiang 453007, China*)

Abstract: This paper is to derive a Schur's lemma for Bakry-Emery Ricci curvature on Kähler manifolds. That is, the equation $R_{i\bar{j}} + f_{i\bar{j}} = \lambda g_{i\bar{j}}$ with two smooth real-valued functions f, λ is studied on Kähler manifolds. By the Bianchi identity, we obtain that λ must be a constant.

Keywords: Kähler manifolds; Schur's lemma; Kähler-Ricci soliton

2010 MR Subject Classification: 53C21; 53C25

Document code: A **Article ID:** 0255-7797(2018)02-0214-03

1 Introduction

The Ricci soliton is a natural generalization of Einstein metrics, which is a self-similar solution to Hamiltion's Ricci flow. In [1], Pigola, Rigoli, Rimoldi and Setti introduced the gradient Ricci almost soliton. That is, if there exist two smooth functions f, λ such that

$$R_{ij} + f_{ij} = \lambda g_{ij}, \tag{1.1}$$

then (M^n, g) is called a gradient Ricci almost soliton, where $R_{ij} + f_{ij}$ is called the ∞-dimensional Bakry-Emery Ricci tensor. Clearly, a gradient Ricci soliton is a special case of the gradient Ricci almost soliton when λ is a constant. In particular, if $\lambda = \rho R + \mu$, where R is the scalar curvature and ρ, μ are two constants, then (1.1) is called the gradient ρ-Einstein soliton defined in [2] which is a self-similar solution to the following geometric flow first considered by Bourguignon in [3]

$$\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} - \rho R g_{ij}). \tag{1.2}$$

For some study with respect to the gradient Ricci almost soliton, the interested reader can refer to [1, 4–7] for more details.

Note that if f given in (1.1) satisfies $f_{ij} = 0$, then (1.1) becomes

$$R_{ij} = \lambda g_{ij} \tag{1.3}$$

* **Received date:** 2016-06-12 **Accepted date:** 2016-09-22

Foundation item: Supported by NSFC (11371018; 11671121); Henan Provincial Key Teacher (2013GGJS-057); IRTSTHN (14IRTSTHN023).

Biography: Huang Guangyue (1976–), male, born at Puyang, Henan, associate professor, major in geometrical analysis.

and the classical Schur's lemma states that the scalar curvature $R = n\lambda$ must be a constant when $n \geq 3$. However, there exist gradient Ricci almost solitons with nontrivial function f such that λ is not a constant. A natural question is to consider whether can one find manifolds satisfying (1.1) with nontrivial function f on which λ is constant. In this paper, we consider this problem on Kähler manifolds and prove the following results.

Theorem 1.1 Let $(M^n, g_{i\bar{j}})$ be an n-dimensional Kähler manifold with $n \geq 2$. If there exist two smooth real-valued functions f, λ satisfying the equation

$$R_{i\bar{j}} + f_{i\bar{j}} = \lambda g_{i\bar{j}}, \tag{1.4}$$

then λ must be a constant.

Therefore, by virtue of Theorem 1.4 of Chen and Zhu in [8], we obtain the following.

Corollary 1.2 Let $(M^n, g_{i\bar{j}})$ be an n-dimensional $(n \geq 2)$ complete Kähler manifold with harmonic Bochner tensor. If there exist two smooth real-valued functions f, λ satisfying (1.4) with $f_{ij} = 0$(that is, ∇f is a holomorphic vector field), then we have

1) if the function $\lambda > 0$, then $(M^n, g_{i\bar{j}})$ is isometric to the quotient of $N^k \times \mathbb{C}^{n-k}$, where N^k is a k-dimensional Kähler-Einstein manifold with positive scalar curvature;

2) if the function $\lambda < 0$, then $(M^n, g_{i\bar{j}})$ is isometric to the quotient of $N^k \times \mathbb{C}^{n-k}$, where N^k is a k-dimensional Kähler-Einstein manifold with negative scalar curvature.

Remark If λ defined in (1.4) is a constant, then it is called a Kähler-Ricci soliton. For the classification of the Kähler-Ricci soliton, we refer to [8, 9].

2 Proof of Results

Using the concepts as in [8], under the Kähler metric $g = (g_{i\bar{j}})$, the Ricci curvature and the scalar curvature defined by

$$R_{i\bar{j}} = R_{i\bar{j}k\bar{k}}, \quad R = R_{i\bar{i}} = R_{i\bar{i}j\bar{j}},$$

respectively. By the first Bianchi identity, we have

$$R_{i\bar{j},j} = R_{i\bar{j}l\bar{l},j} = -(R_{\bar{j}jl\bar{l},i} + R_{jil\bar{l},\bar{j}}) = -R_{\bar{j}jl\bar{l},i} = R_{j\bar{j}l\bar{l},i} = R_{,i}. \tag{2.1}$$

By virtue of (1.4), we obtain

$$R = \mathrm{tr}_g(R_{i\bar{j}}) = n\lambda - \Delta f. \tag{2.2}$$

Therefore, from (2.3), we obtain

$$\begin{aligned} (n\lambda)_{,i} &= R_{,i} + (\Delta f)_{,i} = R_{,i} + f_{j\bar{j},i} = R_{,i} + f_{i\bar{j},j} \\ &= R_{,i} + [\lambda g_{i\bar{j}} - R_{i\bar{j}}]_{,j} = R_{,i} + \lambda_{,i} - R_{i\bar{j},j} = \lambda_{,i}, \end{aligned} \tag{2.3}$$

where in the third equality we used

$$f_{j\bar{j},i} = f_{\bar{j}j,i} = f_{\bar{j}i,j} - f_l R_{\bar{l}\bar{j}j,i} = f_{\bar{j}i,j} = f_{i\bar{j},j} \tag{2.4}$$

and in the last equality we used (2.1). Therefore, from (2.4), it is easy to see that

$$(n-1)\lambda_{,i}=0 \text{ for all } i, \tag{2.5}$$

which shows that λ is a constant.

We complete the proof of Theorem 1.1.

References

[1] Pigola S, Rigoli M, Rimoldi M, Setti A. Ricci almost solitons[J]. Ann. Sc. Norm. Super. Pisa Cl. Sci., 2011, 10(4): 757–799.

[2] Catino G, Mazzieri L. Gradient Einstein solitons[J]. Nonl. Anal., 2016, 132: 66–94.

[3] Bourguignon J P. Ricci curvature and Einstein metrics[M]. Notes Math., Vol. 838, Berlin: Springer, 1981, 42–63.

[4] Barros A, Ribeiro JR E. Some characterizations for compact almost Ricci solitons[J]. Proc. Amer. Math. Soc., 2012, 140(3): 1033–1040.

[5] Barros A, Ribeiro JR E. Characterizations and integral formulae for generalized m-quasi-Einstein metrics[J]. Bull. Braz. Math. Soc. (N.S.), 2014, 45(2): 325–341.

[6] Huang G Y, Wei Y. The classification of (m,ρ)-quasi-Einstein manifolds[J]. Ann. Glob. Anal. Geom., 2013, 44(3): 269–282.

[7] Zeng F, Ma B. The classification of gradient Ricci almost solitons[J]. J. Math., 2014, 34(2): 251–258.

[8] Chen Q, Zhu M. On rigidity of gradient Kähler-Ricci solitons with harmonic Bochner tensor[J]. Proc. Amer. Math. Soc., 2012, 140(11): 4017–4025.

[9] Su Y, Zhang K. On the Kähler-Ricci solitons with vanishing Bochner-Weyl tensor[J]. Acta Math. Sci. Ser. B Engl. Ed., 2012, 32(3): 1239–1244.

Kahler流形上有关Bakry-Emery曲率的Schur引理

黄广月

(河南师范大学数学系, 河南 新乡 453007)

摘要: 本文研究了Kähler流形上有关Bakry-Emery曲率的Schur引理. 即在Kähler流形上考虑方程 $R_{i\bar{j}}+f_{i\bar{j}}=\lambda g_{i\bar{j}}$, 其中$f,\lambda$是光滑实值函数. 利用Bianchi恒等式, 得到了λ是常数.

关键词: Kahler流形; Schur引理; Kähler-Ricci 孤立子

MR(2010)**主题分类号:** 53C21; 53C25 **中图分类号:** O186.12