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A NEW CHARACTERIZATION OF SIMPLE K_3-GROUPS

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Abstract: In this paper, we study the influence of $sn(G)$ on simple K_3-groups. Through the analysis of the subgroups and chief factor of the finite group G, we give a characterization of simple K_3-groups and the results are as follows. If $|G| = p^2qr$ and $sn(G) = \{r, pr, pq\}$, where $p < q < r$ are different primes, then $G \cong A_5$. And the similar conclusions hold for all the other simple K_3-groups.

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1 Introduction

Let G be a finite group and p a prime. The number $n_p(G)$ of Sylow p-subgroups of G is an important invariant pertaining to G and we call it the Sylow p-number of G. By a Sylow number for G, we mean an integer which is a Sylow p-number of G for some prime p. The Sylow number was investigated by many authors such as Hall, Brauer, Hall, Zhang, and Moretó (see for instance [1, 3, 4, 6–9]). Zhang [9] launched a systematic study on the influence of arithmetical properties on the group structure.

We set $sn(G) = \{n_p(G)|p \mid |G|\}$. Zhang [9] posed the following problem, namely what can we see about the finite groups G in terms of $|sn(G)|$? And he made the following claim: it seems true that G is solvable if $|sn(G)| = 2$. The above claim was proved in [7]. Now we consider the influence of $sn(G)$ on simple K_3-groups.

A finite simple group G is called a simple K_3-group if $|G|$ has exactly three distinct prime divisors. We know that $|A_5| = 2^2 \cdot 3 \cdot 5$ and $sn(A_5) = \{5, 2 \cdot 5, 2 \cdot 3\}$. So the following problem is interesting: if $|G| = p^2qr$ and $sn(G) = \{r, pr, pq\}$, where $p < q < r$ are different primes, then $G \cong A_5$ holds? The answer of the problem is yes. In this paper, we get the following results by using an elementary and skillful method of applying Sylow's theorem.

Main Theorem (1) Let $|G| = p^2qr$ and $sn(G) = \{r, pr, pq\}$, where $p < q < r$ are different primes, then $G \cong A_5$.

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(2)Let $|G| = p^3q^2r$ and $sn(G) = \{q^2r, pr, p^2q^2\}$, where $p < q < r$ are different primes, then $G \cong A_6$.

(3) Let $|G| = p^3qr$ and $sn(G) = \{qr, p^2r, p^3\}$, where $p < q < r$ are different primes, then $G \cong L_2(7)$.

(4) Let $|G| = p^3q^2r$ and $sn(G) = \{q^2, p^2r, p^2q^2\}$, where $p < q < r$ are different primes, then $G \cong L_2(8)$.

(5) Let $|G| = p^4q^2r$ and $sn(G) = \{q^2r, p^3r, pq^2\}$, where $p < q < r$ are different primes, then $G \cong L_2(17)$.

(6) Let $|G| = p^4q^3r$ and $sn(G) = \{q^3r, p^2r, p^4q^2\}$, where $p < q < r$ are different primes, then $G \cong L_3(3)$.

(7) Let $|G| = p^5q^3r$ and $sn(G) = \{q^3r, p^2r, p^5q^2\}$, where $p < q < r$ are different primes, then $G \cong U_3(3)$.

(8) Let $|G| = p^6q^4r$ and $sn(G) = \{r, pr, pq\}$, where $p < q < r$ are different primes, then $G \cong U_4(2)$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard (cf. [2] for example).

2 Preliminaries

We need the following two simple lemmas to show our results.

Lemma 2.1 (see [5]) If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.

Lemma 2.2 (see [9]) Let G be a finite group and M a normal subgroup of G, then the product of $n_p(G)$ and $n_p(G/M)$ divides $n_p(G)$.

3 Proof of Main Theorem

Now we will prove the main theorem case by case.

Proof (1) If G is solvable, then G has an elementary abelian minimal normal subgroup N. Note that $sn(G) = \{r, pr, pq\}$, thus $|N| = p$ and $|G/N| = pqr$. And it follows that G/N is supersolvable. Therefore G is supersolvable and $n_r(G) = 1$, which is a contradiction. And so G is unsolvable and $G \cong A_5$ by Lemma 2.1.

(2) Assume that G is solvable, then G has a $\{q, r\}$-Hall subgroup H and $|H| = q^2r$. By Sylow's theorem, we know that $n_r(H) \mid q^2$. If $n_r(H) = q$, then $q \equiv 1 \pmod r$, which is a contradiction since $q < r$. If $n_r(H) = q^2$, then $q^2 \equiv 1 \pmod r$, which implies that $r \mid q+1$. Consequently $q = 2$ and $r = 3$, a contradiction since $p < q$. Thus $n_r(H) = 1$. Note that $|G : N_G(H)| \mid p^3$, thus $n_r(G)$ is at most p^3, which is impossible since $n_r(G) = p^2q^2$. Therefore G is unsolvable.

We obtain that G has a chief factor H/N such that $H/N \cong A_5$, A_6, $L_2(7)$ or $L_2(8)$ by Lemma 2.1, where N is a maximal solvable normal subgroup of G. Set $\overline{H} := H/N \cong A_5$,

$\overline{G} := G/N$, we have

$$A_5 \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \lesssim \mathrm{Aut}(\overline{H}).$$

Let $K := \{x \in G | xN \in C_{\overline{G}}(\overline{H})\}$, then $A_5 \leq G/K \leq \mathrm{Aut}(A_5) \cong S_5$. Hence $G/K \cong A_5$ or $G/K \cong S_5$. If $G/K \cong A_5$, then $|K| = 6$ and so $K = N$. Let $P_5 \in Syl_5(G)$, it follows that $|G/N : N_G(P_5)N/N| = 6$. Note that $|N_G(P_5)| = 10$ and $|N_G(P_5)N/N| = 10$, thus $N \bigcap N_G(P_5) = 1$ and consequently $n_5(N_G(P_5)N) = 6$. Since $N_G(P_5)N$ is of order 60 and not 5-closed, we get that $N_G(P_5)N \cong A_5$, which is contradict to the solvability of $N_G(P_5)N$. If $G/K \cong S_5$, similarly we can get a contradiction. If $H/N \cong L_2(7)$, then $n_3(H/N) \mid n_3(G)$ by Lemma 2.2, namely $28 \mid 14$, a contradiction. In fact 14 is not a Sylow 3-number. Similarly $H/N \ncong L_2(8)$. If $H/N \cong A_6$, then by Lemma 2.1, we have $N = 1$ and $H = G \cong A_6$ since $G = p^3q^2r$ and $|A_6| = 2^3 \cdot 3^2 \cdot 5$.

(3) If G is solvable, then G has a maximal subgroup M such that $M \trianglelefteq G$ and $|G : M|$ is a prime. If $|M| = p^3q$, then $n_q(G) = n_q(M) \mid p^3$, a contradiction since $n_q(G) = p^2r$. By the same reason $|M| \neq p^3r$. Hence $|M| = p^2qr$. Let N be a minimal normal subgroup of M, then $|N| = p$ or p^2. If $|N| = p$, then $|M/N| = pqr$ and so M/N is supersovable. Therefore M is supersolvable, which implies $n_r(G) = n_r(M) = 1$, a contradiction. If $|N| = p^2$, then N Char M and so $N \trianglelefteq G$. Since $|G/N| = pqr$, we obtain that G/N is supersolvable. Let $R \in Syl_r(G)$, then $RN/N \trianglelefteq G/N$. Since $N_{G/N}(RN/N) = N_G(R)N/N = G/N$, we have $G = N_G(R)N$. Note that $|G| = p^3qr$ and $|N| = p^2$, we get that $p \mid |N_G(R)|$, contradict to $n_r(G) = p^3$. Therefore G is unsolvable.

By Lemma 2.1, it follows that G has a chief factor H/N such that $H/N \cong A_5$ or $L_2(7)$. If $H/N \cong A_5$, then $n_r(G) = n_5(G) = p^3 = 8$, which is a contradiction. If $H/N \cong L_2(7)$, then by Lemma 2.1, we have $N = 1$ and $H = G \cong L_2(7)$ since $G = p^3qr$ and $|L_2(7)| = 2^3 \cdot 3 \cdot 7$.

(4) Suppose that G is solvable, then G has a $\{q, r\}$-Hall subgroup H and $|G : N_G(H)| \mid p^3$. It is easy to show that $n_r(H) = 1$ by Sylow's theorem. Therefore $n_r(G)$ is at most p^3, contradict to $n_r(G) = p^2q^2$. So G is unsolvable.

By Lemma 2.1 G has a chief factor H/N such that $H/N \cong A_5$, A_6, $L_2(7)$ or $L_2(8)$. If $H/N \cong A_5$, then $n_2(G) = n_p(G) = q^2 = 9$. By Lemma 2.2, we get that $n_2(H/N) \mid n_2(G)$, namely $5 \mid 9$, which is a contradiction. By the same reason, $H/N \ncong A_6$, $L_2(7)$. If $H/N \cong L_2(8)$, then by Lemma 2.1 we have $N = 1$ and $H = G \cong L_2(8)$ since $G = p^3q^2r$ and $|L_2(8)| = 2^3 \cdot 3^2 \cdot 7$.

(5) Suppose that G is solvable, then G has a maximal subgroup M such that $M \trianglelefteq G$ and $|G : M|$ is a prime. If $|G : M| = q$, then the Sylow p-subgroup P of M is also the the Sylow p-subgroup of G. Since $n_p(G) = q^2r$ we have $N_G(P) = P \leq M$, which implies that $N_G(M) = M$, contradict to $M \trianglelefteq G$. Similarly $|G : M| \neq r$. Consequently $|G : M| = p$ and $|M| = p^3q^2r$. Now we consider the $\{q, r\}$-Hall subgroup N of M. It is easy to show that $n_r(N) = 1$. Therefore $n_r(M)$ is at most p^3, a contradiction since $n_r(M) = n_r(G) = pq^2$. So G is unsolvable.

We get that G has a chief factor H/N such that $H/N \cong A_5$, A_6, $L_2(7)$, $L_2(8)$ or $L_2(17)$

by Lemma 2.1. Similarly to the above, we can show that $G \cong L_2(17)$.

(6) Suppose that G is solvable, then there exist a maximal subgroup M of G such that $M \trianglelefteq G$ and $|G:M|$ is a prime. If $|G:M| = r$, then $n_q(G) = n_q(M) \mid p^4$ by Sylow's theorem, contradict to $n_q(G) = p^2r$. If $|G:M| = q$, then $n_p(G) = n_p(M) =\mid q^2r$, a contradiction since $n_p(G) = q^3r$. If $|G:M| = p$, then $n_r(G) = n_r(M) \mid p^3q^3$, which is impossible since $n_r(G) = p^4q^2$. Therefore G is unsolvable.

We can see from Lemma 2.1 that G has a chief factor H/N such that $H/N \cong A_5$, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$ or $L_3(3)$. Now similarly to the above, we can show that $G \cong L_3(3)$.

(7) Assume that G is solvable, then G has a maximal subgroup M such that $M \trianglelefteq G$ and $|G:M|$ is a prime. If $|G:M| = r$, then $n_q(G) = n_q(M) \mid p^5$ by Sylow's theorem, which is contradict to $n_q(G) = p^2r$. If $|G:M| = q$, then $n_p(G) = n_p(M) \mid q^2r$, a contradiction since $n_p(G) = q^3r$. If $|G:M| = p$, then $n_r(G) = n_r(M) \mid p^4q^3$, which is impossible since $n_r(G) = p^5q^2$. Therefore G is unsolvable.

It is easy to see that G has a chief factor H/N such that $H/N \cong A_5$, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$ or $U_3(3)$ by Lemma 2.1. Now similarly to the above, we can show that $G \cong U_3(3)$.

(8) Suppose that G is solvable, then G has a maximal subgroup M such that $M \trianglelefteq G$ and $|G:M|$ is a prime. If $|G:M| = r$, then $n_p(G) = n_p(M) \mid q^4$ by Sylow's theorem, which is contradict to $n_p(G) = q^3r$. If $|G:M| = q$, then $n_r(G) = n_r(M) \mid p^6q^3$, a contradiction since $n_r(G) = p^4q^4$. It follows that $|G:M| = p$ and $|M| = p^5q^4r$. Now we consider a $\{q,r\}$-Hall subgroup H of M. It is evident that H is also a $\{q,r\}$-Hall subgroup of G. Note that $n_r(G) = p^4q^4$ and $|G:N_G(H)| \mid p^5$, thus $n_r(H) = q^4$ by Sylow's theorem. In fact, if $n_r(H) \leq q^3$, then $n_r(G)$ is at most p^5q^3, a contradiction since $p^5q^3 < n_r(G) = p^4q^4$. Hence $r \mid q^4 - 1 = (q^2+1)(q^2-1)$. If $r \mid q^2 - 1$, then $r \mid q+1$ since $q < r$. Consequently $q = 2$ and $r = 3$, which is contradict to $p < q < r$. Therefore $r \mid q^2 + 1$. Since $n_r(G) = p^4q^4$, we get that $r \mid p^4q^4 - 1$. Hence $r \mid p^4 - 1$ since $r \mid q^4 - 1$. Therefore $r \mid p^2 + 1$. And it follows that $r \mid (q^2+1) - (p^2+1)$, namely $r \mid (q-p)(q+p)$, which implies that $r \mid p+q$. Now we get a contradiction since $p + q < 2r$. Therefore G is unsolvable.

By Lemma 2.1, we know that G has a chief factor H/N such that $H/N \cong A_5$, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$. Now similarly to above, we can show that $G \cong U_4(2)$.

Now the proof of the theorem is complete.

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单K_3-群的新刻画

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摘 要: 本文研究了$sn(G)$对单K_3-群的影响. 通过对有限群G的子群和主因子的分析, 给出了单K_3-群的一个刻画, 结果如下: 如果$|G| = p^2qr$且$sn(G) = \{r, pr, pq\}$, 这里$p < q < r$为不同素数, 那么$G \cong A_5$. 类似的结论对其它单K_3-群都成立.

关键词: Sylow子群; Sylow数; 单K_3-群

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