Vol. 38 (2018)
No. 1

数 学 杂 志
J. of Math. (PRC)

STABILITY OF COMPLETENESS FOR EXPONENTIAL SYSTEMS IN THE WEIGHTED BANACH SPACES

YANG Xiang-dong, WANG Yue

(*Department of Mathematics, Kunming University of Science and Technology, Kunming 650093, China*)

Abstract: In this paper, we study the stability of completeness of exponential systems in weighted Banach spaces. Using the method of perturbation of analytic functions, we obtain several new results on stability. Our results can be regarded as generalization of some existing classical theorems which consider the same problem on Banach spaces over finite intervals.

Keywords: stability; exponential system; completeness; Banach space

2010 MR Subject Classification: 30B60; 42A10

Document code: A **Article ID:** 0255-7797(2018)01-0103-10

1 Introduction

Let $\alpha(t)$ be a nonnegative continuous function defined on $\mathbb{R}$ such that

$$\lim_{|t|\to\infty} \frac{\alpha(t)}{|t|} = \infty. \tag{1.1}$$

The weighted Banach space C_α consists of complex continuous functions f defined on the real axis $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, equipped with the norm

$$\|f\|_\alpha = \sup\{|f(t)\exp(-\alpha(t))| : t \in \mathbb{R}\}.$$

Let $\mathbf{M}(\Lambda)$ denote the set of functions which are finite linear combinations of exponential system $\{e^{\lambda t} : \lambda \in \Lambda\}$ with exponents $\Lambda = \{\lambda_n : n = 1, 2, \cdots\}$ which is a sequence of complex numbers. Condition (1.1) guarantees that $\mathbf{M}(\Lambda)$ is a subspace of C_α.

The completeness of $\mathbf{M}(\Lambda)$ in C_α was studied in many settings. Much was written about the properties of completeness and the closure of $\mathbf{M}(\Lambda)$ (see [1–6] and [11–15], for example). There was an interest in the study of stability of the completeness for the exponential systems in the weighted Banach spaces. We are aware of many results in this direction are on the stability of the completeness for exponential systems in the spaces $C[-a, a]$ and $L^p(-a, a)$ (see [7, 12, 13]). Motivated by the work of B. N. Khabibullin (see [7–10]), our purpose here is to study the stability of the completeness for exponential systems in C_α. Our approach to the problem is different from theirs. Combination of Deng's work on the completeness of

* **Received date:** 2016-06-21 **Accepted date:** 2016-08-31

Foundation item: Supported by National Natural Science Foundation of China (11261024).

Biography: Yang Xiangdong (1976–), male, Bai nationality, born at Lijiang, Yunnan, professor, major in complex analysis.

exponential systems in C_α (see [1]) and Khabibullin's work on subharmonic interpretation of shifts of zeros of entire functions allows us to characterize the stability of the completeness of $\mathbf{M}(\Lambda)$ in C_α.

We now give a brief description of the results obtained. In Section 2, we show that under certain nearness conditions on $\Lambda = \{\lambda_n\}$ and $\Gamma = \{\gamma_n\}$ which are complex sequences with gaps in the right half plane, $\mathbf{M}(\Lambda)$ and $\mathbf{M}(\Gamma)$ are complete or incomplete in C_α simultaneously. In Section 3, we characterize the stability of $\mathbf{M}(\Lambda)$ in C_α wherever $\Lambda = \{\lambda_n, \mu_n\}$ is a positive sequence, wherever each λ_n appearing μ_n times with μ_n not necessarily bounded.

Notations For a complex sequence $\Gamma = \{\gamma_n\}$, the finite linear combination of $\{e^{\gamma t} : \gamma \in \Gamma\}$ is denoted by $\mathbf{M}(\Gamma)$. The symbol $D(a,t)$ is used to denote the disk $|z-a| < t$. The letter A denotes positive constants and it may be different at each occurrence. Throughout this paper, the right half-plane $\{z = x + iy : x > 0\}$ is denoted by $\mathbb{C}_+$.

2 The Exponents with No Multiplicity

In this section, we will consider the stability problem wherever the gap exponents with no multiplicity. More precisely, the exponents is defined as follows. Let $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \cdots\}$ be a sequence of complex numbers satisfying

$$\sup\{|\theta_n| : n = 1, 2, \cdots\} < \frac{\pi}{2} \tag{2.1}$$

and

$$\inf\{|\lambda_{n+1}| - |\lambda_n| : n = 1, 2, \cdots\} > 0. \tag{2.2}$$

The main result of this section is described as follows.

Theorem 2.1 Let $\alpha(t)$ be a nonnegative convex on $\mathbb{R}$ satisfying (1.1). If two sequences of complex numbers $\Lambda = \{\lambda_n\}$ and $\Gamma = \{\gamma_n\}$ satisfy (2.1) and (2.2), furthermore, there exist a decreasing function $\beta : [0, +\infty) \to (0, +\infty)$ and a positive sequence $\{t_n\}$ that is linked with the sequence such that

$$|\lambda_n - \gamma_n| < t_n, n = 1, 2, \cdots \text{ and } \sum_{|\lambda_n| \geq r} t_n < \beta(r), r \geq 0, \tag{2.3}$$

then $\mathbf{M}(\Lambda)$ and $\mathbf{M}(\Gamma)$ are complete or incomplete in C_α simultaneously.

We will make use of the following result from [1].

Lemma 2.1 (see [1]) If $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \cdots\}$ is a complex sequence satisfying (2.1) and (2.2), then the function

$$G(z) = \prod_{n=1}^{\infty} \Big(\frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\overline{\lambda_n}}} \Big) e^{(\frac{z}{\lambda_n} + \frac{z}{\overline{\lambda_n}})} \tag{2.4}$$

is analytic in the right half-plane $\mathbb{C}_+$, and vanishes exactly on the sequence $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \cdots\}$. With $r = |z|$ and $x = \Re z$, for some positive constant A,

$$|G(z)| \leq \exp\{x\lambda(r) + Ax\}, \ z \in \mathbb{C}_+,$$

where $\lambda(r) = 2 \sum\limits_{|\lambda_n| \le r} \frac{\cos\theta_n}{|\lambda_n|}$, if $r \ge |\lambda_1|$; $\lambda(r) = 0$, otherwise.

The crux of the proof of the following proposition is the method applied to shifts of zeros of entire functions. By this approach, we show that for a function analytic in the right half-plane $\mathbb{C}_+$, shifts of its zeros generate another function analytic in the right half-plane whose growth and zeros are very similar to the original one.

Proposition 2.1 Let $G(z)$ be a function analytic in the right half-plane $\mathbb{C}_+$ which is defined in (2.4) and vanishes exactly on the sequence $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \cdots\}$, where Λ satisfies (2.1) and (2.2). Given a decreasing function $\beta : [0, +\infty) \to (0, +\infty)$, we can find a function $G_1(z)$ analytic in the right half-plane $\mathbb{C}_+ = \{z = x+iy : x > 0\}$ with sequence of simple zeros $\Gamma = \{\gamma_n = |\gamma_n|e^{i\vartheta_n} : n = 1, 2, \cdots\}$ satisfies (2.1) and (2.2), furthermore, we can find a positive sequence $\{t_n\}$ that is linked with the sequences such that (2.3) is satisfied and for all $z \in \mathbb{C}_+ \setminus \bigcup\limits_{n=1}^{\infty} D(\lambda_n, t_n)$,

$$|\log|G(z)| - \log|G_1(z)|| \le A\Big(\frac{1}{|z|} + \frac{1}{|z|^2} + x\Big). \tag{2.5}$$

Proof Since the zeros of $G(z)$ have a gap, we can choose a sequence of strict positive numbers $\{t_n\}$ such that the disks $D(\lambda_n, t_n)$ are mutually disjoint and (2.3) is satisfied. We can also select strict positive number d_n such that

$$d_n \le \frac{t_n}{2}, \quad \sum_{n=1}^{\infty} d_n|\lambda_n| < \infty. \tag{2.6}$$

We will estimate the sum of differences

$$\sum_n e_n(z) := \sum_n \Big(\log\Big|\frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\overline{\lambda_n}}}\Big| - \log\Big|\frac{1 - \frac{z}{\gamma_n}}{1 + \frac{z}{\overline{\gamma_n}}}\Big| + \frac{2x\cos\theta_n}{|\lambda_n|} - \frac{2x\cos\vartheta_n}{|\gamma_n|}\Big)$$

for all $z \in \mathbb{C}_+ \setminus \bigcup\limits_{n=1}^{\infty} D(\lambda_n, t_n)$. Firstly, we establish an upper bound for $e_n(z)$:

$$\begin{aligned}
e_n(z) &= \log\Big|\frac{z - \lambda_n}{z + \overline{\lambda_n}}\Big| - \log\Big|\frac{z - \gamma_n}{z + \overline{\gamma_n}}\Big| + \frac{2x\cos\theta_n}{|\lambda_n|} - \frac{2x\cos\vartheta_n}{|\gamma_n|} \\
&= \log\Big|1 + \frac{(\gamma_n - \lambda_n)z + (\overline{\gamma_n} - \overline{\lambda_n})z + (\overline{\lambda_n}\gamma_n - \lambda_n\overline{\gamma_n})}{(z + \overline{\lambda_n})(z - \gamma_n)}\Big| + \frac{2x\cos\theta_n}{|\lambda_n|} - \frac{2x\cos\vartheta_n}{|\gamma_n|} \\
&\le \Big|\frac{(\gamma_n - \lambda_n)z + (\overline{\gamma_n} - \overline{\lambda_n})z + (\overline{\lambda_n}\gamma_n - \lambda_n\overline{\gamma_n})}{(z + \overline{\lambda_n})(z - \gamma_n)}\Big| + \frac{2x|\lambda_n - \gamma_n|}{|\lambda_n||\gamma_n|} \\
&\le \Big|\frac{(\gamma_n - \lambda_n)z + (\overline{\gamma_n} - \overline{\lambda_n})z + (\overline{\lambda_n}\gamma_n - \overline{\lambda_n}\lambda_n + \overline{\lambda_n}\lambda_n - \lambda_n\overline{\gamma_n})}{(z + \overline{\lambda_n})(z - \gamma_n)}\Big| + \frac{2x|\lambda_n - \gamma_n|}{|\lambda_n||\gamma_n|} \\
&\le \frac{(2|z| + 2|\lambda_n|)|\lambda_n - \gamma_n|}{|(z + \overline{\lambda_n})(z - \gamma_n)|} + \frac{2x|\lambda_n - \gamma_n|}{|\lambda_n||\gamma_n|}.
\end{aligned}$$

For $z \in \mathbb{C}_+$ and $\Re\lambda_n > 0$, we have $|z - \lambda_n| \le |z + \overline{\lambda_n}|$. If we choose a positive sequence d_n which satisfies (2.6), then $|z - \gamma_n| \ge |z - \lambda_n| - |\gamma_n - \lambda_n| \ge \frac{|z-\lambda_n|}{2}$, thus

$$e_n(z) \le \frac{4|z|d_n + 4|\lambda_n|d_n}{|z - \lambda_n|^2} + \frac{2xd_n}{|\lambda_n||\gamma_n|}. \tag{2.7}$$

Second, we establish an upper bound for $-e_n(z)$:

$$\begin{aligned}
-e_n(z) &= -\log\left|\frac{z-\lambda_n}{z+\overline{\lambda_n}}\right| + \log\left|\frac{z-\gamma_n}{z+\overline{\gamma_n}}\right| - \frac{2x\cos\theta_n}{|\lambda_n|} + \frac{2x\cos\vartheta_n}{|\gamma_n|} \\
&= \log\left|1+\frac{(\lambda_n-\gamma_n)z+(\overline{\lambda_n}-\overline{\gamma_n})z+(\lambda_n\overline{\gamma_n}-\overline{\lambda_n}\gamma_n)}{(z-\lambda_n)(z+\overline{\gamma_n})}\right| - \frac{2x\cos\theta_n}{|\lambda_n|} + \frac{2x\cos\vartheta_n}{|\gamma_n|} \\
&\leq \left|\frac{(\lambda_n-\gamma_n)z+(\overline{\lambda_n}-\overline{\gamma_n})z+(\lambda_n\overline{\gamma_n}-\overline{\lambda_n}\gamma_n)}{(z-\lambda_n)(z+\overline{\gamma_n})}\right| + \frac{2x|\lambda_n-\gamma_n|}{|\lambda_n||\gamma_n|} \\
&\leq \left|\frac{(\lambda_n-\gamma_n)z+(\overline{\lambda_n}-\overline{\gamma_n})z+(\lambda_n\overline{\gamma_n}-\lambda_n\overline{\lambda_n}+\lambda_n\overline{\lambda_n}-\overline{\lambda_n}\gamma_n)}{(z-\lambda_n)(z+\overline{\gamma_n})}\right| + \frac{2x|\lambda_n-\gamma_n|}{|\lambda_n||\gamma_n|} \\
&\leq \frac{(2|z|+2|\lambda_n|)|\lambda_n-\gamma_n|}{|(z-\lambda_n)(z+\overline{\gamma_n})|} + \frac{2x|\lambda_n-\gamma_n|}{|\lambda_n||\gamma_n|}.
\end{aligned}$$

For $z\in\mathbb{C}_+$ and $\Re\lambda_n>0$, we have $|z-\gamma_n|\leq|z+\overline{\gamma_n}|$. If we choose a positive sequence d_n which satisfies (2.6), then $|z-\gamma_n|\geq\frac{|z-\lambda_n|}{2}$ and we have

$$-e_n(z)\leq\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2}+\frac{2xd_n}{|\lambda_n||\gamma_n|}. \tag{2.8}$$

Combine (2.7) with (2.8), for all $z\in\mathbb{C}_+\backslash\bigcup\limits_{n=1}^{\infty}D(\lambda_n,t_n)$, we have

$$|e_n(z)|\leq\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2}+\frac{2xd_n}{|\lambda_n||\gamma_n|}$$

and

$$\sum_n|e_n(z)|\leq\sum_n\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2}+\sum_n\frac{2xd_n}{|\lambda_n||\gamma_n|}. \tag{2.9}$$

By (2.2), (2.3) and (2.6), we have

$$\sum_{n=1}^{\infty}\frac{2d_n}{|\lambda_n||\gamma_n|}<\infty. \tag{2.10}$$

Fix a point $z\in\mathbb{C}_+\backslash\bigcup\limits_{n=1}^{\infty}D(\lambda_n,t_n)$,

$$\begin{aligned}
\sum_n\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2} &\leq \sum_{|\lambda_n|\geq\frac{|z|}{2}}\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2}+\sum_{|\lambda_n|<\frac{|z|}{2}}\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2} \\
&\leq \sum_{|z-\lambda_n|\geq\frac{|z|}{2}}\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2}+\sum_{|\lambda_n|\geq\frac{|z|}{2}}\frac{4|z|d_n+4|\lambda_n|d_n}{t_n^2}.
\end{aligned}$$

Hence, we can estimate the first sum as

$$\sum_{|z-\lambda_n|\geq\frac{|z|}{2}}\frac{4|z|d_n+4|\lambda_n|d_n}{|z-\lambda_n|^2}\leq\sum_n\frac{2^4|z|d_n+2^4|\lambda_n|d_n}{|z|^2},$$

and the second sum as

$$\sum_{|\lambda_n|\geq\frac{|z|}{2}}\frac{4|z|d_n+4|\lambda_n|d_n}{t_n^2}\leq\sum_n\frac{2^5|\lambda_n|d_n+2^4|\lambda_n|d_n+2^3d_n^2}{|z|^2}.$$

Thus combine with (2.6), (2.9) and (2.10), we eventually get the esitmate

$$\sum_n|e_n(z)|\leq A\Big(\frac{1}{|z|}+\frac{1}{|z|^2}+x\Big),\ z\in\mathbb{C}_+\backslash\bigcup_{n=1}^{\infty}D(\lambda_n,t_n),\tag{2.11}$$

where $A=\max\{\sum\limits_{n=1}^{\infty}\frac{2d_n}{|\lambda_n||\gamma_n|},2^5\sum\limits_{n=1}^{\infty}d_n|\lambda_n|\}$.

To conclude the proof, we use the representation (2.4) of $G(z)$. By Lemma 2.1 and the conditions imposed on Γ, we can define a function $G_1(z)$ in the form of (2.4),

$$G_1(z)=\prod_{n=1}^{\infty}\Big(\frac{1-\frac{z}{\gamma_n}}{1+\frac{z}{\overline{\gamma_n}}}\Big)e^{(\frac{z}{\gamma_n}+\frac{z}{\overline{\gamma_n}})},$$

which is analytic in the right half-plane $\mathbb{C}_+$ and vanishes exactly on the sequence $\Gamma=\{\gamma_n: n=1,2,\cdots\}$. According to (2.11), estimate (2.5) is satisfied for $G(z)$ and $G_1(z)$.

We now get ready to prove Theorem 2.1.

Proof of Theorem 2.1 In order to prove $\mathbf{M}(\Lambda)$ and $\mathbf{M}(\Gamma)$ are complete or incomplete in C_α simultaneously, it suffices to prove that incompleteness of any of the two systems implies incompleteness of the other. To achieve this, we recall the proof in [1].

We assume that $\mathbf{M}(\Lambda)$ is incomplete in C_α. From [1], we know that the incompleteness of $\mathbf{M}(\Lambda)$ in C_α is equivalent to the existence of a non-trivial function $g(z)$ analytic in the right half-plane $\mathbb{C}_+$, which vanishes on some sequence $\Upsilon\supseteq\Lambda$ and is defined by

$$g(z)=\frac{G(z)}{(1+z)^N}\exp\{-g_1(z)-Nz-N\},\tag{2.12}$$

where N is a large positive integer, $G(z)$ is defined by (2.4), and $g_1(z)$ is analytic in the right half-plane, satisfying

$$\Re g_1(z)=\frac{x}{\pi}\int_{-\infty}^{+\infty}\frac{\varphi(t)}{x^2+(y-t)^2}\mathrm{d}t,\tag{2.13}$$

$\varphi(t)$ is an even function such that $\varphi(t)=\alpha(\lambda_\Lambda(t)-a)$ for some $a\in\mathbb{R}$ and all $t\geq 0$, where $\lambda_\Lambda(t)$ is defined in Lemma 2.1.

Suppose (2.3) is satisfied for some complex sequences Γ with (2.1) and (2.2) imposed. By (2.10), we know that we can find a non-trivial analytic in the right half-plane whose real part is defined in (2.13). Replace $G(z)$ with $G_1(z)$ which is defined in Proposition 2.1, we can get a function analytic in the right half-plane, which vanishes on Γ and satisfies (2.12), wherever the function $\lambda_\Gamma(t)$ defined in Lemma 2.1 has the same growth as $\lambda_\Lambda(t)$. This implies the incompleteness of $\mathbf{M}(\Gamma)$ in C_α.

3 The Exponents with Multiplicity

In this section, we will consider the stability problem wherever the exponent is a positive multiplicity sequence $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$, here $\mu_n\to\infty$ is allowed. We need some definitions and auxiliary results from [11].

Definition 3.1 If a real positive sequence $\mathbf{A}=\{a_n\}$ satisfies for some positive constant c the spacing condition $a_{n+1}-a_n\geq c$ for all $n\geq 1$ and $\lim\limits_{n\to\infty}\frac{n}{a_n}=D\geq 0$, then we say it belongs to the class $\mathbf{L(c,\ D)}$.

Definition 3.2 Let the sequence $\mathbf{A}\in\mathbf{L(c,\ D)}$ and α, β be real positive numbers such that $\alpha+\beta<1$. We say that a sequence $\mathbf{B}=\{b_n\}_{n=1}^{\infty}$ with real positive terms b_n, not necessarily in an increasing order, belongs to the class $\mathbf{A}_{\alpha,\beta}$ if for all n $\in\mathbb{N}$, we have

$$b_n\in\{z:|z-a_n|\leq a_n^{\alpha}\},$$

and for all $m\neq n$ one of the following holds

(i) $b_m=b_n$.

(ii) $|b_m-b_n|\geq\max\{e^{-a_m^{\beta}},e^{-a_n^{\beta}}\}$.

We may write $\mathbf{B}$ in the form of a multiplicity sequence $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$, by grouping together all those terms that have the same modulus, and ordering them so that $\lambda_n<\lambda_{n+1}$, this form of $\mathbf{B}$ is called as $\{\lambda,\mu\}$ reordering (see [11]).

Theorem 3.1 Let $\alpha(t)$ be a nonnegative convex on $\mathbb{R}$ satisfying (1.1). Suppose two sequences of positive numbers $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$ and $\Gamma=\{\gamma_n,\mu_n\}_{n=1}^{\infty}$ are $\{\lambda,\mu\}$ reordering of two positive sequence defined in Definition 3.1 and Definition 3.2, furthermore, suppose there exist a decreasing function $\beta:[0,+\infty)\to(0,+\infty)$ and a positive sequence $\{t_n\}$ that is linked with the sequences such that

$$|\lambda_n-\gamma_n|<t_n, n=1,2,\cdots \quad\text{and}\quad \sum_{|\lambda_n|\geq r}\mu_n t_n<\beta(r), r\geq 0, \tag{3.1}$$

then $\mathbf{M}(\Lambda)$ and $\mathbf{M}(\Gamma)$ are complete or incomplete in C_{α} simultaneously.

Before we prove Theorem 3.1, we should establish a proposition which is similar to Proposition 2.1. And we will use the following result from [11].

Lemma 3.1 (see [11]) Let $\mathbf{A}\in\mathbf{L(c,\ D)}$, $\mathbf{B}\in\mathbf{A}_{\alpha,\beta}$ and $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$ be its $\{\lambda,\mu\}$ reordering. Then the function

$$G(z)=\prod_{n=1}^{\infty}\Big(\frac{b_n-z}{b_n+z}\Big)e^{\frac{2z}{b_n}}=\prod_{n=1}^{\infty}\Big(\frac{\lambda_n-z}{\lambda_n+z}\Big)^{\mu_n}e^{\frac{2z\mu_n}{\lambda_n}} \tag{3.2}$$

is analytic in the right half-plane $\mathbb{C}_+$, and vanishes exactly on the sequence $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$. With $r=|z|$ and $x=\Re z$, for some positive constant A, we have

$$|G(z)|\leq\exp\{x\sigma_{\Lambda}(r)+Ax\}, \tag{3.3}$$

where $\sigma_{\Lambda}(r)=2\sum\limits_{\lambda_n\leq r}\frac{\mu_n}{\lambda_n}$. The following proposition is a modified version of Proposition 2.1.

Proposition 3.1 Let $G(z)$ be a function analytic in the right half-plane $\mathbb{C}_+$ which is defined in (3.2) and vanishes exactly on the sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$, which is the $\{\lambda, \mu\}$ reordering of some positive sequence defined in Definition 3.1 and Definition 3.2. Given a decreasing function $\beta : [0, +\infty) \to (0, +\infty)$, we can find a function $G_1(z)$ analytic in the right half-plane $\mathbb{C}_+$ with the sequence of zeros $\Gamma = \{\gamma_n, \mu_n\}_{n=1}^{\infty}$ which is the $\{\lambda, \mu\}$ reordering of a positive sequence satisfying Definition 3.1 and Definition 3.2, furthermore, we can find a positive sequence $\{t_n\}$ that is linked with the sequence such that (3.1) is satisfied and for all $z \in \mathbb{C}_+ \setminus \bigcup_{n=1}^{\infty} D(\lambda_n, t_n)$

$$|\log |G(z)| - \log |G_1(z)|| \leq A\Big(\frac{1}{|z|} + \frac{1}{|z|^2} + x\Big). \tag{3.4}$$

Proof The proof is a modification of the one for Proposition 2.1. By the properties of the zeros of $G(z)$, we can choose a sequence of strict positive numbers $\{t_n\}$ such that the disks $D(\lambda_n, t_n)$ are mutually disjoint and (3.4) is satisfied. We can also select strict positive number d_n such that

$$d_n \leq \frac{t_n}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \mu_n d_n \lambda_n < \infty. \tag{3.5}$$

We will estimate the sum of differences $z \in \mathbb{C}_+ \setminus \bigcup_{n=1}^{\infty} D(\lambda_n, t_n)$,

$$\sum_n \mu_n e_n(z) := \sum_n \mu_n \Big(\log \Big| \frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\lambda_n}} \Big| - \log \Big| \frac{1 - \frac{z}{\gamma_n}}{1 + \frac{z}{\gamma_n}} \Big| + \frac{2x}{\lambda_n} - \frac{2x}{\gamma_n} \Big).$$

An upper bound for $e_n(z)$ is obtained as follows:

$$\begin{aligned} e_n(z) &= \log \Big| \frac{z - \lambda_n}{z + \lambda_n} \Big| - \log \Big| \frac{z - \gamma_n}{z + \gamma_n} \Big| + \frac{2x}{\lambda_n} - \frac{2x}{\gamma_n} \\ &= \log \Big| 1 + \frac{2(\gamma_n - \lambda_n)z}{(z + \lambda_n)(z - \gamma_n)} \Big| + \frac{2x}{\lambda_n} - \frac{2x}{\gamma_n} \\ &\leq \Big| \frac{2(\gamma_n - \lambda_n)z}{(z + \lambda_n)(z - \gamma_n)} \Big| + \frac{2x|\lambda_n - \gamma_n|}{\lambda_n \gamma_n}. \end{aligned}$$

For $z \in \mathbb{C}_+$ and $\lambda_n > 0$, we have $|z + \lambda_n| \geq |z - \lambda_n|$. If we choose a positive sequence $2d_n \leq t_n$, then $|z - \gamma_n| \geq |z - \lambda_n| - |\gamma_n - \lambda_n| \geq \frac{|z - \lambda_n|}{2}$, thus

$$e_n(z) \leq \frac{4|z|d_n}{|z - \lambda_n|^2} + \frac{2x d_n}{\lambda_n \gamma_n}. \tag{3.6}$$

A similar upper bound for $-e_n(z)$ is also obtained in the following estimate

$$\begin{aligned} -e_n(z) &= -\log \Big| \frac{z - \lambda_n}{z + \lambda_n} \Big| + \log \Big| \frac{z - \gamma_n}{z + \gamma_n} \Big| - \frac{2x}{\lambda_n} + \frac{2x}{\gamma_n} \\ &= \log \Big| 1 + \frac{2(\lambda_n - \gamma_n)z}{(z - \lambda_n)(z + \gamma_n)} \Big| - \frac{2x}{\lambda_n} + \frac{2x}{\gamma_n} \\ &\leq \Big| \frac{2(\lambda_n - \gamma_n)z}{(z - \lambda_n)(z + \gamma_n)} \Big| + \frac{2x|\lambda_n - \gamma_n|}{\lambda_n \gamma_n}. \end{aligned}$$

For $z \in \mathbb{C}_+$ and $\gamma_n > 0$, we have $|z+\gamma_n| \geq |z-\gamma_n|$. If we choose a positive sequence $2d_n \leq t_n$, then $|z-\gamma_n| \geq |z-\lambda_n| - |\gamma_n - \lambda_n| \geq \frac{|z-\lambda_n|}{2}$, thus

$$-e_n(z) \leq \frac{4|z|d_n}{|z-\lambda_n|^2} + \frac{2xd_n}{\lambda_n\gamma_n}. \tag{3.7}$$

By (3.6) and (3.7), for all $z \in \mathbb{C}_+ \setminus \bigcup\limits_{n=1}^{\infty} D(\lambda_n, t_n)$,

$$\sum_n \mu_n |e_n(z)| \leq \sum_n \frac{4|z|\mu_n d_n}{|z-\lambda_n|^2} + \sum_n \frac{2x\mu_n d_n}{\lambda_n\gamma_n}. \tag{3.8}$$

By (3.5), we have

$$\sum_{n=1}^{\infty} \frac{2\mu_n d_n}{\lambda_n\gamma_n} < \infty. \tag{3.9}$$

The first term in (3.8) can be estimated as follows

$$\begin{aligned}\sum_n \frac{4|z|\mu_n d_n}{|z-\lambda_n|^2} &\leq \sum_{|\lambda_n| \geq \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{|z-\lambda_n|^2} + \sum_{|\lambda_n| < \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{|z-\lambda_n|^2} \\ &\leq \sum_{|z-\lambda_n| \geq \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{|z-\lambda_n|^2} + \sum_{|\lambda_n| \geq \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{t_n^2}.\end{aligned}$$

By

$$\begin{aligned}\sum_{|z-\lambda_n| \geq \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{|z-\lambda_n|^2} &\leq \sum_{|z-\lambda_n| \geq \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{(\frac{|z|}{2})^2} \leq \sum_n \frac{2^4\mu_n d_n}{|z|}, \\ \sum_{|\lambda_n| \geq \frac{|z|}{2}} \frac{4|z|\mu_n d_n}{t_n^2} &\leq \sum_n \frac{2^5|\lambda_n|\mu_n d_n}{|z|^2},\end{aligned}$$

(3.8) and (3.9), we have

$$\sum_n \mu_n |e_n(z)| \leq A\Big(\frac{1}{|z|} + \frac{1}{|z|^2} + x\Big),\ z \in \mathbb{C}_+ \setminus \bigcup_{n=1}^{\infty} D(\lambda_n, t_n), \tag{3.10}$$

where $A = \max\{\sum\limits_{n=1}^{\infty} \frac{2\mu_n d_n}{\lambda_n\gamma_n}, 2^5 \sum\limits_{n=1}^{\infty} \mu_n d_n \lambda_n\}$.

To conclude the proof, we use representation (3.2) of $G(z)$. By Lemma 3.1 and the conditions imposed on Γ, we can define a function $G_1(z)$ in the form of (3.2)

$$G_1(z) = \prod_{n=1}^{\infty} \Big(\frac{z-\gamma_n}{z+\gamma_n}\Big)^{\mu_n} e^{\frac{2z\mu_n}{\gamma_n}},$$

which is analytic in the right half-plane $\mathbb{C}_+ = \{z = x+iy : x > 0\}$ and vanishes exactly on the sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$. According to (3.10), estimate (3.4) is satisfied for $G(z)$ and $G_1(z)$.

With an application of Proposition 3.1, we can prove Theorem 3.1. The proof of Theorem 3.1 is similar to Theorem 2.1 which is again the proof of the main result in [1].

Proof of Theorem 3.1 To prove $\mathbf{M}(\Lambda)$ and $\mathbf{M}(\Gamma)$ are complete or incomplete in C_α simultaneously, it suffices to prove that incompleteness of any of the two systems implies incompleteness of the other. To achieve this, we just need to repeat the proof in [1] word by word.

We assume that $\mathbf{M}(\Lambda)$ is incomplete in C_α. From [1], we know that the incompleteness of $\mathbf{M}(\Lambda)$ in C_α is equivalent to the existence of a non-trivial function $g(z)$ analytic in the right half-plane $\mathbb{C}_+$, which vanishes on some sequence $\Upsilon \supseteq \Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ and is defined by

$$g(z) = \frac{G(z)}{(1+z)^N} \exp\{-g_1(z) - Nz - N\}, \tag{3.11}$$

where N is a large positive integer, $G(z)$ is defined by (3.2), and $g_1(z)$ is analytic in the right half-plane, satisfying

$$\Re g_1(z) = \frac{x}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(t)}{x^2 + (y-t)^2} \mathrm{d}t, \tag{3.12}$$

$\varphi(t)$ is an even function such that $\varphi(t) = \alpha(\sigma_\Lambda(t) - a)$ for some $a \in \mathbb{R}$ and all $t \geq 0$, where $\sigma_\Lambda(t)$ is defined in Lemma 3.1.

Suppose (3.1) is satisfied for some positive sequence $\Gamma = \{\gamma_n, \mu_n\}_{n=1}^{\infty}$ which is the $\{\lambda, \mu\}$ reordering of a positive sequence defined in Definition 3.1 and Definition 3.2. By (3.8), we know that we can find a non-trivial analytic in the right half-plane whose real part is defined in (3.12). Replace $G(z)$ with $G_1(z)$ which is defined in Proposition 3.1, we can get a function analytic in the right half-plane, which vanishes on $\Gamma = \{\gamma_n, \mu_n\}_{n=1}^{\infty}$ and satisfies (3.11), wherever the function $\sigma_\Gamma(t)$ defined in Lemma 3.1 has the same growth as $\lambda_\Lambda(t)$. This implies the incompleteness of $\mathbf{M}(\Gamma)$ in C_α.

References

[1] Deng G T. Incompleteness and closure of a linear span of exponential system in a weighted Banach space[J]. J. Approx. The., 2003, 125(1): 1–9.

[2] Deng G T. Weighted exponential polynomial approximation[J]. Sci. China, 2003, 46(2): 280–287.

[3] Deng G T. On weighted polynomial approximation with gaps[J]. Nagoya Math. J., 2005, 178: 55–61.

[4] Deng G T. Incompleteness and minimality of complex exponential system[J]. Sci. China, 2007, 50(10): 1–10.

[5] Khabibullin B N. Sets of uniqueness in spaces of entire functions of a single variable[J]. Math. USSR Izvestiya, 1992, 39(2): 1063–1084.

[6] Khabibullin B N. Completeness of sets of complex exponentials in convex sets: open problems, proceeding of the NATO advanced study institute[J]. Math. Phys. Chem., 2001: 371–373.

[7] Khabibullin B N. Stability of completeness for exponential systems on compact convex sets in $\mathbb{C}$[J]. Math. Notes, 2002, 72(3): 542–550.

[8] Khabibullin B N. An approximation theorem for entire functions of exponential type and the stability of zero sequences[J]. Mat. Sb., 2004, 195(1): 143–156.

[9] Khabibullin B N. Spectral synthesis for the intersection of invariant subspaces of holomorphic functions[J]. Mat. Sb., 2005, 196(3): 119–142.

[10] Kudasheva E G, Khabibullin B N. Variation of subharmonic function under transformation its Rieze measure[J]. www. math. bsunet. ru/khb-e, 2007.

[11] Zikkos E. Completeness of an exponential system in weighted Banach spaces and closure of its linear span[J]. J. Approx. The., 2007, 146(1): 115–148.

[12] Sedletskii A M. Nonharmonic analysis[J]. J. Math. Sci., 2003, 116(5): 3551–3619.

[13] Sedletskii A M. Fourier transforms and approximations[M]. Russia: Gordon and Breach Science Publishers, 2000.

[14] Yang X D. Incompleteness of exponential system in the weighted Banach space[J]. J. Approx. The., 2008, 153(1): 73–79.

[15] Zhang Y H, Deng G T. Growth properties of generalized Poission integral in the half space[J]. J. Math., 2013, 33(3): 473–478.

加权Banach空间中指数函数系完备的稳定性

杨向东, 王 越

(昆明理工大学数学系, 云南 昆明 650093)

摘要: 本文研究了加权Banach空间中指数函数系完备的稳定性问题. 利用解析函数扰动的方法, 获得了加权Banach空间中指数函数系完备的稳定性若干结果, 推广了现有有限区间Banach空间上的经典结果.

关键词: 稳定性; 指数函数系; 完备性; Banach 空间

MR(2010)**主题分类号**: 30B60; 42A10 **中图分类号**: O174.5