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SOME PROPERTIES IN THE GENERALIZED MORREY SPACES ON HOMOGENOUS CARNOT GROUPS

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Abstract: In this paper, the behaviors for the Riesz potential or fractional maximal operator in the generalized Morrey spaces on the Heisenberg group and the Lebesgue spaces on the Carnot group are studied. According to the methods of abstract harmonic analysis in Heisenberg group and the representation formula of solution of Dirichlet problem for subLaplacian, we mainly give some characterizations for the boundedness of the weighted Hardy operator, fractional maximal operator and fractional potential operator in the vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ on homogenous Carnot group $\mathbb{G}$. Moreover, we also obtain the embedding inequality for Morrey potentials in such these spaces without vanishing norm. All these results above generalize the related ones in the generalized Morrey spaces on the Heisenberg group and the Lebesgue spaces on the Carnot group.

Keywords: Carnot group; weighted Hardy operator; fractional maximal operator; potential operator; generalized Morrey space

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1 Introduction

In the paper, we are mainly concerned with some properties in the generalized Morrey spaces on homogenous Carnot group. As is now well known to all, Morrey space is the classical generalization for Lebesgue space in function space theories. Since the classical Morrey spaces were introduced by Morrey in [26] (or refer to [40]), there were many variants and a great deal of progress in the aspect. The classical Morrey spaces together with the weighted Lebesgue spaces, were applied to deal with the local regularity properties of solutions of partial differential equations (refer to [22]). In the local Morrey (or Morrey type) spaces and the global Morrey (or Morrey type) spaces the boundednesses of various classical operators were largely considered, for example, maximal, potential, singular, Hardy operators and commutators and others, here we may refer to Adams [1], Akbulut et al. [2],

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Adams and Xiao [3–6], Burenkov et al. [9, 11], Guliyev et al. [12, 15], Chiarenza and Frasca [13], Kurata et al. [23], Komori and Shirai [24], Lukkassen et al. [25], Nakai et al. [27, 28], Persson et al. [30], Softova [35], Sugano and Tanaka [36] and references therein. In the classical harmonic analysis, the vanishing Morrey space was firstly introduced by Vitanza [38] to discuss the regularity results for elliptic partial differential equations, and later Ragusa [31] and Samko et al. (see [32, 34] and references therein) together systematically studied the boundedness of various classical operators in such these type of spaces. For the characterizations for classical operator in the abstract harmonic analysis, we may refer to some books by Folland and Stein [14], Varopoulos et al. [39] and Thangavelu [37]. Guliyev et al. (see [17, 18]) studied Riesz potential and fractional maximal operator in the generalized Morrey spaces on the Heisenberg group. As for the properties of Lebesgue space on Carnot group in abstract potential theory, we may refer to Bonfiglioli et al. (see [7, 8]), Gafofalo and Rotz [19] and Han Yazhou et al. [21]. In fact, we know little about the properties of the generalized Morrey space on Carnot group (see only [16] and [29]). Stimulated by the above statements, we continue to study the boundedness of some operators from Samko (see [32–34]) in the generalized Morrey spaces on Carnot group and simultaneously develop the results from Bonfiglioli et al. (see [8]) on Carnot group. To be exact, our aim is to character the boundedness of the weighted Hardy operator, fractional maximal operator and fractional potential operator in the vanishing generalized Morrey spaces on Carnot group, and simultaneously consider the Morrey-Sobolev type embedding theorems in the generalized Morrey spaces on Carnot group. To establish our results on Carnot group, at first we will recall some notations, classical operators and basic properties on Carnot group below.

A Carnot group is a simply connected nilpotent Lie group $\mathbb{G} \equiv (\mathbb{R}^N, \circ)$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is to say, there exist linear subspaces $V_1, \cdots, V_k$ of $\mathcal{G}$ so that the direct sum vector space decomposition below

$$\mathcal{G} = V_1 \bigoplus \cdots \bigoplus V_k,\ [V_1, V_i] = V_{i+1} \text{ for } i = 1, 2, \cdots, k-1 \text{ and } [V_1, V_k] = 0$$

holds, where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

The dilations $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N (\lambda > 0)$ is a family of automorphisms of group $\mathbb{G}$ satisfying

$$\delta_\lambda(x_1, \cdots, x_N) = (\lambda^{\alpha_1} x_1, \cdots, \lambda^{\alpha_N} x_N),$$

here $1 = \alpha_1 = \cdots = \alpha_m < \alpha_{m+1} \leq \cdots \leq \alpha_N$ are integers and $m = \dim(V_1)$.

The subLaplacian operator $\mathcal{L} = \sum\limits_{j=1}^{m} X_j^2$ is the second-order partial differential operator on $\mathbb{G}$ and its intrinsic gradient $\nabla_{\mathcal{L}}$ associated with $\mathcal{L}$ can be written as follows

$$\nabla_{\mathcal{L}} = (X_1, \cdots, X_m),$$

where $\{X_1, \cdots, X_m\}$ is a family of vector fields to form a linear basis of the first layer of $\mathcal{G}$.

The curve $\gamma : [a, b] \to \mathbb{G}$ is called horizontal if $\gamma(a) = x, \gamma(b) = y \in \mathbb{G}$ and $\gamma'(t) \in V_1$ for all t. Define the Carnot-Caratheodory distance between x and y by

$$d_{CC}(x, y) = \inf_{\gamma} \int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}} dt,$$

where the infimum is taken over all horizontal curves γ connecting to x and y. Accordingly, the Carnot-Caratheodory ball is denoted by $B_{CC}(x, r) = \{y \in \mathbb{G} : d_{CC}(x, y) < r\}$. By the left invariant properties, we see that

$$d_{CC}(zx, zy) = d_{CC}(x, y),\ B_{CC}(x, r) = xB_{CC}(e, r),\ \ \forall x, y, z \in \mathbb{G} \text{ and } r > 0$$

and

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x, y),\ \ \forall x, y \in \mathbb{G} \text{ and } \forall \lambda > 0.$$

For $x \in \mathbb{G}$ and $r > 0$, we denote by $B(x, r) = \{y \in \mathbb{G} : \rho(y^{-1} \circ x) =| y^{-1} \circ x |< r\}$ the $\mathbb{G}$-ball with x and radius r, and by $B(e, r) = \{y \in \mathbb{G} : \rho(y) < r\}$ the open ball centered at the identity element e of $\mathbb{G}$ with radius r. Here the continuous function $\rho : \mathbb{G} \to [0, \infty)$ is a homogenous norm on $\mathbb{G}$ and satisfies $\rho(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ for all $x \in \mathbb{G}$. Moreover, there exists a constant $c \geq 1$ such that $\rho(xy) \leq c(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$. We remark that the pseudometric $\rho(x, y) =| x^{-1} \circ y |$ is equivalent to the metric d_{CC} in the following sense

$$C^{-1}\rho(x, y) \leq d_{CC}(x, y) \leq C\rho(x, y),\ \ \forall x, y \in \mathbb{G} \text{ and } \forall\, C > 1,$$

and satisfies

$$\rho(zx, zy) = \rho(x, y),\ D(x, r) = xD(e, r),\ \ \forall x, y, z \in \mathbb{G} \text{ and } r > 0,$$

where $D(x, r) = \{y \in \mathbb{G} : \rho(x, y) < r\}$ is the metric ball associated with ρ. For convenience, we will use d and $B(x, r)$ instead of d_{CC} and $B_{CC}(x, r)$, respectively.

According to the left translation and dilation, it is clearly to know that

$$| B(x, r) |= r^Q | B(x, 1) |= r^Q | B(0, 1) |,$$

where the homogeneous dimension Q of $\mathbb{G}$ is equivalent to $Q = \sum_{j=1}^{m} j \dim(V_j)$.

The classical generalized Morrey type space $\mathcal{L}^{p,\varphi}(\mathbb{G})$ on $\mathbb{G}$ is defined by the following norm

$$\| f \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})}:= \sup_{x\in\mathbb{G}, r>0} \varphi(x, r)^{-\frac{1}{p}} \| f \|_{L^p(B(x,r))}< \infty$$

for $0 \leq \lambda \leq Q$ and $1 \leq p \leq \infty$. Here $\varphi(x, r)$ belongs to the class $\beth = \beth(\mathbb{G} \times (0, \infty))$ of non-negative measurable functions on $\mathbb{G} \times [0, \infty)$, which are positive on $\mathbb{G} \times (0, \infty)$. If $\varphi(x, r) = r^\lambda$, then $\mathcal{L}^{p,\varphi}(\mathbb{G})$ is exactly the classical Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{G})$ for $0 \leq \lambda \leq Q$. For $\lambda = 0$ and $\lambda = Q$, we know that $\mathcal{L}^{p,0}(\mathbb{G}) = L^p(\mathbb{G})$ and $\mathcal{L}^{p,Q}(\mathbb{G}) = L^\infty(\mathbb{G})$, respectively. As for $\lambda < 0$ and $\lambda > Q$, we know $\mathcal{L}^{p,\lambda}(\mathbb{G}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$. Note that this definition of generalized Morrey type space $\mathcal{L}^{p,\varphi}(\mathbb{G})$ is slightly different from the Guliyev's one (refer to [16–18]).

Denote by $W\mathcal{L}^{p,\varphi}(\mathbb{G})$ the generalized weak Morrey space of all functions $f \in L^p_{\rm loc}(\mathbb{G})$ via

$$\| f \|_{W\mathcal{L}^{p,\varphi}(\mathbb{G})} := \sup_{x\in\mathbb{G}, r>0} \varphi(x,r)^{-\frac{1}{p}} \| f \|_{WL^p(B(x,r))} < \infty,$$

where $WL^p(B(x,r))$ is the weak L^p-space of measurable functions f on $B(x,r)$ with the norm

$$\begin{aligned} \| f \|_{WL^p(B(x,r))} \equiv \| f\chi_{B(x,r)} \|_{WL^p(\mathbb{G})} &:= \sup_{t>0} t \mid \{y \in B(x,r) : \mid f(y) \mid > t\} \mid^{\frac{1}{p}} \\ &= \sup_{t>0} t^{\frac{1}{p}} (f\chi_{B(x,r)})^*(t) < \infty, \end{aligned}$$

where g^* denotes the non-increasing rearrangement of the function g.

The vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ is defined as the spaces of all functions $f \in \mathcal{L}^{p,\varphi}(\mathbb{G})$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{G}} \varphi(x,r)^{-\frac{1}{p}} \| f \|_{L^p(B(x,r))} = 0. \tag{1.1}$$

Correspondingly, the vanishing generalized weak Morrey space $VW\mathcal{L}^{p,\varphi}(\mathbb{G})$ is defined as the spaces of all functions $f \in W\mathcal{L}^{p,\varphi}(\mathbb{G})$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{G}} \varphi(x,r)^{-\frac{1}{p}} \| f \|_{WL^p(B(x,r))} = 0.$$

Obviously, it is natural to impose on $\varphi(x,r)$ with the following conditions

$$\lim_{r\to 0} \sup_{x\in\mathbb{G}} \frac{r^Q}{\varphi(x,r)} = 0 \tag{1.2}$$

and

$$\inf_{r>1} \sup_{x\in\mathbb{G}} \varphi(x,r) > 0. \tag{1.3}$$

From conditions (1.2) and (1.3), we easily know that the bounded functions with compact support belong to $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ and $VW\mathcal{L}^{p,\varphi}(\mathbb{G})$.

In the paper, we firstly consider the multi-dimensional weighted Hardy operators as follows

$$\mathrm{H}^\alpha_\omega f(x) = \mid x \mid^{\alpha - Q} \omega(\mid x \mid) \int_{|y|<|x|} \frac{f(y)dy}{\omega(\mid y \mid)}, \ \ \mathcal{H}^\alpha_\omega f(x) = \mid x \mid^{\alpha} \omega(\mid x \mid) \int_{|y|>|x|} \frac{f(y)dy}{\mid y \mid^Q \omega(\mid y \mid)},$$

where $\alpha \geq 0$. In the sequel $\mathbb{G}$ with $Q = 1$, the Hardy operators above may be read with the versions

$$\mathrm{H}^\alpha_\omega f(x) = x^{\alpha-1}\omega(x) \int_0^x \frac{f(y)dy}{\omega(y)}, \ \ \mathcal{H}^\alpha_\omega f(x) = x^{\alpha}\omega(x) \int_x^\infty \frac{f(y)dy}{y\omega(y)}, \ x > 0.$$

If $\omega(t) = t^\beta$, then the operators above are denoted by

$$\mathrm{H}^\alpha_\beta f(x) =| x |^{\alpha+\beta-Q} \int_{|y|<|x|} \frac{f(y)dy}{| y |^\beta}, \ \mathcal{H}^\alpha_\beta f(x) =| x |^{\alpha+\beta} \int_{|y|>|x|} \frac{f(y)dy}{| y |^{\beta+Q}},$$

and the one-dimensional by

$$\mathrm{H}^\alpha_\beta f(x) = x^{\alpha+\beta-1} \int_0^x \frac{f(y)dy}{y^\beta}, \ \mathcal{H}^\alpha_\beta f(x) = x^{\alpha+\beta} \int_x^\infty \frac{f(y)dy}{y^{\beta+1}}, \ x > 0.$$

Besides, we also consider some operators as follows.

(1) For $f \in L^1_{\mathrm{loc}}(\mathbb{G})$, the fractional maximal operator $\mathcal{M}^\alpha f$ with order α of the function f is defined by

$$\mathcal{M}^\alpha_{\mathcal{L}} f = \sup_{r>0} | B(\cdot, r) |^{\frac{\alpha}{Q}-1} \int_{B(\cdot,r)} | f(y) | dy, \ 0 \le \alpha < Q,$$

where the supremum is taken over all the balls $B(\cdot, r)$ in $\mathbb{G}$. When $\alpha = 0$, $\mathcal{M}^\alpha$ is the centered Hardy-Littlewood maximal operator $\mathcal{M}$.

(2) The potential type operator with order α is denoted by

$$I^\alpha f = \int_{\mathbb{G}} I(\cdot, y) f(y) dy, \ 0 < \alpha < Q,$$

here $I(\cdot, y) = d(\cdot, y)^{\alpha-Q}$. Here we also call $I^\alpha f$ the $\mathbb{G}$-fractional integral with order α of f.

Let f be a non-negative function on $[0, \ell]$. If there exists a constant $C \ge 1$ such that $f(x) \le Cf(y)$ for all $x \le y$ or $x \ge y$, then f is named almost increasing or decreasing. Moreover, if the two almost increasing or decreasing functions f and g satisfy $c_1 f \le g \le c_2 f$ for $c_1, c_2 > 0$, then they are equivalent.

Definition 1.1 Let $0 < \ell \le \infty$.

Denote by $W = W([0, \ell])$ the class of continuous and positive functions $\phi(r)$ on $(0, \ell]$ such that the limit $\lim\limits_{r\to 0} \phi(r)$ exists and is finite.

Denote by $W_0 = W_0([0, \ell])$ the class of almost increasing functions $\phi(r) \in W$ on $(0, \ell)$.

Denote by $\overline{W} = \overline{W}([0, \ell])$ the class of functions $\phi(r) \in W$ such that $r^a \phi(r) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}$.

Denote by $\underline{W} = \underline{W}([0, \ell])$ the class of functions $\phi(r) \in W$ such that $r^{-b}\phi(r)$ is almost decreasing for some $b \in \mathbb{R}$.

In the rest of this paper, we will make some arrangement as follows. In Section 2, we will introduce some necessary lemmas. In Section 3, we will discuss our main theorems and their proofs.

2 Some Necessary Lemmas

In the section, we have something in mind to list the related lemmas. At first we provide two results with similar ones from Persson and Samko (see [30, Proposition 3.6, 3.8]) as well as Euclidean setting.

Lemma 2.1 For $1 \le p < \infty$, $0 < s \le p$ and $1 \le \ell \le \infty$, let $\nu(t) \in \overline{W}([0,\ell])$, $\nu(2t) \le C\nu(t)$, $\frac{\varphi^{\frac{s}{p}}(x,t)}{\nu} \in \underline{W}([0,\ell])$ for $x \in \mathbb{G}$. Then

$$\left(\int_{|z|<|y|} \frac{| f(z) |^s}{\nu(| z |)} dz\right)^{\frac{1}{s}} \le C\mathcal{A}(| y |) \| f \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})}, \quad 0 <| y |\le \ell,$$

where $C > 0$ does not depend on y and f, and

$$\mathcal{A}(r) = \left(\int_0^r t^{Q(1-\frac{s}{p})-1} \frac{\varphi^{\frac{s}{p}}(x,t)}{\nu(t)} dt\right)^{\frac{1}{s}} \quad \text{for } x \in \mathbb{G}.$$

Lemma 2.2 For $1 \le p < \infty$ and $0 \le s \le p$, let $\varphi(r) \ge Cr^Q$ and $\nu(t) \in \overline{W}(\mathbb{R}_+)$. Then

$$\left(\int_{|z|>|y|} | f(z) |^s \nu(| z |) dz\right)^{\frac{1}{s}} \le C\mathcal{B}(| y |) \| f \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})}, \quad y \ne 0,$$

where $C > 0$ does not depend on y and f, and

$$\mathcal{B}(r) = \left(\int_r^\infty t^{Q(1-\frac{s}{p})-1} \varphi^{\frac{s}{p}}(x,t)\nu(t) dt\right)^{\frac{1}{s}} \quad \text{for } x \in \mathbb{G}.$$

Next we will introduce the Hardy-Littlewood-Sobolev theorem for subLaplacians, which was proved by Bonfiglioli et al. in [8].

Lemma 2.3 (see [8], Theorem 5.9.1) Let $\mathcal{L}$ be a subLaplacian on the homogeneous Carnot group $\mathbb{G}$ and d be an $\mathcal{L}$-gauge. Suppose $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then there exists a positive constant $C = C(\alpha, p, \mathbb{G}, d, \mathcal{L})$ such that

$$\| I^\alpha f \|_{\mathcal{L}^q(\mathbb{G})} \le C \| f \|_{\mathcal{L}^p(\mathbb{G})} \quad \text{for every } f \in L^p(\mathbb{G}),$$

here the notation $\| \cdot \|_{L^r}$ denotes the L^r norm in $\mathbb{G} = \mathbb{R}^N$ with respect to the Lebesghe measure.

3 Statements of Main Results

In the section we start to sate our main theorems. Firstly we consider the boundedness of weighted Hardy operator in the vanishing Morrey type space.

Theorem 3.1 Let $1 \le p, q < \infty$ and φ satisfy conditions (1.1)–(1.3).

(I) Suppose that $\omega \in \overline{W}([0,\ell])$, $\omega(2t) \le C\omega(t)$, $\frac{1}{\omega} \in \underline{W}([0,\ell])$. If

$$\sup_{x\in\mathbb{G}, r>0} \frac{1}{\varphi(x,r)} \int_{B(x,r)} \omega^q(| y |) | y |^{q(\alpha-Q)} \left(\int_0^{|y|} \frac{t^{\frac{Q}{p'}-1}\varphi^{\frac{1}{p}}(y,r)}{\omega(t)} dt\right)^q dy < \infty, \tag{3.1}$$

then the operator H^α_ω is bounded from $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ to $V\mathcal{L}^{q,\varphi}(\mathbb{G})$.

(II) Suppose that $\omega \in \overline{W}([0,\ell])$ and $\omega(2t) \le C\omega(t)$ or $\frac{1}{\omega} \in \underline{W}([0,\ell])$. If

$$\sup_{x\in\mathbb{G}, r>0} \frac{1}{\varphi(x,r)} \int_{B(x,r)} \omega^q(| y |) | y |^{q\alpha} \left(\int_{|y|}^\infty \frac{t^{-\frac{Q}{p}-1}\varphi^{\frac{1}{p}}(y,r)}{\omega(t)} dt\right)^q dy < \infty, \tag{3.2}$$

then the operator $\mathcal{H}^{\alpha}_{\omega}$ is bounded from $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ to $V\mathcal{L}^{q,\varphi}(\mathbb{G})$.

Proof Put $s=1$ and $\nu(t)=\omega(t)$ in Lemma 2.1. Then

$$\mid \mathrm{H}^{\alpha}_{\omega}f(y)\mid\leq C\omega(\mid y\mid)\mid y\mid^{\alpha-Q}\int_0^{|y|}\frac{t^{\frac{Q}{p'}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}dt\parallel f\parallel_{\mathcal{L}^{p,\varphi}(\mathbb{G})}$$

for $y\in B(x,r)$, and we obtain

$$\parallel \mathrm{H}^{\alpha}_{\omega}f\parallel^q_{L^q(B(x,r))}\lesssim\int_{B(x,r)}\omega^q(\mid y\mid)\mid y\mid^{q(\alpha-Q)}\left(\int_0^{|y|}\frac{t^{\frac{Q}{p'}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}dt\right)^q dy. \tag{3.3}$$

That is to say

$$\parallel \mathrm{H}^{\alpha}_{\omega}f\parallel^q_{\mathcal{L}^{q,\varphi}(\mathbb{G})}\lesssim\sup_{x\in\mathbb{G},r>0}\frac{1}{\varphi(x,r)}\int_{B(x,r)}\omega^q(\mid y\mid)\mid y\mid^{q(\alpha-Q)}\left(\int_{|y|}^{\infty}\frac{t^{\frac{Q}{p'}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}dt\right)^q dy. \tag{3.4}$$

Hence $\mathrm{H}^{\alpha}_{\omega}f\in\mathcal{L}^{q,\varphi}(\mathbb{G})$.

On the other hand, by inequality (3.3) and conditions (1.1)–(1.2), we get that

$$\lim_{r\to 0}\sup_{x\in\mathbb{G}}\varphi^{-\frac{1}{q}}(x,t)\parallel \mathrm{H}^{\alpha}_{\omega}f\parallel_{L^q(B(x,r))}=0,$$

which implies $\mathrm{H}^{\alpha}_{\omega}f\in V\mathcal{L}^{q,\varphi}(\mathbb{G})$, i.e., the operator $\mathrm{H}^{\alpha}_{\omega}$ is bounded from $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ to $V\mathcal{L}^{q,\varphi}(\mathbb{G})$.

Similarly, applying Lemma 2.2 into $\mathcal{H}^{\alpha}_{\omega}$, we have

$$\mid \mathcal{H}^{\alpha}_{\omega}f(y)\mid\leq C\omega(\mid y\mid)\mid y\mid^{\alpha}\int_{|y|}^{\infty}\frac{t^{-\frac{Q}{p}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}dt\parallel f\parallel_{\mathcal{L}^{p,\varphi}(\mathbb{G})}$$

for $y\in B(x,r)$, and we know that

$$\parallel \mathcal{H}^{\alpha}_{\omega}f\parallel^q_{L^q(B(x,r))}\lesssim\int_{B(x,r)}\omega^q(\mid y\mid)\mid y\mid^{q\alpha}\left(\int_{|y|}^{\infty}\frac{t^{-\frac{Q}{p}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}dt\right)^q dy. \tag{3.5}$$

Therefore

$$\parallel \mathcal{H}^{\alpha}_{\omega}f\parallel^q_{\mathcal{L}^{q,\varphi}(\mathbb{G})}\lesssim\sup_{x\in\mathbb{G},r>0}\frac{1}{\varphi(x,r)}\int_{B(x,r)}\omega^q(\mid y\mid)\mid y\mid^{q\alpha}\left(\int_{|y|}^{\infty}\frac{t^{-\frac{Q}{p}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}dt\right)^q dy, \tag{3.6}$$

and it follows $\mathcal{H}^{\alpha}_{\omega}f\in\mathcal{L}^{q,\varphi}(\mathbb{G})$. Moreover, with inequality (3.5) and conditions (1.1)–(1.3), we obtain that $\lim\limits_{r\to 0}\sup\limits_{x\in\mathbb{G}}\varphi^{-\frac{1}{q}}(x,t)\parallel \mathcal{H}^{\alpha}_{\omega}f\parallel_{L^q(B(x,r))}=0$, and imply $\mathcal{H}^{\alpha}_{\omega}f\in V\mathcal{L}^{q,\varphi}(\mathbb{G})$. Then we conclude the operator $\mathcal{H}^{\alpha}_{\omega}$ is also bounded from $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ to $V\mathcal{L}^{q,\varphi}(\mathbb{G})$.

Second, we will deal with the boundedness of fractional maximal operator and potential operator in the vanishing generalized Morrey space. When $\varphi(x,r)=r^{\lambda}$ and $\psi(x,r)=r^{\mu}$, we may obtain Corollary 3.1.

Theorem 3.2 Let $\mathcal{L}$ be a subLaplacian on the homogeneous Carnot group $\mathbb{G}$ and d be an $\mathcal{L}$-gauge. Suppose $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$ and $\varphi,\psi\in\daleth$. If

$$C_\delta:=\int_\delta^\infty \frac{\sup_{x\in\mathbb{G}}\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}<\infty \tag{3.7}$$

for every $\delta>0$ and

$$\int_r^\infty \frac{\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}dt\le C_0\frac{\psi^{\frac{1}{q}}(x,r)}{r^{\frac{Q}{q}}}, \tag{3.8}$$

where C_0 doesn't depend on $x\in\mathbb{G}$ and $r>0$, then there exists a positive constant $C=C(\alpha,p,\mathbb{G},d,\mathcal{L})$ such that $\parallel \mathcal{M}^\alpha_{\mathcal{L}}f\parallel_{\mathcal{L}^{q,\psi}(\mathbb{G})}\le C\parallel f\parallel_{\mathcal{L}^{p,\varphi}(\mathbb{G})}$, $\parallel I^\alpha f\parallel_{\mathcal{L}^{q,\psi}(\mathbb{G})}\le C\parallel f\parallel_{\mathcal{L}^{p,\varphi}(\mathbb{G})}$. Moreover, if φ and ψ also satisfy conditions (1.1)–(1.3), then the operators $\mathcal{M}^\alpha_{\mathcal{L}}$ and I^α are bounded from $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ to $V\mathcal{L}^{q,\psi}(\mathbb{G})$.

Here we firstly recall the definition of $\mathcal{L}$-gauge d. If d is a homogeneous symmetric norm being smooth out of the origin and satisfying $\mathcal{L}(d^{2-Q})=0$ in $\mathbb{G}\setminus\{0\}$, then we call d $\mathcal{L}$-gauge on $\mathbb{G}$ (see Section 5.4 in [8]).

Proof As is well known, $M^\alpha f\le CI^\alpha(\mid f\mid)$, and we only consider the case for I^α. At first we divide the function f into the expression $f=f_1+f_2$ so that $I^\alpha f=I^\alpha f_1+I^\alpha f_2$, where $f_1=f\chi_{B(x,2r)}$ and $f_2=f\chi_{\mathbb{G}\setminus B(x,2r)}$. From Lemma 2.3, we see that

$$\parallel I^\alpha f_1\parallel_{L^q(B(x,r))}\le\parallel I^\alpha f_1\parallel_{L^q(\mathbb{G})}\le C\parallel f_1\parallel_{L^p(\mathbb{G})}=C\parallel f\parallel_{L^p(B(x,2r))} . \tag{3.9}$$

Then

$$\parallel I^\alpha f_1\parallel_{L^q(B(x,r))}\lesssim r^{\frac{Q}{q}}\int_r^\infty\frac{\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}dt\parallel f\parallel_{\mathcal{L}^{p,\varphi}(\mathbb{G})} . \tag{3.10}$$

Since there exist two constants $c_1,c_2\ge 1$ so that the inequality $\frac{d(y,z)}{c_1}\le d(x,y)\le c_2d(y,z)$ holds for $z\in B(x,r)$ and $y\in\mathbb{G}\setminus B(x,2r)$, and therefore

$$\parallel I^\alpha f_2\parallel_{L^q(B(x,r))}\le C\int_{\mathbb{G}\setminus B(x,2r)}\frac{\mid f(y)\mid dy}{d(x,y)^{Q-\alpha}}\parallel\chi_{B(x,r)}\parallel_{L^q(\mathbb{G})} .$$

Put $\gamma>\frac{Q}{q}$. Since $\parallel\chi_{B(x,R)}\parallel_{L^p(\mathbb{G})}\sim R^{\frac{Q}{p}}$, by the Hölder inequality and Fubini's theorem, it follows that

$$\begin{aligned}\int_{\mathbb{G}\setminus B(x,2r)}\frac{\mid f(y)\mid dy}{d(x,y)^{Q-\alpha}}&\le\gamma\int_{\mathbb{G}\setminus B(x,2r)}\frac{\mid f(y)\mid dy}{d(x,y)^{Q-\alpha-\gamma}}\int_{d(x,y)}^\infty t^{-\gamma-1}dt\\&=\gamma\int_{2r}^\infty t^{-\gamma-1}dt\int_{\{y\in\mathbb{G}:2r\le d(x,y)\le t\}}\frac{\mid f(y)\mid dy}{d(x,y)^{Q-\alpha-\gamma}}\\&\le C\int_{2r}^\infty t^{-\gamma-1}\parallel f\parallel_{L^p(B(x,t))}\parallel d(x,\cdot)^{\alpha-Q+\gamma}\parallel_{L^{p'}(B(x,t))}dt\\&\le C\int_r^\infty t^{-\frac{Q}{q}-1}\parallel f\parallel_{L^p(B(x,t))}dt,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, which implies

$$\| I^{\alpha}f_2 \|_{L^q(B(x,r))}\leq Cr^{\frac{Q}{q}}\int_r^{\infty} t^{-\frac{Q}{q}-1} \| f \|_{L^p(B(x,t))} dt.$$

Hence

$$\| I^{\alpha}f_2 \|_{L^q(B(x,r))}\lesssim r^{\frac{Q}{q}}\int_r^{\infty} \frac{\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}dt \| f \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})} . \tag{3.11}$$

From inequalities (3.10) and (3.11), we see that

$$\| I^{\alpha}f \|_{L^q(B(x,r))}\lesssim r^{\frac{Q}{q}}\int_r^{\infty} \frac{\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}dt \| f \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})} .$$

By inequalities (3.7)–(3.8) and conditions (1.1)–(1.3), it follows that $\| I^{\alpha}f \|_{\mathcal{L}^{q,\psi}(\mathbb{G})}\leq C \| f \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})}$, and the potential operator I^{α} is bounded in the vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}(\mathbb{G})$ to another vanishing generalized Morrey space $V\mathcal{L}^{p,\psi}(\mathbb{G})$.

Corollary 3.1 Let $\mathcal{L}$ be a subLaplacian on the homogeneous Carnot group $\mathbb{G}$ and d be an $\mathcal{L}$-gauge. Suppose $0<\alpha<Q$, $1<p<\frac{Q-\lambda}{\alpha}$ and $0<\lambda<Q-\alpha p$. If $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, then the operators $\mathcal{M}_{\mathcal{L}}^{\alpha}$ and I^{α} are bounded from $V\mathcal{L}^{p,\lambda}(\mathbb{G})$ to $V\mathcal{L}^{q,\mu}(\mathbb{G})$, where $\frac{\mu}{q}=\frac{\lambda}{p}$.

In Section 5.3 in [8], the function Γ is defined as the fundamental solution for subLaplacian $\mathcal{L}$ on homogeneous Carnot group. That is to say, $-\mathcal{L}(\Gamma(y^{-1}\circ\cdot))=\mathrm{Dirac}_y$ holds in the weak sense of distribution, where Dirac_y is the dirac measure supported at y. Now we intend to study the Sobolev-Stein embedding theorem and accordingly give the the Morrey-Sobolev-Stein embedding theorem in generalized Morrey space on homogenous Carnot group.

Theorem 3.3 Let $\mathcal{L}$ be a subLaplacian on the homogeneous Carnot group $\mathbb{G}$ of homogenous dimension Q and d be an $\mathcal{L}$-gauge. Suppose $0<\alpha<Q$ and $\varphi,\psi\in\daleth$. If

$$C_{\delta}:=\int_{\delta}^{\infty}\frac{\sup_{x\in\mathbb{G}}\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}<\infty \tag{3.12}$$

for every $\delta>0$ and

$$\int_r^{\infty}\frac{\varphi^{\frac{1}{p}}(x,t)dt}{t^{1+\frac{Q}{q}}}dt\leq C_0\frac{\psi^{\frac{1}{q}}(x,r)}{r^{\frac{Q}{q}}},$$

where C_0 doesn't depend on $x\in\mathbb{G}$ and $r>0$, then there exists a positive constant $C=C(\alpha,p,\mathbb{G},d,\mathcal{L})$ such that

$$\| u \|_{\mathcal{L}^{q,\psi}(\mathbb{G})}\leq C \| \nabla_{\mathcal{L}}u \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})} \text{ for each } u\in C_0^{\infty}(\mathbb{G},\mathbb{R}),$$

where $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$.

Proof Applying the representation formula of solution of Dirichlet problem for sub-Laplacian to $u\in\mathcal{C}_0^{\infty}(\mathbb{G},\mathbb{R})$, by integrating by parts, we see that

$$u(x)=\int_{\mathbb{G}}(\nabla_{\mathcal{L}}\Gamma)(x^{-1}\circ y)\nabla_{\mathcal{L}}u(y)dy.$$

Since $\nabla_{\mathcal{L}}$ is smooth in $\mathbb{G} \setminus \{0\}$ and δ-homogeneous of degree zero, there exists a suitable constant C depending only on $\mathcal{L}$ so that

$$| \nabla_{\mathcal{L}}\Gamma |=| \beta_d \nabla_{\mathcal{L}}(d^{2-Q}) | \leq C d^{1-Q},$$

where β_d is a constant depended on d. Consequently,

$$| u(x) | \leq C \int_{\mathbb{G}} | \nabla_{\mathcal{L}} u(y) | \, d^{1-Q}(x,y) dy = C I^1(| \nabla_{\mathcal{L}} u |)(x).$$

Therefore, from Theorem 3.2, we obtain that

$$\| u \|_{\mathcal{L}^{q,\psi}(\mathbb{G})} \leq C \| I^1(| \nabla_{\mathcal{L}} u |) \|_{\mathcal{L}^{q,\psi}(\mathbb{G})} \leq C \| \nabla_{\mathcal{L}} u \|_{\mathcal{L}^{p,\varphi}(\mathbb{G})},$$

which is exactly the desired results to prove.

Set $\varphi(x,r) = r^{\lambda}$ and $\psi(x,r) = r^{\mu}$ in Theorem 3.3. It is known that $C_0^{\infty}(\mathbb{G},\mathbb{R})$ is sense in $\mathcal{L}_0^{p,\lambda}(\mathbb{G},\mathbb{R})$ but not $\mathcal{L}^{p,\lambda}(\mathbb{G},\mathbb{R})$. Hence by Theorem 3.3, we may easily infer the next corollary.

Corollary 3.2 Let $\mathcal{L}$ be a subLaplacian on the homogeneous Carnot group $\mathbb{G}$ and d be an $\mathcal{L}$-gauge. Suppose $0 < \alpha, \lambda < Q$ and $1 < p < \frac{Q-\lambda}{\alpha}$. Then there exists a positive constant $C = C(\alpha, p, \mathbb{G}, d, \mathcal{L})$ such that

$$\| u \|_{\mathcal{L}^{q,\mu}(\mathbb{G})} \leq C \| \nabla_{\mathcal{L}} u \|_{\mathcal{L}^{p,\lambda}(\mathbb{G})} \text{ for each } u \in \mathcal{L}_0^{p,\lambda}(\mathbb{G},\mathbb{R}),$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q} \text{ and } \frac{\mu}{q} = \frac{\lambda}{p}.$$

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关于齐次Carnot群上广义Morrey 空间中一些性质

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摘要: 本文研究了关于Heisenberg群上的广义Morrey空间和Carnot群上的Lebesgue空间中Riesz位势算子或者分数阶极大算子的行为. 根据Heisenberg群中抽象调和分析方法以及subLaplacian算子的Dirichlet问题解的表示公式, 本文主要给出了关于齐次Carnot群$\mathbb{G}$上消失的广义Morrey 空间$V\mathcal{L}^{p,\varphi}(\mathbb{G})$ 中的加权Hardy算子、分数阶极大算子和分数阶位势算子的有界性刻画. 进而也得到无消失模的广义Morrey空间上Morrey位势的浸入不等式. 所有这些结果推广了关于Heisenberg群上的广义Morrey空间和Carnot群上的Lebesgue空间中的相关结论.

关键词: Carnot群; 加权Hardy算子; 分数阶极大算子; 分数阶位势算子; 广义Morrey空间

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